Engineering Mathematics - III

S. K. Kate
S. D. Navare
S. R. Bandewar
A. R. Tambe
R. S. Acharya
V. D. Pawar

TECHNICAL PUBLICATIONS PUNE

Technical Publications Pune™

Strictly According to the Revised Syllabus of

PUNE UNIVERSITY - 2003 Course

Engineering Mathematics - III

[207002] S.E. (Mech./SW Production/SW Industrial Engg.) - Semester I
Metallurgy Engineering - Semester II

S. K. Kate
Sr. Lecturer, Mathematics
Vishwakarma Institute of
Information Technology
Pune

A. R. Tambe
Lecturer (Selection Grade)
Pravara Rural Engg. College
Loni

Mrs. S. D. Navare
Head, First Year Engineering
Sinhagad College of Engineering
Pune

R. S. Acharya
HOD, First Year Incharge
Asst. Prof., Mathematics Dept.
Modern College of Engineering
Pune

S. R. Bandewar
Sr. Lecturer, Mathematics
Vishwakarma Institute of Technology
Pune

V. D. Pawar
M.Sc.
Lecturer, Mathematics
All India Shri Shivaji Memorial Society's
Women's College of Engineering
Pune

Price Rs. 500/-

Visit us at : **www.vtubooks.com**

Technical Publications Pune

Engineering Mathematics - III

First Edition : August 2006

ISBN 81 - 8431 - 060 - 9

Published by :

Technical Publications Pune®

#1, Amit Residency, 412, Shaniwar Peth,
Pune - 411 030, India.
Phone : (020) 24495496 , Tele/Fax : (020) 24495497
Email : technical@vtubooks.com

Printer :

Alert DTPrinters
Omkar Compound,
Sr.no. 10/3,
Pune Sinhagad Road,
Pune - 411 041

Head office

Technical Publications
1, Amit Residency, 412,Shaniwar Peth, Pune - 411030, India
(020) 24495496, 66023931,Tele/Fax : (020) 24495497

Bangalore

Pragati Book House
1676-14th, Main Road, Prakash Nagar,
Bangalore - 560021, India
Phone : (080) 23324037, Fax : 23324848

Jalgaon

Pragati Books Pvt. Ltd.
34, V. V. Golani Market, Navi Peth,
Jalgaon - (02587) 425 001, India
Phone : 220395

Mumbai

Pragati Books Pvt. Ltd.
385, S.V.P. Road, Rasdhara,
Co-op. Society, Girgaum, Mumbai - 400 004
India. Phone : (022) 23869976, 23856339

Nagpur

Pratibha Book Distributors
Shop No. 3, 1st Floor Lokratna, Commercial Complex,
Zansi Rani Square, Sitabuldi,
Nagpur - 440012, India. Phone : (0712) 547129

Hyderabad

Nirali Book House
22,Shivam Enclave, 4-5 - 947,
Badi Chavadi, Hyderabad - 500095
India. Phone : (040) 5545313

Nasik

Pragati Books Pvt. Ltd.
741, Gaydhani Sankul, First Floor,
Raviwar Karanja, Nasik - 422001
India. Phone : (0253) 2506438

Chennai

Pragati Books
9/1, Montieth Road, Behind Taas Mahal,
Egmore, Chennai - 600 008,
India. Ph. - (044) 5518 3535 Mobile - 94440 01782

Pune

Pragati Books Pvt. Ltd.
119, Budhwar Peth,
Jogeshwari Mandir lane, Pune - 411 002, India
Phone : (020) 24452044

Technical Books Distributor
B - Ground floor, 'QUANTA ANNAM'
24th Street, H-965/966, Opp. U C MAS
Anna Nagar (W), Chennai - 600040, India.
Phone : (044) 26161903, Mobile No. : 9840324419

Pragati Impex (Export Division)
21, Amit Residency, 412, Shaniwar Peth,
Pune - 411030, India
+919845021552, +919370286000

This edition can be exported from India only.

Table of Contents

Chapter 3 : Applications of Differential Equations (3 - 1) to (3 - 34)

Chapter 4 : Applications of Partial Differential Equations (4 - 1) to (4 - 68)

Chapter 5 : Matrices (5 - 1) to (5 - 32)

Chapter 6 : Laplace Transform (6 - 1) to (6 - 88)

Chapter 12 : Vector Differentiation (12 - 1) to (12 - 134)

1 Linear Differential Equations with Constant Coefficients

1.1 Linear Differential Equations With Constant Coefficients

An equation involving derivatives is known as the differential equation.

The general form of linear differential equation with constant coefficient is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = \phi(x) \qquad \ldots (1)$$

where $a_0, a_1, a_2 \ldots a_n$ are all constants and $\phi(x)$ is any function of x.

Let $D = \frac{d}{dx}$ then equation becomes

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \ldots + a_{n-1} D_y + a_n y = \phi(x)$$

$$(a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n) y = \phi(x)$$

$$f(D)\, y = X \qquad \ldots (2)$$

where f (D) is a polynomial in D of degree n and X is any function of x.

f (D) y = 0 is known as Associated equation

and f (D) = 0 is known as Auxiliary equation.

The solution of equation (2) involves two parts

1) Complementary Function (C.F.)

2) Particular Integral (P.I.)

y = C.F. + P.I. gives the complete solution of the differential equation.

1.2 Methods for Finding C.F.

To find C.F. find the roots of Auxiliary equation f (D) = 0.

I) If the roots are real and distinct :

If $m_1, m_2, m_3 \ldots$ are the real roots then $CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots$

II) If the roots are real and repeated

If m_1, m_1 are repeated roots of f (D) = 0 then $CF = (C_1 + C_2 x)\, e^{m_1 x}$

If m_1, m_1, m_1 are the roots of f (D) = 0 then $CF = (C_1 + C_2 x + C_3 x^2)\, e^{m_1 x}$

III) If the roots are complex : (complex roots always occur in conjugate pair)

i.e. if $\alpha + i\beta$ is one root then $\alpha - i\beta$ will be another root, then

C.F. = $e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x]$

IV) If the roots are complex repeated

i.e. $\alpha \pm i\beta$ and $\alpha \pm i\beta$ are the roots then

C.F. = $e^{\alpha x}\left[(C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x\right]$

Note :

1) The no. of arbitrary constants present in the C.F. they must be equal in number with the order of the differential equation.

2) If $()^2$ is present in f (D) then the root is repeated twice.

If $()^3$ is present in f (D) then the root is repeated thrice.

3) For the cubic equation apply the synthetic division.

4) For the 4^{th} power equation we may get all complex roots, in such a case synthetic division is not applicable. Here adjust the perfect square.

5) For $aD^2 + bD + C = 0$ (quadratic)

Use $D = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

6) Rules for synthetic division : If sum of all coefficient is zero then D = 1 is the root. If sums of alternate coefficients are equal then D = – 1 is the root.

If D = 1 and – 1 are not the roots then try by the divisors of the last number.

7) Synthetic division is the method applicable only for finding real roots.

1.3 Find the C.F. of the following f (D) = 0

1) $D^3 - 7D - 6 = 0$

– 1	1	0	– 7	– 6
		– 1	1	6
	1	– 1	– 6	$\boxed{0}$

Sums of alternates are equal $\therefore D = -1$.

$\therefore (D+1)(D^2 - D - 6) = 0$

$\therefore (D+1)(D-3)(D+2) = 0$

$\therefore D = -1,\ D = 3,\ D = -2$ are real roots.

$\therefore \quad \text{C.F.} = C_1 e^{-x} + C_2 e^{3x} + C_2 e^{-2x}$

2) $D^3 + 7D^2 + 16D + 10 = 0$

Sums of alternates are equal $\therefore$ D = – 1.

$$\therefore \qquad (D+1)(D^2+6D+10) = 0$$

– 1	1	7	16	10
		– 1	– 6	– 10
	1	6	10	$\boxed{0}$

$D + 1 = 0,\ D^2+6D+10 = 0$

$$D = \frac{-6\pm\sqrt{36-40}}{2}$$

$$D = \frac{-6\pm\sqrt{-4}}{2}$$

$$= -3\pm i$$

$$\alpha = -3,\ \beta = 1$$

$\therefore$ One root is real and two roots are complex.

$$\text{C.F.} = C_1e^{-x}+e^{-3x}(C_2\cos x+C_3\sin x)$$

3) $$D^4-a^4 = 0$$

$$(D^2-a^2)(D^2+a^2) = 0$$

$$(D+a)(D-a)(D^2+a^2) = 0$$

$$D = -a,\ D = a$$

$$D^2 = -a^2$$

$$D = \pm ia$$

$$\alpha = 0,\ \beta = a$$

Two roots are real and two roots are complex.

$$\text{C.F.} = C_1e^{-ax}+C_2e^{ax}+e^{0x}(C_3\cos ax+C_4\sin ax)$$

4) $D^3+3D^2+3D+1 = 0$

Sums of alternates are equal $\therefore$ D = – 1

– 1	1	3	3	1
		– 1	– 2	– 1
	1	2	1	$\boxed{0}$

$$(D+1)^3 = 0$$

$$(D+1)(D^2+2D+1) = 0$$

$$(D+1)(D+1)^2 = 0$$

$$D = -1, -1, -1$$

$$\text{C.F.} = (C_1+C_2x+C_3x^2)\, e^{-x}$$

5) $D^3+D^2+D+1 = 0$

Sums of alternates are equal $\therefore$ D = – 1

– 1	1	1	1	1
		– 1	0	– 1
	1	0	1	$\boxed{0}$

$$(D+1)\ (D^2+1) = 0$$

$$D = -1,\ D^2 = -1 \Rightarrow D = -1,\ D = \pm i$$

One root is real and two roots are complex.

$$\text{C.F.} = C_1 e^{-x} + e^{0x}\ (C_1 \cos x + C_2 \sin x)$$

6) $D^6 - 6D^5 + 12D^4 - 6D^3 - 9D^2 + 12D - 4 = 0$

Sum of all coefficients is zero $\therefore$ D = 1

1	1	– 6	12	– 6	– 9	12	– 4
		1	– 5	7	1	– 8	4
1	1	– 5	7	1	– 8	4	$\boxed{0}$
		1	– 4	3	4	– 4	
1	1	– 4	3	4	– 4	$\boxed{0}$	
		1	– 3	0	+ 4		
– 1	1	– 3	0	4	$\boxed{0}$		
		– 1	+ 4	– 4			
	1	– 4	+ 4	$\boxed{0}$			

$$(D-1)^3\ (D+1)\ (D^2-4D+4) = 0$$

$$(D-1)^3\ (D+1)\ (D-2)^2 = 0$$

$$D = 1, 1, 1, -1, 2, 2$$

$$\therefore \quad \text{C.F.} = (C_1+C_2x+C_3x^2)\, e^{x} + C_4 e^{-x} + (C_5+C_6x)\, e^{2x}$$

7) $D^3+D^2-D-1 = 0$

$$(D-1)(D^2+2D+1) = 0$$

$$(D-1)(D+1)^2 = 0$$

1	1	1	− 1	− 1
		1	2	1
	1	2	1	$\boxed{0}$

$$D = 1, -1, -1$$

$$\text{C.F.} = C_1 e^x + (C_2 + C_3 x)\ e^{-x}$$

8) $$D^3 - 5D^2 + 8D - 4 = 0$$

$$(D-1)(D^2-4D+4) = 0$$

$$(D-1)(D-2)^2 = 0$$

1	1	− 5	8	− 4
		1	− 4	4
	1	− 4	4	$\boxed{0}$

$$D = 1, 2, 2$$

$$\text{C.F.} = C_1 e^x + (C_2 + C_3 x)\, e^{2x}$$

9) $$D^4 - 1 = 0$$

$$(D^2+1)(D^2-1) = 0$$

$$D^2 = -1, \quad D^2 = 1$$

$$D = \pm i \qquad D = \pm 1$$

Two roots are real and two roots are complex.

$\therefore$ $$\text{C.F.} = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$$

10) $$(D-1)^2 (D^2+1)^2 = 0$$

$$D = 1, 1, \qquad D^2 = -1$$

$$D = \pm i \ \text{ repeated}$$

$$\text{C.F.} = (C_1 + C_2 x)\, e^x + \left[(C_3 + C_4 x) \cos x + (C_5 + C_6 x) \sin x\right]$$

11) $$D^4 + 8D^2 + 16 = 0$$

$$(D^2+4)^2 = 0$$

$$D^2 + 4 = 0 \qquad \text{twice}$$

$$D^2 = -4$$

$$D = \pm 2i \quad \text{repeated complex}$$

$$\text{C.F.} = e^{0x}\left[(C_1 + C_2x)\cos 2x + (C_3 + C_4x)\sin 2x\right]$$

12) $D^4 - 2D^3 + D^2 = 0$

$$D^2(D^2 - 2D + 1) = 0$$

$$D^2(D-1)^2 = 0$$

$$D^2 = 0,\ (D-1)^2 = 0$$

$$D = 0, 0, 1, 1$$

$$\text{C.F.} = (C_1 + C_2x)\,e^{0x} + (C_3 + C_4x)\,e^x$$

13) $D^3 - 2D + 4 = 0$

$$(D+2)\,(D^2 - 2D + 2) = 0$$

− 2	1	0	− 2	4
		− 2	4	− 4
	1	− 2	2	$\boxed{0}$

$$D = -2,\ D = \frac{+2 \pm \sqrt{4-8}}{2}$$

$$D = -2,\ D = 1 \pm i$$

$$\text{C.F.} = C_1e^{-2x} + e^{+x}\,(C_2\cos x + C_3\sin x)$$

14) $D^5 - D = 0$

$$D\,(D^4 - 1) = 0$$

$$D = 0,\ (D^2 - 1)\ (D^2 + 1) = 0$$

$$(D-1)\,(D+1)\,(D^2 + 1) = 0$$

$$D = 0,\ D = 1,\ D = -1,\ D^2 = -1$$

$$\text{C.F.} = C_1e^{0x} + C_2e^x + C_3e^{-x} + e^{0x}\,(C_4\cos x + C_5\sin x)$$

15) $D^3 - 3D^2 + 4$

− 1	1	− 3	0	4
		− 1	− 4	− 4
	1	− 4	4	$\boxed{0}$

$$(D-2)^2\,(D+1) = 0$$

$$\text{C.F.} = C_1e^{-x} + (C_2 + C_3x)\,e^{2x}$$

16) $D^3 - D^2 + 3D + 5 = 0$

$$(D+1)(D^2 - 2D + 5) = 0$$

– 1	1	– 1	3	5
		– 1	2	– 5
	1	– 2	5	$\boxed{0}$

$$D = \frac{+2 \pm \sqrt{4-20}}{2}$$

$$D = -1,\ D = 1 \pm i2$$

$$\text{C.F.} = C_1 e^{-x} + e^{x}(C_2 \cos 2x + C_3 \sin 2x)$$

17) $D^3 + 7D^2 + 16D + 10 = 0$

$$(D+1)\ (D^2 + 6D + 10) = 0$$

– 1	1	7	16	10
		– 1	– 6	– 10
	1	6	10	$\boxed{0}$

$$D^2 + 6D + 10 = 0$$

$$D = \frac{-6 \pm \sqrt{36-40}}{2}$$

$$= -3 \pm i$$

$$\therefore \quad D = -1, \quad D = -3 \pm i$$

$$\text{C.F.} = C_1 e^{-x} + e^{-3x}\ (C_2 \cos x + C_2 \sin x)$$

18) $D^3 - 3D^2 + 3D - 1 = 0$

$$(D-1)^3 = 0$$

$$D = 1,\ 1,\ 1$$

$$\text{C.F.} = (C_1 + C_2 x + C_3 x^2)\ e^{+x}$$

19) $D^4 - 7D^2 + 12 = 0$

$$(D^2 - 3)\ (D^2 - 4) = 0$$

$$(D + \sqrt{3})\ (D - \sqrt{3})\ (D-2)\ (D+2) = 0$$

$$\text{C.F.} = C_1 e^{-\sqrt{3}x} + C_2 e^{\sqrt{3}x} + C_3 e^{2x} + C_4 e^{-2x}$$

20) $D^4 - 10D^2 + 9 = 0$

$$(D^2 - 9)(D^2 - 1) = 0$$

$$(D-3)(D+3)(D-1)(D+1) = 0$$

$$D = 3, -3, 1, -1$$

$$\text{C.F.} = C_1 e^{-3x} + C_2 e^{3x} + C_3 e^{x} + C_4 e^{-x}$$

21) $D^4 - 2D^3 + 4D^2 + 2D - 5 = 0$

− 1	1	− 2	4	2	− 5
		− 1	3	− 7	5
+ 1	1	− 3	7	− 5	$\boxed{0}$
		+ 1	− 2	5	
	1	− 2	5	$\boxed{0}$	

$$(D-1)(D+1)(D^2 - 2D + 5) = 0$$

$$D = 1, -1, \qquad D = \frac{2 \pm \sqrt{4-20}}{2}$$

$$D = 1, -1, \qquad D = 1 \pm 2i$$

$$\text{C.F.} = C_1 e^{x} + C_2 e^{-x} + e^{x}\,(C_3 \cos 2x + x_4 \sin 2x)$$

Note : For problem Nos. 22 to 33 as all the roots are complex synthetic division is not applicable so the roots are obtained by adjustment.

22) $D^4 + 10D^2 + 9 = 0$

$$(D^2 + 1)(D^2 + 9) = 0$$

$$D^2 = -1, D^2 = -9$$

$$D = \pm i \quad D = \pm 3i$$

All the roots are complex.

$$\text{C.F.} = C_1 \cos x + C_2 \sin x + C_3 \cos 3x + C_4 \sin 3x$$

23) $D^4 + 1 = 0$

$$D^4 + 2D^2 + 1 - 2D^2 = 0 \qquad \leftarrow \text{Note this step.}$$

$$(D^2 + 1)^2 - (\sqrt{2}\,D)^2 = 0$$

$$(D^2 + 1 + \sqrt{2}\,D)(D^2 + 1 - \sqrt{2}\,D) = 0$$

$$D^2 + \sqrt{2}\,D + 1 = 0, \qquad D^2 - \sqrt{2}\,D + 1 = 0$$

$$\therefore \quad D = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2} \qquad D = \frac{\sqrt{2} \pm \sqrt{2-4}}{2}$$

$$\therefore \quad D = \frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}} \qquad D = \frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$$

$$\text{C.F.} = e^{x/\sqrt{2}}\left[C_1 \cos\frac{x}{\sqrt{2}} + C_2 \sin\frac{x}{\sqrt{2}}\right] + e^{-x/2}\left[C_3 \cos\frac{x}{\sqrt{2}} + C_4 \sin\frac{x}{\sqrt{2}}\right]$$

24) $D^4 + 8D^2 + 16 = 0$

$$(D^2+4)^2 = 0$$

$$D^2 = -4 \quad \text{twice}$$

$$D = \pm 2i$$

$$\text{C.F.} = e^{0x}\left[(C_1 + C_2x)\cos 2x + (C_3 + C_4x)\sin 2x\right]$$

25) $D^4 + m^4 = 0$

$$D^4 + 2m^2D^2 + m^4 - 2m^2D^2 = 0 \quad \leftarrow \text{note this step.}$$

$$(D^2+m^2)^2 - (\sqrt{2}\,mD)^2 = 0$$

$$(D^2 + m^2 + \sqrt{2}\,m\,D)\ (D^2 + m^2 - \sqrt{2}\,m\,D) = 0$$

$$D^2 + \sqrt{2}\,m\,D + m^2 = 0 \quad D^2 - \sqrt{2}\,m\,D + m^2 = 0$$

$$D = \frac{-\sqrt{2m} \pm \sqrt{2m^2 - 4m^2}}{2}$$

$$D = -\frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}} \qquad D = \frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}$$

$$\text{C.F.} = e^{-\frac{mx}{\sqrt{2}}}\left[C_1 \cos\frac{mx}{\sqrt{2}} + C_2 \sin\frac{mx}{\sqrt{2}}\right] + e^{\frac{mx}{\sqrt{2}}}\left[C_3 \cos\frac{mx}{\sqrt{2}} + C_4 \sin\frac{mx}{\sqrt{2}}\right]$$

26) $D^4 + D^2 + 1 = 0$

$$D^4 + 2D^2 + 1 - D^2 = 0 \quad \leftarrow \text{note this step.}$$

$$(D^2+1)^2 - D^2 = 0$$

$$(D^2 + D + 1)\ (D^2 - D + 1) = 0$$

$$D^2 + D + 1 = 0 \qquad D^2 - D + 1 = 0$$

$$D = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$D = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2} \qquad D = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\text{C.F.} = e^{-x/2}\left[C_1\cos\frac{\sqrt{3}x}{2}+C_2\sin\frac{\sqrt{3}x}{2}\right]+e^{x/2}\left[C_3\cos\frac{\sqrt{3}x}{2}+C_4\sin\frac{\sqrt{3}x}{2}\right]$$

27) $D^4+5D^2+4 = 0$

$$(D^2+1)(D^2+4) = 0$$

$$D^2 = -1,\ D^2 = -4$$

$$D = \pm i \quad D = \pm 2i$$

$$\alpha = 0, \beta = 1, \alpha = 0, \beta = 2$$

$$\text{C.F.} = e^{0x}(C_1\cos x+C_2\sin x)+e^{0x}(C_3\cos 2x+C_4\sin 2x)$$

28) $D^4+2D^3+3D^2+2D+1 = 0$

$(D^2)^2+2D^2D+D^2+2D^2+2D+1 = 0$ ← note this step.

$$(D^2+D)^2+2(D^2+D)+1 = 0$$

$$(D^2+D+1)^2 = 0$$

$D = \dfrac{-1\pm i\sqrt{3}}{2}$ repeated

$$\text{C.F.} = e^{-x/2}\left[(C_1+C_2x)\cos\frac{\sqrt{3}x}{2}+(C_3+C_4x)\sin\frac{\sqrt{3}x}{2}\right]$$

29) $D^4-4D^3+8D^2-8D+4 = 0$

$(D^2)^2-2D^2\cdot 2D+(2D)^2+4D^2-8D+4 = 0$ ← note this step.

$$(D^2-2D)^2+4(D^2-2D)+4 = 0$$

$$(D^2-2D+2)^2 = 0$$

$$D = \frac{+2\pm\sqrt{4-8}}{2}$$

$$D = 1\pm i$$

$$\text{C.F.} = e^{x}\left[(C_1+C_2x)\cos x+(C_3+C_4x)\sin x\right]$$

30) $D^5-D^4+2D^3-2D^2+D-1 = 0$

$$D^4(D-1)+2D^2(D-1)+D-1 = 0$$

$$(D-1)(D^4+2D^2+1) = 0$$

$$(D-1)(D^2+1)^2 = 0$$

$$D = 1, \; D^2 = -1 \quad \text{repeated.}$$

$$\text{C.F.} = C_1 e^x + e^{0x}\left[(C_2 + C_3 x)\cos x + (C_4 + C_5 x)\sin x\right]$$

1.4 Inverse Operator

f (D) is simple operator and $\frac{1}{f(D)}$ is the inverse operator of f (D).

If f (D) = D – a then

f (D) y = X gives

(D – a) y = X

$\therefore$ Particular integral $y = \frac{1}{D-a} X$

$\therefore$ (D – a) y = X

$D = \frac{d}{dx}$

$\therefore \frac{dy}{dx} - ay = X$

which is the linear differential equation of 1st order and 1st degree. Whose general solution is given by

$$y \, e^{\int -a dx} = \int X e^{\int -a dx} dx + C$$

$$y \, e^{-ax} = \int X \, e^{-ax} \, dx + C$$

$$y = e^{ax} \int X \, e^{-ax} \, dx + C \, e^{ax}$$

As all the arbitrary constants are involved in complementary function, particular integral will be independent of arbitrary constants.

$$\therefore \quad y = e^{ax} \int X \, e^{-ax} \, dx$$

$$\therefore \quad \frac{1}{D-a} X = e^{ax} \int X \, e^{-ax} \, dx$$

Replacing a by – a

$$\frac{1}{D+a} X = e^{-ax} \int X \, e^{+ax} \, dx$$

Replacing a by 0

$$\frac{1}{D} X = \int X \, dx$$

Thus Particular Integral P.I. $= \frac{1}{f(D)} X$ is given by

$$\boxed{\frac{1}{D} X = \int X \, dx}$$

$$\boxed{\frac{1}{D-a} X = e^{ax} \int X e^{-ax} \, dx}, \quad \boxed{\frac{1}{D+a} X = e^{-ax} \int X e^{-ax} \, dx}$$

1.5 Shortcut Methods for Finding Particular Integral of Special Functions

Type 1 : If $X = e^{ax}$

1) As $D e^{ax} = a e^{ax}$ i.e. while finding derivative of e^{ax}, we replace D by a to get the answer. Generalising this for any polynomial f (D), we get Formula F_1.

F_1) $$\boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \quad \text{if } f(a) \neq 0}$$

2) Case of failure if f (a) = 0 i.e. we get zero in the denominator after the replacement then we apply the basic formula for Particular Integral.

i.e. $$\frac{1}{D-a} e^{ax} = e^{ax} \int e^{-ax} \cdot e^{ax} \, dx$$

$$= e^{ax} \cdot x$$

Similarly we can derive

$$\frac{1}{(D-a)^2} e^{ax} = \frac{x^2}{2!} e^{ax}$$

Generalising this we get Formula F_2

F_2) $$\boxed{\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}}$$

3) If there is a combination of zero and non zero factors then use formula F_1 and F_2 separately.

F_3) $$\boxed{\frac{1}{\phi(D) \cdot (D-a)^r} e^{ax} = \frac{1}{\phi(a)} \cdot \frac{x^r}{r!} \cdot e^{ax}}$$

4) For constant k we can write $k \cdot e^{0x}$. Thus for constant we can replace D by zero.

F_4) $$\boxed{\frac{1}{f(D)} k = k \cdot \frac{1}{f(D)} e^{0x} = \frac{k}{f(0)}, \; f(0) \neq 0}$$

5) For a^x we can write $e^{x \log a}$. Thus for a^x we can replace D by log a.

F5) $$\boxed{\frac{1}{f(D)} a^x = \frac{1}{f(\log a)} a^x \left\{a^k = e^{\log a^k} = e^{k \log a}\right\}}$$

Note :

1) Here take the factors of f (D) for finding P.I.
2) Replace D by a only in the non zero factor.
3) For the zero factor use the formula F_2.
4) If X = constant then replace D by zero.
5) If X = a^x then replace D by 'log a'.

P.I. term which becomes zero after the replacement	Ans
$\frac{1}{(D-a)^r} e^{ax}$	$\frac{x^r}{r!} e^{ax}$
$\frac{1}{(D+a)^r} e^{-ax}$	$\frac{x^r}{r!} e^{-ax}$
$\frac{1}{D-a} e^{ax}$	$x e^{ax}$
$\frac{1}{D+a} e^{-ax}$	$x e^{-ax}$
$\frac{1}{(D-a)^2} e^{ax}$	$\frac{x^2}{2!} e^{ax}$
$\frac{1}{(D-a)^3} e^{ax}$	$\frac{x^3}{3!} e^{ax}$

Procedure for Type 1

Step 1 : Use P.I. formula.

Step 2 : Use factors of f (D) for finding P.I. and if necessary use $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$.

Step 3 : Separate all the terms and consider PI_1, PI_2, PI_3 ....

For non zero factor	For mixed factors zero and non zero	For X = constant	For X = a^x
Step 4 : Replace D by a only in non zero factor.	Step 4 : Replace D by a only in non zero factor.	Step 4 : X = constant then replace D by zero.	Step 4 : X = a^x then replace D by Log a.
Step 5 : Simplify.	Step 5 : Simplify.	Step 5 : Simplify.	Step 5 : Simplify.
	Step 6 : For zero factor use F_2.		
	Step 7 : Simplify.		

1.6 Problems on Type 1

Example 1.1 : $(D^3 + 3D)\, y = \cosh 2x \sinh 3x$

Solution : Auxiliary equation is $D^3 + 3D = 0$

Take D common.

$$D(D^2 + 3) = 0$$

Separate the two factors.

$$D = 0 \qquad D^2 + 3 = 0$$

$$D = 0 \qquad D^2 = -3$$

Square of any real term is always positive.

$$\therefore \quad D = \pm i\sqrt{3}$$

$$= \alpha \pm i\beta$$

$$D = 0, \quad \alpha = 0, \beta = \sqrt{3}$$

$\therefore$ One root is real and two roots are complex.

$$\text{C.F.} = C_1 e^{0x} + e^{0x}(C_1 \cos x\sqrt{3} + C_2 \sin x\sqrt{3})$$

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 1 : Use P.I. formula.

$$= \frac{1}{D^3 + 3D} \cosh 2x \ \sinh 3x$$

Step 2 : Take the factors of f (D) for finding P.I.

$$= \frac{1}{D(D^2+3)} \left(\frac{e^{2x} + e^{-3x}}{2}\right)\left(\frac{e^{3x} - e^{-3x}}{2}\right)$$

Step 3 : Separate all the terms.

$$= \frac{1}{D(D^2+3)} \frac{1}{4} [e^{5x} - e^{-5x} + e^{x} - e^{-x}]$$

Step 4 : Simplify.

$$= \frac{1}{4}\left\{\frac{1}{D(D^2+3)} e^{5x} - \frac{1}{D(D^2+3)} e^{-5x} + \frac{1}{D(D^2+3)} e^{x} - \frac{1}{D(D^2+3)} e^{-x}\right\}$$

Step 5 : Replace D by a only in non zero factor.

$$= \frac{1}{4}\left\{\frac{1}{(5)(28)}e^{5x} - \frac{1}{(-5)(28)}e^{-5x} + \frac{1}{(1)(4)}e^{x} - \frac{1}{(-1)(4)}e^{-x}\right\}$$

Step 6 : Simplify.

$$= \frac{1}{4}\left\{\frac{e^{5x}+e^{-5x}}{14} + \frac{e^{x}+e^{-x}}{4}\right\}$$

$$= \frac{1}{280}\left(\frac{e^{5x}+e^{-5x}}{2}\right) + \frac{1}{8}\left(\frac{e^{x}+e^{-x}}{2}\right)$$

Step 7 : Use hyperbolic formulae.

$$= \frac{1}{280}\cosh 5x + \frac{1}{8}\cosh x$$

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^{0x} + C_2 \cos x\sqrt{3} + C_3 \sin x\sqrt{3} + \frac{1}{280}\cosh 5x + \frac{1}{8}\cos h\, x$$

Example 1.2 : $(D^4 - 1)\, y = cosh x\, sinh x$

Solution : Auxiliary equation is $D^4 - 1 = 0$

i.e. $(D^2 - 1)(D^2 + 1) = 0$

i.e. $(D-1)(D+1)(D^2+1) = 0$

$\therefore$ $D = \pm 1, \quad D^2 = -1$

$D = \pm 1, \quad D = \pm i$

$\therefore$ Two roots are real and two roots are complex.

$\therefore$ $C.F = C_1 e^{x} + C_2 e^{-x} + e^{0x}(C_3 \cos x + C_4 \sin x)$

Step 1 : Use P.I formula.

$$P.I = \frac{1}{D^4 - 1}\cosh x \sinh x$$

Step 2 : Use factors of f (D).

$$P.I = \frac{1}{(D^2-1)(D^2+1)}\left(\frac{\sinh 2x}{2}\right)$$

Step 3 : Separate all terms.

$$P.I = \frac{1}{2}\,\frac{1}{(D^2-1)(D^2+1)}\left(\frac{e^{2x}-e^{-2x}}{2}\right)$$

$$= \frac{1}{4}\left\{\frac{1}{(D^2-1)(D^2+1)}\ e^{2x} - \frac{1}{(D^2-1)(D^2+1)}\ e^{2x}\right\}$$

Put D = 2 Put D = – 2

Step 4 : Replace D by a only in non zero factor.

$$= \frac{1}{4}\left\{\frac{1}{(4-1)(4+1)}\ e^{2x} - \frac{1}{(4-1)(4+1)}\ e^{-2x}\right\}$$

Step 5 : Simplify.

$$= \frac{1}{4}\left\{\frac{e^{2x}}{15} - \frac{e^{-2x}}{15}\right\}$$

$$= \frac{1}{30}\left(\frac{e^{2x}-e^{-2x}}{2}\right)$$

$$= \frac{1}{30}\ \sinh 2x$$

$y = C.F + P.I$ is the complete solution.

$$\therefore \quad y = C_1 e^{x} + C_2 e^{-x} + C_3 \cos x + C_4 \sin x + \frac{1}{30}\sinh 2x$$

Example 1.3 : $(D^3+D^2-D-1)\,y = \sinh x$

Solution : $C.F. = C_1 e^{x} + (C_2 + C_3 x)\,e^{-x}$

[Refer solved Problem No. 7 in problems on C.F.]

Step 1 : Use P.I. formula.

$$P.I. = \frac{1}{f(D)}\ X$$

$$P.I. = \frac{1}{D^3+D^2-D-1}\ \sinh x$$

Step 2 : Use factors of f (D) for finding P.I.

$$P.I. = \frac{1}{(D-1)(D+1)^2}\left(\frac{e^{x}-e^{-x}}{2}\right)$$

Step 3 : Separate all the terms.

$$P.I. = \frac{1}{2}\left[\frac{1}{(D-1)(D+1)^2}\ e^{x} - \frac{1}{(D-1)(D+1)^2}\ e^{-x}\right]$$

Step 4 : Replace D by a only in non zero factor. Here a = 1 and a = – 1 respectively.

$$\text{P.I.} = \frac{1}{2}\left[\frac{1}{(D-1)\,(1+1)^2}\,e^x - \frac{1}{(-1-1)\,(D+1)^2}\,e^{-x}\right]$$

Step 5 : Simplify.

$$\text{P.I.} = \frac{1}{2}\left[\frac{1}{4}\cdot\frac{1}{D-1}\,e^x + \frac{1}{2}\cdot\frac{1}{(D+1)^2}\,e^{-x}\right]$$

Step 6 : For zero factor use formula F_2.

$$\text{P.I.} = \frac{1}{2}\left[\frac{1}{4}\cdot x\,e^x + \frac{1}{2}\cdot\frac{x^2}{2\,!}e^{-x}\right]$$

Step 7 : Simplify.

$$\text{P.I.} = \frac{x}{8}e^x + \frac{x^2}{8}e^{-x}$$

$$y = \text{C.F.} + \text{P.I. is the complete solution.}$$

Example 1.4 : $(D^3 - 5D^2 + 8D - 4)\,y = e^{2x} + 2e^x$

Solution : $\text{C.F.} = C_1 e^x + (C_2 + C_3 x)\,e^{2x}$

[Refer solved Problem No. 8 in problems on C.F.]

$$\text{P.I.} = \frac{1}{f(D)}X$$

Step 1 : Use P.I. formula.

$$= \frac{1}{D^3 - 5D^2 + 8D - 4}\left(e^{2x} + 2e^x\right)$$

Step 2 : Use factors of f (D) for finding P.I.

$$= \frac{1}{(D-1)\,(D-2)^2}\left(e^{2x} + 2e^x\right)$$

Step 3 : Separate all the terms.

$$= \frac{1}{(D-2)^2(D-1)}\,e^{2x} + \frac{1}{(D-1)\,(D-2)^2}\,2e^x$$

Step 4 : Replace D by a only in non zero factor i.e. put D = 2 and D = 1 respectively.

$$= \frac{1}{(2-1)\,(D-2)^2}\,e^{2x} + \frac{2}{(D-1)\,(1-2)^2}\,e^x$$

Step 5 : Simplify.

$$= \frac{1}{(D-2)^2}\,e^{2x} + \frac{2}{1}\cdot\frac{1}{D-1}\,e^x$$

Step 6 : For the zero factor use formula F_2.

$$= \frac{x^2}{2!} e^{2x} + 2 \cdot \frac{x}{1} e^x$$

Step 7 : y = C.F. + P.I. is the complete solution.

Example 1.5 : $(D^3 + D^2 + D + 1)\, y = \cosh x$

Solution : Auxiliary equation i.e. $D^3 + D^2 + D + 1 = 0$

$$\text{C.F.} = C_1 e^{-x} + C_2 \cos x + C_3 \sin x$$

[Refer Problem No. 5 in problems on C.F.]

Step 1 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{D^3 + D^2 + D + 1} \cosh x$$

Step 2 : Use factors of f (D) for finding P.I.

$$= \frac{1}{(D+1)(D^2+1)} \left(\frac{e^x + e^{-x}}{2} \right)$$

Step 3 : Separate all terms.

$$= \frac{1}{2} \left\{ \frac{1}{(D+1)(D^2+1)} e^x + \frac{1}{(D+1)(D^2+1)} e^{-x} \right\}$$

Step 4 : Replace D by a only in non zero factor i.e. D = 1 and D = – 1 respectively.

$$= \frac{1}{2} \left\{ \frac{1}{(1+1)\,(1+1)} e^x + \frac{1}{(D+1)\,(1+1)} e^{-x} \right\}$$

Step 5 : Simplify.

$$= \frac{1}{2} \left\{ \frac{e^x}{4} + \frac{1}{2} \frac{1}{(D+1)} e^{-x} \right\}$$

Step 6 : For zero factor use formula F_2.

$$= \frac{1}{2} \left\{ \frac{e^x}{4} + \frac{1}{2} \cdot \frac{x^1}{1} e^{-x} \right\}$$

$$= \frac{e^x}{8} + \frac{x}{4} e^{-x}$$

$$y = \text{C.F} + \text{P.I}$$

$$= C_1 e^{-x} + C_2 \cos x + C_3 \sin x + \frac{e^x}{8} + \frac{x}{4} e^{-x}$$

is the complete solution of the Differential equation.

Mixed Problems

Note : In such a case find all the particular integrals separately PI_1, PI_2, PI_3, PI_4 and so on.

Example 1.6 : $(D^6 - 6D^5 + 12D^4 - 6D^3 - 9D^2 + 12D - 4)\, y = e^x + 1$

Solution : $$\text{C.F.} = (C_1 + C_2 x + C_3 x^2)\, e^x + C_4 e^{-x} + (C_5 + C_6 x)\, e^{2x}$$

[Refer Problem No. 6 in problems on C.F.]

Step 1 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 2 : Use factors of f (D).

$$\text{P.I.} = \frac{1}{(D-1)^3 (D+1)(D-2)^2} (e^x + 1)$$

Step 3 : Separate all terms.

$$= \left\{ \frac{1}{(D-1)^3\, (D+1)\, (D-2)^2} e^x + \frac{1}{(D-1)^3 (D+1)\, (D-2)^2} 1 \right\}$$

Consider PI_1

$$PI_1 = \frac{1}{(D-1)^3 (D+1)(D-2)^2} e^x$$

Step 4 : Replace D by a i.e. D = 1.

$$PI_1 = \frac{1}{(D-1)^3 (1+1)(1-2)^2} e^x$$

Step 5 : Simplify.

$$PI_1 = \frac{1}{2\,(D-1)^3} e^x$$

Consider PI_2

$$PI_2 = \frac{1}{(D-1)^3 (D+1)(D-2)^2} 1$$

Step 4 : As X is a constant then replace D by 0.

$$PI_2 = \frac{1}{(0-1)^3 (0+1)(0-2)^2}$$

Step 5 : Simplify.

$$PI_2 = \frac{1}{-4}$$

Step 6 : Use formula F_2.

$$= \frac{1}{2} \cdot \frac{x^3}{3\,!} e^x$$

Step 7 : Simplify.

$$= \frac{x^3}{12} e^x$$

Thus $P.I = PI_1 + PI_2$

$$y = C.F + P.I$$

is the complete solution of the differential equation.

Example 1.7 : $(D^3 - 5D^2 + 8D - 4)\, y = 4e^{2x} + e^x + 2^x + 3$

Solution : $\quad C.F. = C_1 e^x + (C_2 + C_3 x)\, e^{2x}$

[Refer Problem No. 8 in problems on C.F]

Step 1 : Use P.I. Formula.

$$P.I. = \frac{1}{f(D)} X$$

Step 2 : Use factors of f (D).

$$P.I. = \frac{1}{(D-1)(D-2)^2} 4e^{2x} + e^x + 2^x + 3$$

Step 3 : Separate all the terms.

$$P.I. = \frac{1}{(D-1)(D-2)^2} 4e^{2x} + \frac{1}{(D-1)(D-2)^2} e^x + \frac{1}{(D-1)(D-2)^2} 2^x + \frac{1}{(D-1)(D-2)^2} 3$$

Consider

$$PI_1 = \frac{1}{(D-1)(D-2)^2} 4e^{2x}$$

Step 4 : Replace D by a in non zero factor. i.e. D = 2

$$PI_1 = \frac{1}{(2-1)(D-2)^2} 4e^{2x}$$

Step 5 : Simplify.

$$= \frac{4}{1} \cdot \frac{1}{(D-2)^2} e^{2x}$$

Step 6 : For zero factor use F_2.

$$= \frac{4}{1} \cdot \frac{x^2}{2!} e^{2x}$$

Step 7 : Simplify.

$$PI_1 = 2x^2 e^{2x}$$

Consider

$$PI_2 = \frac{1}{(D-1)(D-2)^2} e^x$$

Step 4 : Replace D by a in non zero factor i.e. D = 1.

$$PI_2 = \frac{1}{(D-1)(1-2)^2} e^x$$

Step 5 : Simplify.

$$= \frac{1}{1 \cdot (D-1)} e^x$$

Step 6 : For zero factor use F_2.

$$= \frac{x}{1} e^x$$

Consider

$$PI_3 = \frac{1}{(D-1)(D-2)^2} 2^x$$

Step 4 : For a^x replace D by log a, (Here D = log 2)

$$PI_3 = \frac{1}{(\log 2-1)(\log 2-2)} 2^x$$

Consider

$$PI_4 = \frac{1}{(D-1)(D-2)^2} 3$$

Step 4 : For X = constant replace D by 0.

$$PI_4 = \frac{1}{(0-1)(0-2)^2} 3$$

Step 5 : Simplify.

$$= -\frac{3}{4}$$

Thus

$$\text{P.I.} = PI_1 + PI_2 + PI_3 + PI_4$$

$$= 2x^2 e^x + xe^x + \frac{2^x}{(\log 2-1)(\log 2-2)^2} - \frac{3}{4}$$

and y = C.F + P.I is the complete solution.

Exercise 1.1

1. $(D^2+4D+4)y = e^{-2x} + 2^x + 3$ **[Ans. : $y = (C_1 + C_2x)\, e^{-2x} + \frac{x^2}{2} e^{-2x} + \frac{2^x}{(\log 2 + 2)^2} + \frac{3}{4}$]**

2. $(D^3 - 3D^2 + 4)y = \cosh 2x$ **[Ans. : $y = C_1e^{-x} + (C_2 + C_3x)e^{2x} + \frac{x^2}{12} e^{2x} - \frac{e^{-2x}}{32}$]**

3. $(D^2 - 4)y = (1+e^x)^2 + 3$ **[Ans. : $y = C_1e^{2x} + C_2e^{-2x} + \frac{x\,e^{2x}}{4} - \frac{2e^x}{3} - 1$]**

4. $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2e^x + 3e^{-x} + 2$

 [Ans. : $y = C_1e^x + (C_2 + C_3x)e^{2x} + \frac{x^2}{2} e^{2x} + 2x\,e^x - \frac{e^{-x}}{6} - \frac{1}{2}$]

5. $(D^2 + 13D + 36)y = e^{-4x} + \sinh x$ **[Ans. : $y = C_1e^{-4x} + C_2e^{-9x} + \frac{x}{5} e^{-4x} + \frac{e^x}{100} - \frac{e^{-x}}{48}$]**

1.7 Type 2 : If X = sin ax or cos ax Use

Let $x = \sin(ax+b)$ or $\cos(ax+b)$

We know

$$D\cos(ax+b) = -a\sin(ax+b)$$

$$D^2\cos(ax+b) = -a^2\cos(ax+b)$$

$$D^3\cos(ax+b) = +a^2\cdot a\sin(ax+b)$$

$$D^4\cos(ax+b) = a^4\cos(ax+b)$$

$$\text{i.e} \left(D^2\right)^2 \cos(ax+b) = \left(-a^2\right)^2 \cos(ax+b)$$

Similarly $\left(D^2\right)^2 \sin(ax+b) = \left(-a^2\right)^2 \sin(ax+b)$

i.e. For sin ax or cos ax we replace $D^2 = -a^2$.

Generalising this we get formula F_1.

$$F_1)\ \frac{1}{f(D^2)} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax+b) = \frac{1}{f(-a^2)} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax+b) \text{ if } f(-a^2) \neq 0$$

i.e. For sin (ax + b) or cos (ax + b) replace $D^2 = -a^2$ if the denominator is non zero.

Case of failure : If $f(-a^2) = 0$

We know that $e^{iax} = \cos ax + i\sin ax$

$\therefore$ $\sin ax = \text{Img } e^{iax}$

$\cos ax = \text{Real } e^{iax}$

$$\therefore \quad \frac{1}{f(D^2)} \sin ax = \text{Img} \frac{1}{f(D^2)} e^{iax}$$

$$= \text{Img} \frac{x}{f'(D^2)} e^{iax}$$

Again put $D^2 = -a^2$

$$= \text{Img} \frac{x}{f'(-a^2)} e^{iax}$$

$$\frac{1}{f(D^2)} \sin ax = \frac{x}{f'(-a^2)} \sin ax \quad \text{if } f'(-a^2) \neq 0$$

i.e. For the zero factor multiply by x to the numerator and differentiate the denominator w.r.t D and then put $D^2 = -a^2$ again.

Generalising this we get formula F_2

$$\boxed{F_2) \ \frac{1}{(D^2+a^2)^r} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax+b) = \left(-\frac{x}{2a}\right)^r \cdot \frac{1}{r\,!} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\left(ax+b+\frac{r\pi}{2}\right)}$$

For example :

A) $\dfrac{1}{D^2+a^2} \sin ax$

The denominator becomes zero if we put $D^2 = -a^2$ $\therefore$ multiply by x to the numerator and differentiate the denominator.

$$= x \cdot \frac{1}{2D} \sin ax$$

$$\left\{ \text{Use } \frac{1}{D} X = \int X\,dx \quad \int \sin ax\,dx = \frac{-\cos ax}{a} \right\}$$

$$= x \cdot \frac{1}{2} \cdot \frac{-\cos ax}{a}$$

$$= \frac{-x}{2a} \cos ax$$

B) $\dfrac{1}{D^2+a^2} \cos ax$

$D^2 = -a^2$ gives zero $\therefore$ Multiply by x to numerator and differentiate the denominator.

$$= x \cdot \frac{1}{2D} \cos ax$$

$$\left\{\text{Use } \frac{1}{D} X = \int X\,dx \quad \int \cos ax\,dx = \frac{\sin ax}{a}\right\}$$

$$= \frac{x}{2} \cdot \frac{\sin ax}{a}$$

$$= \frac{x}{2a} \sin ax$$

C) $\frac{1}{(D^2+a^2)^2} \sin ax$

$D^2 = -a^2$ gives zero $\therefore$

$$= x \cdot \frac{1}{2(D^2+a^2) \cdot 2D} \cdot \sin ax$$

$$\left\{\text{Use } \frac{1}{D} X = \int X\,dx\right\}$$

$$= \frac{x}{4} \cdot \frac{1}{D^2+a^2} \cdot \left(\frac{-\cos ax}{a}\right)$$

$$= \frac{-x}{4a} \cdot \frac{1}{D^2+a^2} \cos ax$$

Using above formula (B).

$$= \frac{-x}{4a} \cdot \frac{x}{2a} \sin ax$$

$$= -\frac{x^2}{8a^2} \sin ax$$

D) $\frac{1}{(D^2+a^2)^2} \cos ax$

$D^2 = -a^2$ gives zero.

$$= x \cdot \frac{1}{2(D^2+a^2)\, 2D} \cos ax$$

$$\left\{\text{Use } \frac{1}{D} X = \int X\,dx\right\}$$

$$= \frac{x}{4} \cdot \frac{1}{D^2+a^2} \cdot \frac{\sin ax}{a}$$

$$= \frac{x}{4a} \frac{1}{D^2+a^2} \sin ax$$

Using above formula (A).

$$= \frac{x}{4a} \cdot \frac{-x}{2a} \cos ax$$

$$= \frac{-x^2}{8a^2} \cos ax$$

Thus for the zero factor.

A) $\frac{1}{D^2 + a^2} \sin ax = -\frac{x}{2a} \cos ax$

B) $\frac{1}{D^2 + a^2} \cos ax = \frac{x}{2a} \sin ax$

C) $\frac{1}{(D^2 + a^2)^2} \sin ax = -\frac{x^2}{8a^2} \sin ax$

D) $\frac{1}{(D^2 + a^2)^2} \cos ax = -\frac{x^2}{8a^2} \cos ax$

Note : Here powers and multiplication of trigonometric terms are not allowed thus we should separate them using the following formulae.

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}, \cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$2 \sin A \cos B = \sin (A + B) + \sin(A - B)$$

$$2 \cos A \cos B = \cos(A - B) + \cos(A + B)$$

$$2 \sin A \sin B = \cos (A - B) - \cos(A + B)$$

Term	Its zero factor
sinx, cosx	$D^2 + 1$
sin2x, cos2x	$D^2 + 4$
sin3x, cos3x	$D^2 + 9$
sinax, cosax	$D^2 + a^2$

Procedure for Type 2

Step 1 : Use P.I. formula.

Step 2 : Separate all the terms and consider PI_1, PI_2, PI_3 separately.

Step 3 : Observe the trigonometric ratio sin ax or cos ax and note the zero factor for that ratio. By using the factors which are present in complementary function we can notice that the zero factor is present in f (D) or not.

If the zero factor is not present then follow procedure (2A).

If the zero factor is present then follow procedure (2B).

Procedure (2A)

Step 4 : Don't take the factors if f (D) for finding P.I. (i.e. keep f (D) in polynomial form).

Step 5 : Replace $D^2 = -a^2$, $D^3 = D^2 \cdot D = -a^2 \cdot D$, $D^4 = (D^2)^2 = a^4$

(Don't replace D by $\sqrt{-a^2}$ it is wrong)

If the denominator reduces to a constant then P.I. is complete.

Step 6 : Simplify.

Step 7 : If the term involving D remains in the denominator then rationalise to get D^2 in denominator.

Step 8 : Simplify (i.e. multiply the two brackets in the denominator).

Step 9 : Put $D^2 = -a^2$ in denominator and simplify.

Step 10 : Open the bracket in the numerator and take the derivatives.

Step 11 : Simplify.

1.8 Problems on Type 2

Example 1.8 : $(D^3 + D^2 - D - 1)\,y = 2 \sin x \cos x$

Solution : $\text{C.F.} = C_1 e^x + (C_2 + C_3 x)\, e^{-x}$

(Refer solved Problem No. 7 in problems on C.F.)

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 1 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{D^3 + D^2 - D - 1} 2 \sin x \cos x$$

Step 2 : Separate the terms.

$$\text{P.I.} = \frac{1}{D^3 + D^2 - D - 1} \sin 2x$$

Step 3 : The zero factor for $\sin 2x$ is $D^2 + 4$ which is not present in f(D) $\therefore$ Apply procedure (2A).

Step 4 : Don't take the factors of f (D).

$$\therefore \quad \text{P.I.} = \frac{1}{D^3 + D^2 - D - 1} \sin 2x$$

Step 5 : Replace $D^2 = -a^2$, $D^3 = -a^2 D$, here $a = 2 \therefore D^2 = -4$, $D^3 = -4D$

$$\text{P.I.} = \frac{1}{-4D-4-D-1} \sin 2x$$

Step 6 : Simplify.

$$\text{P.I.} = \frac{1}{-5(D+1)} \sin 2x$$

Step 7 : If the term involving D remains in the denominator then rationlise.

$$= \frac{(D-1)}{-5(D+1)(D-1)} \sin 2x$$

Step 8 : Simplify.

$$= \frac{(D-1)}{-5(D^2-1)} \sin 2x$$

Step 9 : $D^2 = -a^2$ i.e. $D^2 = -4$ in denominator

$$= \frac{(D-1)}{-5(-4-1)} \sin 2x$$

Step 10 : Simplify.

$$= \frac{1}{25}(D-1) \sin 2x$$

Step 11 : Take derivatives

$$= \frac{1}{25}(2\cos 2x - \sin 2x)$$

$$\{\because D \sin 2x = 2\cos 2x\}$$

Step 12 : Simplify.

$$\text{P.I.} = \frac{1}{25}\{2\cos 2x - \sin 2x\}$$

$$y = \text{C.F.} + \text{P.I.}$$

i.e.
$$y = C_1 e^x + (C_2 + C_3 x) e^{-x} + \frac{1}{25}(2\cos 2x - \sin 2x)$$

is the complete solution.

Example 1.9 : $(D^3 + D^2 + D + 1)y = \cos^2 x$

Hint : $\cos^2 x = \dfrac{1+\cos 2x}{2}$

Solution : $\text{C.F} = C_1 e^{-x} + C_2 \cos x + C_3 \sin x$

(Refer Problem No. 5 in problems on CF).

Step 1 : Use P.I formula.

$$\text{P.I} = \frac{1}{D^3+D^2+D+1} \frac{1+\cos 2x}{2}$$

Step 2 : Separate PI_1, PI_2

$$= \frac{1}{2}\left\{\frac{1}{D^3+D^2+D+1}\, 1 + \frac{1}{D^3+D^2+D+1} \cos 2x\right\}$$

Step 3 : For constant put D = 0 and for cos 2x put $D^2 = -4$, $D^3 = -4D$.

$$= \frac{1}{2}\left\{\frac{1}{0+1} + \frac{1}{-4D-4+D+1} \cos 2x\right\}$$

Step 4 : Simplify.

$$= \frac{1}{2}\left\{1 + \frac{1}{-3\,(D+1)} \cos 2x\right\}$$

Step 5 : Rationalise.

$$= \frac{1}{2}\left\{1 - \frac{1}{3} \frac{D-1}{D^2-1} \cos 2x\right\}$$

Step 6 : Again $D^2 = -4$

$$= \frac{1}{2}\left\{1 - \frac{1}{3} \frac{(D-1)}{(-4-1)} \cos 2x\right\}$$

Step 7 : Simplify.

$$= \frac{1}{2}\left\{1 + \frac{1}{15} (D-1) \cos 2x\right\}$$

Step 8 : Take derivatives.

$$= \frac{1}{2}\left\{1 + \frac{1}{15} (-2 \sin 2x - \cos 2x)\right\}$$

$$y = \text{C.F} + \text{P.I}$$

$$= C_1 e^x + C_2 \cos x + C_3 \sin x + \frac{1}{2} - \frac{1}{30} (2 \sin 2x + \cos 2x)$$

is the complete solution.

Example 1.10 : $(D^4 + m^4)\, y = \sin mx$

Solution :

$$\text{C.F.} = e^{mx/\sqrt{2}}\left[C_1 \cos \frac{mx}{\sqrt{2}} + C_2 \sin \frac{mx}{\sqrt{2}}\right] + e^{-mx/\sqrt{2}}\left[C_3 \cos \frac{mx}{\sqrt{2}} + C_4 \sin \frac{mx}{\sqrt{2}}\right]$$

[Refer solved Problem No. 25 in problems on C.F]

Step 1 : Use P.I formula

$$\text{P.I} = \frac{1}{D^4 + m^4} \sin mx$$

Step 2 : Put $D^2 = -m^2$, $D^4 = m^4$

$$\therefore \quad \text{P.I} = \frac{1}{m^4 + m^4} \sin mx$$

$$= \frac{1}{2m^4} \sin mx$$

$$y = \text{C.F} + \text{P.I} \quad \text{is the complete solution.}$$

$$\therefore \quad y = e^{mx/\sqrt{2}}\left[C_1 \cos\frac{mx}{\sqrt{2}} + C_2 \sin\frac{mx}{\sqrt{2}}\right] + e^{-mx/\sqrt{2}}\left[C_3 \cos\frac{mx}{\sqrt{2}} + C_4 \sin\frac{mx}{\sqrt{2}}\right]$$

$$+ \frac{1}{2\,m^4} \sin\ mx$$

Example 1.11 : $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 10y + 37 \sin 3x = 0$

where $y = 3$, $\dfrac{dy}{dx} = 0$, *when* $x = 0$ *Find* y *when* $x = \dfrac{\pi}{2}$.

Solution : Let $D = \dfrac{d}{dx}$ $\therefore$ The equation becomes.

$$D^2y + 2Dy + 10y = -37 \sin 3x$$

$$(D^2 + 2D + 10)\, y = -37 \sin 3x$$

$$\text{A.E.} \quad D^2 + 2D + 10 = 0$$

Use $aD^2 + bD + c = 0$ $\quad D = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\therefore \quad D = \frac{-2 \pm \sqrt{4 - 40}}{2}$$

$$= \frac{-2 \pm \sqrt{-36}}{2}$$

$$= -1 \pm 3i = -1 \pm i\,3$$

$$\alpha \pm i\beta$$

$$\therefore \quad \text{C.F.} = e^{-x}(C_1 \cos 3x + C_2 \sin 3x)$$

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 1 : Apply P.I. formula.

$$\text{P.I.} = \frac{1}{D^2+2D+10}(-37)\sin 3x$$

Step 2 : D^2+9 is the zero factor for sin 3x. which is not present in f(D) ∴ Don't take factors of f(D) for finding P.I.

Step 3 : Replace $D^2 = -a^2$ Here a = 3, $D^2 = -9$

$$\text{P.I.} = \frac{1}{-9+2D+10}(-37)\sin 3x$$

Step 4 : Simplify.

$$= (--37)\cdot\frac{1}{2D+1}\sin 3x$$

Step 5 : Rationalise.

$$= (-37)\frac{2D-1}{(2D+1)(2D-1)}\sin 3x$$

Step 6 : Simplify.

$$= (-37)\cdot\frac{2D-1}{4D^2-1}\sin 3x$$

Step 7 : Again $D^2 = -a^2$ i.e. $D^2 = -9$

$$= -37\frac{(2D-1)}{(4(-9)-1)}\sin 3x$$

Step 8 : Simplify.

$$= (2D-1)\sin 3x$$

Step 9 : Take the derivatives.

$$= 2(3\cos 3x) - \sin 3x$$

Step 10 : Simplify.

$$= 6\cos 3x - \sin 3x$$

$$\therefore \quad y = \text{C.F.} + \text{P.I.}$$

$$= e^{-x}(C_1\cos 3x + C_2\sin 3x) + 6\cos 3x - \sin 3x$$

The conditions y = 3, $\frac{dy}{dx} = 0$ at x = 0 are given to find the values of C_1 and C_2

$\therefore$ We must find $\frac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = -e^{-x}(C_1 \cos 3x + C_2 \sin 3x) + e^{-x}(-3C_1 \sin 3x + 3C_2 \cos 3x)$$
$$-18 \sin 3x - 3\cos 3x$$

y = 3 when x = 0 gives

$$3 = e^0(C_1 + 0) + 6 - 0 \qquad \{\text{As } e^0 = 1, \cos 0 = 1, \sin 0 = 0\}$$

$\Rightarrow C_1 = -3$

Also $\frac{dy}{dx} = 0$ when x = 0 gives

$$0 = -e^0(C_1 + 0) + e^0(0 + 3C_2) + 0 - 3$$
$$0 = -C_1 + 3C_2 - 3$$
$$0 = 3 + 3C_2 - 3$$
$$\Rightarrow \quad 3C_2 = 0$$
$$\Rightarrow \quad C_2 = 0$$

Substituting C_1 and C_2 in y we get

$$y = e^{-x}(-3\cos 3x + 0) + 6\cos 3x - \sin 3x$$

at $x = \frac{\pi}{2}$

$$(y)_{\pi/2} = e^{-\pi/2}\left[-3\cos\frac{3\pi}{2}\right] + 6\cos\frac{3\pi}{2} - \sin\frac{3\pi}{2}$$

We know $\left\{\cos\frac{3\pi}{2} = 0, \sin\frac{3\pi}{2} = -1\right\}$

$$\therefore \quad (y)_{x=\pi/2} = 0 + 0 - (-1)$$
$$= 1$$

Example 1.12 : $\frac{d^2x}{dt^2} + 9x = 4\cos\left(t + \frac{\pi}{3}\right)$

when $x = 0$ *at* $t = 0$ *and* $x = 2$ *at* $t = \frac{\pi}{6}$.

Solution : Let $D = \frac{d}{dt}$ $\therefore$ The equation becomes

$$(D^2 + 9)x = 4\cos\left(t + \frac{\pi}{3}\right)$$

A.E. $\quad D^2+9 = 0$

$$D^2 = -9$$

$$D = \pm 3i$$

$$\text{C.F.} = C_1 \cos 3t + C_2 \sin 3t$$

As the independent variable is 't' $\therefore$ C.F. will involve t.

Now $\quad \text{P.I.} = \frac{1}{D^2+9} 4\cos\left(t+\frac{\pi}{3}\right)$

Put $D^2 = -1$

$$= \frac{1}{-1+9} 4\cos\left(t+\frac{\pi}{3}\right)$$

$$= \frac{4}{8}\cos\left(t+\frac{\pi}{3}\right)$$

$\therefore \quad x = \text{C.F.} + \text{P.I.}$

$$x = C_1 \cos 3t + C_2 \sin 3t + \frac{1}{2}\cos\left(t+\frac{\pi}{3}\right) \qquad \text{... (1)}$$

Given x = 0, at t = 0 substituting in (1)

$$0 = C_1 + 0 + \frac{1}{2}\cos\left(\frac{\pi}{3}\right)$$

$$0 = C_1 + \frac{1}{2}\cdot\frac{1}{2} \Rightarrow C_1 = \frac{-1}{4}$$

Also $x = 2$ at $t = \frac{\pi}{6}$ substituting in (2).

$$2 = C_1 \cos\frac{3\pi}{6} + C_2 \sin\frac{3\pi}{6} + \frac{1}{2}\cos\left(\frac{\pi}{6}+\frac{\pi}{3}\right)$$

$$2 = C_1 \cos\frac{3\pi}{2} + C_2 \sin\frac{3\pi}{2} + \frac{1}{2}\cos\frac{\pi}{2}$$

$$2 = 0 + C_2(-1) + 0$$

$\Rightarrow \quad C_2 = 2$

Substituting C_1 and C_2 in (1).

$$x = -\frac{1}{4}\cos 3t + 2\sin 3t + \frac{1}{2}\cos\left(t+\frac{\pi}{3}\right)$$

Example 1.13 : $cosec\, x \dfrac{d^4y}{dx^4} + cosec\, xy = sin\, 2x.$

Solution : Divide by cosec x $\therefore$ we get $\dfrac{d^4y}{dx^4} + y = \sin 2x \cdot \sin x$

Let $\quad D = \dfrac{d}{dx}$

$$(D^4+1)\,y = \sin 2x \cdot \sin x$$

Now $\quad \sin 2x \sin x = \dfrac{1}{2} \cos(2x-x) + \cos(2x+x)$

$\{2 \sin A \sin B = \cos(A-B) - \cos(A+B)\}$

$$= \frac{1}{2}[\cos x - \cos 3x]$$

$$(D^4+1)\,y = \frac{1}{2}[\cos x - \cos 3x]$$

A.E. is $\quad D^4 + 1 = 0 \quad$ [Refer Problem No. 23 in problems on C.F]

$$\text{C.F.} = e^{x/\sqrt{2}}\left[C_1 \cos\frac{x}{\sqrt{2}} + C_2 \sin\frac{x}{\sqrt{2}}\right] + e^{-x/\sqrt{2}}\left[C_3 \cos\frac{x}{\sqrt{2}} + C_4 \sin\frac{x}{\sqrt{2}}\right]$$

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 1 : Apply P.I. formula.

$$= \frac{1}{D^4+1} \frac{1}{2}[\cos x - \cos 3x]$$

Step 2 : Separate the terms and consider PI_1, PI_2 ...

$$= \frac{1}{2}\left\{\frac{1}{D^4+1}\cos x\right\} - \frac{1}{2}\left\{\frac{1}{D^4+1}\cos 3x\right\}$$

Step 3 : Put $D^2 = -a^2$, $D^4 = a^4$

$$= \frac{1}{2}\left\{\frac{1}{1+1}\cos x\right\} - \frac{1}{2}\left\{\frac{1}{89+1}\cos 3x\right\}$$

Step 4 : Simplify.

$$= \frac{1}{4}\cos x - \frac{1}{164}\cos 3x$$

y = C.F. + P.I. is the complete solution.

Example 1.14 : $\dfrac{d^2x}{dt^2} + 2n\, cos\alpha \dfrac{dx}{dt} + n^2 x = a\, cos\, nt$

given $t = 0, \dfrac{dx}{dt} = 0$ *at* $x = 0.$

Solution : Let $D = \dfrac{d}{dt}$

$$\therefore \quad (D^2 + 2n\cos\alpha\, D + n^2)\, x = a\cos nt$$

A.E.
$$D^2 + 2n\cos\alpha\, D + n^2 = 0$$

$$D = \frac{-2n\cos\alpha \pm \sqrt{4n^2\cos^2\alpha - 4n^2}}{2}$$

$$D = \frac{-2n\cos\alpha \pm 2ni\sqrt{1-\cos^2\alpha}}{2} \qquad \text{As } i = \sqrt{-1}$$

$$D = -n\cos\alpha \pm i\, n\sin\alpha$$

$$\therefore \quad \text{C.F.} = e^{(-n\cos\alpha)t}\left[C_1\cos(n\sin\alpha)t + C_2\sin(n\sin\alpha)t\right]$$

$$\text{P.I.} = \frac{1}{D^2 + 2n\cos\alpha D + n^2}\, a\cos nt$$

Put $D^2 = -n^2$

$$= \frac{1}{-n^2 + 2n\cos\alpha\, D + n^2}\, a\cos nt$$

$$= \frac{a}{2n\cos\alpha}\cdot\frac{1}{D}\cos nt \qquad \left\{\text{Use } \frac{1}{D}X = \int X\,dx\right\}$$

$$= \frac{a}{2n\cos\alpha}\cdot\frac{\sin nt}{n}$$

$$\therefore \quad y = \text{C.F.} + \text{P.I.}$$

$$x = e^{(-n\cos\alpha)t}\left[C_1\cos(n\sin\alpha)t + C_2\sin(n\sin\alpha)t\right] + \frac{a}{2n^2\cos\alpha}\sin nt \qquad \text{... (1)}$$

$$\frac{dx}{dt} = (-n\cos\alpha)\, e^{(-n\cos\alpha)t}\left[C_1\cos(n\sin\alpha)t + C_2\sin(n\sin\alpha)t\right]$$
$$+ e^{(-n\cos\alpha)t}(n\sin\alpha)\left[-C_1\sin(n\sin\alpha)t + C_2\cos(n\sin\alpha)t\right]$$
$$+ \frac{a\cos nt}{2n\cos\alpha} \qquad \text{... (2)}$$

Given at $t = 0$, $x = 0$ substituting in (1).

$$0 = e^0[C_1 + 0] + 0 \Rightarrow C_1 = 0$$

Given at $t = 0$, $\dfrac{dx}{dt} = 0$, substituting in (2).

$$0 = (-n\cos\alpha)[C_1+0]+(n\sin\alpha)[0+C_2]+\frac{a}{2n\cos\alpha}$$

$$0 = -n\cos\alpha\cdot C_1+n\sin\alpha\cdot C_2+\frac{a}{2n\cos\alpha}$$

$$\therefore \quad C_2 = \frac{-a}{2n^2\sin\alpha\cos\alpha}=\frac{-a}{n^2\sin 2\alpha}$$

Substituting C_1 and C_2 in (1)

$$x = e^{(-n\cos\alpha)t}\left[\frac{-a}{n^2\sin 2\alpha}\right]+\frac{a}{2n^2\cos\alpha}\sin nt$$

Example 1.15 : $(D^3-D^2+3D+5)\,y=2\sin x\cos x$

Solution : $\quad C.F = C_1 e^{-x}+e^{x}(C_2\cos 2x+C_3\sin 2x)$

[Refer Problem No. 16 in problems on C.F]

Step 1 : Use P.I formula.

$$P.I = \frac{1}{D^3-D^2+3D+5}\sin 2x$$

Step 2 : Put $D^2=-4$, $D^3=-4D$

$$P.I = \frac{1}{-4D+4+3D+5}\sin 2x$$

Step 3 : Simplify.

$$P.I = \frac{1}{(D+9)}\sin 2x$$

Step 4 : Rationalise.

$$P.I = \frac{(D-9)}{D^2-81}\sin 2x$$

Step 5 : Again $D^2=-4$.

$$P.I = \frac{(D-9)}{-4-81}\sin 2x$$

Step 6 : Take the derivatives.

$$P.I = \frac{-1}{85}(2\cos 2x-9\sin 2x)$$

$$y = C.F + P.I \text{ is the complete solution.}$$

$$y = C_1 e^{-x}+e^{x}(C_2\cos 2x+C_3\sin 2x)-\frac{1}{85}(2\cos 2x-9\sin 2x)$$

Procedure 2B

Step 1, 2, 3 are same as 2A.

Step 4 : Take the factors if f (D) for finding P.I. In this case (D^2+a^2) will be one of the factors.

Step 5 : Replace $D^2=-a^2$, $D^3=-a^2D$, $D^4=a^4$ only in non zero factor i.e. apply the procedure 2A for non zero factor keep the zero factor D^2+a^2 as it is

Step 6 : For the zero factor apply formulae (A), (B), (C), (D).

Example 1.16 : $(D^4+10D^2+9)\,y=96\sin 2x\cos x$

Solution :

$$(D^2+9)(D^2+1)\,y = 48\,[2\sin 2x\cos x]$$

$$= 48\,[\sin 3x+\sin x]$$

{As 2 sin A cos B = sin (A + B) + sin (A – B)}

$$\text{A.E.}\ (D^2+9)(D^2+1) = 0$$

$$D^2 = -9,\quad D^2=-1$$

$$D = \pm 3i,\quad D=\pm i$$

$$\therefore\quad \text{C.F.} = e^{0x}(C_1\cos 3x+C_2\sin 3x)+e^{0x}(C_3\cos x+C_4\sin x)$$

$$\text{P.I.} = \frac{1}{f(D)}X$$

Step 1 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{D^4+10D^2+9}48[\sin 3x+\sin x]$$

Step 2 : Separate all terms.

$$\text{P.I.} = 48\left\{\frac{1}{D^4+10D^2+9}\sin 3x+\frac{1}{D^4+10D^2+9}\sin x\right\}$$

Step 3 : Zero factors are present ∴ Procedure 2B.

Step 4 : Take the factors of f (D).

$$PI_1 = 48\frac{1}{(D^2+9)(D^2+1)}\sin 3x$$

Step 5 : Put $D^2=-a^2$ in non zero factor. i.e. $D^2=-9$

$$PI_1 = 48\frac{1}{(D^2+9)(-9+1)}\sin 3x$$

Step 6 : Simplify.

$$PI_1 = \frac{48}{-8}\cdot\frac{1}{D^2+9}\sin 3x$$

Step 7 : For zero factor use formula (A). $\left\{ \text{i.e.} \frac{1}{D^2+a^2} \sin ax = \frac{-x}{2a} \cos ax \right\}$

$$PI_1 = (-6) \cdot \frac{-x}{2 \cdot 3} \cos 3x$$

Step 8 : Simplify.

$$PI_1 = x \cos 3x$$

Also consider

$$PI_2 = 48 \frac{1}{(D^2+9)(D^2+1)} \sin x$$

Step 5 : Replace $D^2 = -a^2$ only in non zero factor. i.e. $D^2 = -1$

$$PI_2 = 48 \frac{1}{(-1+9)(D^2+1)} \sin x$$

Step 6 : Simplify.

$$PI_2 = \frac{48}{8} \cdot \frac{1}{D^2+1} \sin x$$

Step 7 : For zero factor use formula (A). $\left\{ \text{i.e.} \frac{1}{D^2+a^2} \sin ax = \frac{-x}{2a} \cos ax \right\}$

$$PI_2 = 6 \cdot \left(\frac{-x}{2 \cdot 1} \right) \cdot \cos x$$

Step 8 : Simplify.

$$PI_2 = -3x \cos x$$

Now

$\therefore$ $$PI = PI_1 + PI_2$$

$$= x \cos 3x - 3x \cos x$$

Now $y = C.F + P.I$ is the complete solution.

$\therefore$ i.e. $y = C_1 \cos 3x + C_2 \sin 3x + x \cos 3x - 3x \cos x$

Example 1.17 : $(D^4 + 5D^2 + 4)\, y = \cos \frac{x}{2} \cos \frac{3x}{2}$

Solution : $C.F = C_1 \cos x + C_2 \sin x + C_3 \cos 2x + C_4 \sin 2x$

[Refer Problem No. 27 in problems on C.F]

Step 1 : Use P.I formula.

$$P.I = \frac{1}{D^4+5D^2+4} \cos \frac{x}{2} \cos \frac{3x}{2}$$

Now use 2 cos A cos B = cos (A – B) + cos (A + B)

i.e. $$\cos\frac{x}{2}\cos\frac{3x}{2} = \frac{1}{2}\left[\cos\left(\frac{x}{2}-\frac{3x}{2}\right)+\cos\left(\frac{x}{2}+\frac{3x}{2}\right)\right]$$

$$= \frac{1}{2}[\cos(-x)+\cos 2x]$$

$$= \frac{1}{2}[\cos x+\cos 2x]$$

i.e. $$\text{P.I} = \frac{1}{D^4+5D^2+4}\,\frac{1}{2}\,(\cos x+\cos 2x)$$

Step 2 : Separate all terms.

$$= \frac{1}{2}\left\{\frac{1}{D^4+5D^2+4}\cos x+\frac{1}{D^4+5D^2+4}\cos 2x\right\}$$

Step 3 : As the zero factors for cos x and cos 2x are D^2+1 and D^2+4 respectively. ∴ Take the factors of f(D) for finding P.I.

$$= \frac{1}{2}\left\{\frac{1}{(D^2+1)(D^2+4)}\cos x+\frac{1}{(D^2+1)(D^2+4)}\cos 2x\right\}$$

Step 4 : Put $D^2 = -a^2$ only in non zero factor. i.e. $D^2 = -1$ and $D^2 = -4$ respectively.

$$= \frac{1}{2}\left\{\frac{1}{(-1+4)(D^2+1)}\cos x+\frac{1}{(-4+1)(D^2+4)}\cos 2x\right\}$$

Step 5 : Simplify.

$$= \frac{1}{6}\left\{\frac{1}{D^2+1}\cos x-\frac{1}{D^2+4}\cos 2x\right\}$$

Step 6 : Use $\frac{1}{D^2+a^2}\cos ax = \frac{+x}{2a}\sin ax$

$$= \frac{1}{6}\left\{\frac{x}{2}\sin x-\frac{x}{2.2}\sin x\right\}$$

y = C.F + P.I is the complete solution.

$$y = C_1\cos x+C_2\sin x+C_3\cos 2x+C_4\sin 2x+\frac{x}{12}\sin x-\frac{x}{24}\sin 2x.$$

Example 1.18 : $(D^2+4)\,y = \sin x\,\cos 3x$

Solution : A.E. $D^2+4 = 0$

$$D^2 = -4$$

$$D = \pm 2i$$

$$\alpha \pm i\beta$$

$$C.F. = e^{0x}(C_1 \cos 2x + C_2 \sin 2x)$$

$$P.I. = \frac{1}{f(D)} X$$

$\because$ As $\sin x \cos 3x$

$$= \frac{1}{2}[2 \sin x \cos 3x]$$

$$= \frac{1}{2}[\sin 4x + \sin(-2x)]$$

$$= \frac{1}{2}[\sin 4x - \sin 2x]$$

$[2 \sin A \cos B = \sin(A + B) + \sin(A - B)]$

Step 1 : Use P.I. formula.

$$P.I. = \frac{1}{D^2+4} \frac{1}{2} [\sin 4x - \sin 2x]$$

Step 2 : Separate all terms.

$$P.I. = \frac{1}{2}\left\{\frac{1}{D^2+4} \sin 4x - \frac{1}{D^2+4} \sin 2x\right\}$$

Step 3 : Observe that zero factor is not present in 1st P.I. but zero factor is present in 2nd P.I.

Step 4 : Separate

$$PI_1 = \frac{1}{2} \cdot \frac{1}{D^2+4} \sin 4x$$

Step 5 : $D^2 = -a^2$

$$PI_1 = \frac{1}{2} \cdot \frac{1}{-16+4} \sin 4x$$

Step 6 : Simplify.

$$PI_1 = \frac{1}{2} \cdot \frac{1}{-12} \sin 4x$$

$$PI_1 = \frac{-1}{24} \sin 4x$$

$$PI_2 = \frac{1}{2} \cdot \frac{1}{D^2+4} \sin 2x$$

Step 5 : Use formula (A)

$$PI_2 = \frac{1}{2} \cdot \left(\frac{-x}{2 \cdot 2}\right) \cdot \cos 2x$$

Step 6 : Simplify.

$$PI_2 = \frac{-x}{8} \cos 2x$$

$$P.I. = PI_1 + PI_2$$

$$y = C.F. + P.I.$$

$$= C_1 \cos 2x + C_2 \sin 2x - \frac{1}{24} \sin 4x - \frac{x}{8} \cos 2x$$

Example 1.19 : $(D^2+4)y = \cos x \cos 2x \cos 3x$

Hint : Use 2 cosA cosB = cos (A − B) + cos (A + B)

Solution :

$$= \frac{1}{2}[\cos(x-2x)+\cos(x+2x)]$$

Consider

$$\cos x \cos 2x = \frac{1}{2}[\cos(-x)+\cos 3x]$$

$$= \frac{1}{2}[\cos x+\cos 3x] \qquad \text{As } \{\cos(-\theta)=\cos\theta\}$$

Multiply both sides by cos 3x

$$\therefore \cos x \cos 2x \cos 3x = \frac{1}{2}[\cos x+\cos 3x]\cos 3x$$

$$= \frac{1}{2}[\cos 3x \cos x+\cos^2 3x]$$

Use the formula again and $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \frac{1}{2}\left\{\frac{1}{2}[\cos(2x)+\cos(4x)]+\frac{1+\cos 6x}{2}\right\}$$

$$= \frac{1}{4}\{\cos 2x+\cos 4x+1+\cos 6x\}$$

$$= \frac{1}{4}\{1+\cos 2x+\cos 4x+\cos 6x\}$$

To find C.F. put $D^2+4=0$

i.e. $D^2 = -4$

$$D = \pm 2i$$

$$\alpha = 0, \ \beta = 2$$

$$\text{C.F} = e^{0x}(C_1\cos 2x+C_2\sin 2x)$$

Step 1 : Use P.I formula.

$$\text{P.I} = \frac{1}{D^2+4}\frac{1}{4}[1+\cos 2x+\cos 4x+\cos 6x]$$

Step 2 : Separate PI_1, PI_2, PI_3 ....

$$= \frac{1}{4}\left\{\frac{1}{D^2+4}1+\frac{1}{D^2+4}\cos 2x+\frac{1}{D^2+4}\cos 4x+\frac{1}{D^2+4}\cos 6x\right\}$$

Step 3 : Put $D^2 = -a^2$ in non zero factor.

{ For constant D = 0

For zero factor $\frac{1}{D^2+a^2}\cos ax = \frac{+x}{2a}\sin ax$ }

$$\text{P.I.} = \frac{1}{4}\left\{\frac{1}{0+4}+\frac{x}{2.2}\sin 2x+\frac{1}{-16+4}\cos 4x+\frac{1}{-36+4}\cos 6x\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{4}+\frac{x}{4}\sin 2x-\frac{1}{12}\cos 4x-\frac{1}{32}\cos 6x\right\}$$

y = C.F + P.I is the complete solution.

$$y = C_1\cos 2x+C_2\sin 2x+\frac{1}{16}+\frac{x}{16}\sin 2x-\frac{1}{48}\cos 4x-\frac{1}{128}\cos 6x$$

Example 1.20 : $(D-1)^2(D^2+1)^2\,y=\cos^2 x/2$

Solution : A.E. $(D-1)^2\,(D^2+1)^2 = 0$

$$(D-1)^2 = 0,\quad (D^2+1)^2 = 0$$

$$D = 1, 1 \qquad D^2 = -1 \qquad \text{twice}$$

$$D = 1, 1, \qquad D = \pm i \qquad \text{twice}$$

∴ Roots are repeated real and repeated complex

$$\text{C.F.} = e^{0x}(C_1+C_2x)+e^{0x}[(C_3+C_4x)\cos x+(C_5+C_6x)\sin x]$$

$$\text{P.I.} = \frac{1}{f(D)}X$$

Step 1 : Apply P.I. formula.

$$\text{P.I.} = \frac{1}{(D^2+1)^2\,(D-1)^2}\cos^2\frac{x}{2}$$

Step 2 : Separate all terms.

$$\text{P.I.} = \frac{1}{(D^2+1)^2\,(D-1)^2}\left(\frac{1+\cos x}{2}\right)$$

Step 3 : Consider PI_1, PI_2

$$\text{P.I.} = \frac{1}{(D^2+1)^2\,(D^2-2D+1)}\left(\frac{1}{2}+\frac{\cos x}{2}\right)$$

Consider

$$PI_1 = \frac{1}{(D^2+1)^2\,(D-1)^2}\,\frac{1}{2}$$

Step 4 : As X constant replace D by zero.

$$PI_1 = \frac{1}{(D^2+1)^2\,(0-1)^2}\,\frac{1}{2}$$

Step 5 : Simplify.

$$PI_1 = \frac{1}{(1)(1)^2}\,\frac{1}{2}$$

$$= \frac{1}{2}$$

Consider

$$PI_2 = \frac{1}{(D^2+1)^2\,(D-1)^2}\cdot\frac{\cos x}{2}$$

Step 4 : Take the factors of f (D) for finding P.I. as the zero factor of cos x is present.

$$PI_2 = \frac{1}{2}\cdot\frac{1}{(D^2+1)^2\,(D^2-2D+1)}\cos x$$

Step 5 : Replace D^2 by $-a^2$ only in non zero term.

$$PI_2 = \frac{1}{2}\cdot\frac{1}{(D^2+1)^2\,(-1-2D+1)}\cos x$$

Step 6 : Simplify.

$$PI_2 = \frac{1}{2}\cdot\frac{1}{(D^2+1)^2}\cdot\frac{1}{-2D}\cdot\cos x$$

Step 7 : Use $\frac{1}{D}X = \int X\,dx$

$$PI_2 = \frac{-1}{4}\,\frac{1}{(D^2+1)^2}\sin x$$

Step 8 : Use formula 'c'

$$PI_2 = \frac{1}{-4}\left(-\frac{x^2}{8.1}\right)\sin x$$

Step 9 : Simplify.

$$= \frac{x^2}{32}\sin x$$

$$\therefore \quad PI = PI_1 + PI_2$$

$$= \frac{1}{2} + \frac{x^2}{32}\sin x$$

$$y = CF + PI$$

Example 1.21 : $(D^4+8D^2+16)y = cos^2 x$

Solution : AE $(D^2+4)^2 = 0$ $\quad D^2+4 = \sqrt{0} = 0$

$$D^2 = -4$$

$$D = \pm 2i$$

$$\alpha \pm i\beta$$

$$CF = e^{0x}[(c_1+c_2 x)\cos 2x + (c_3+c_4 c)\sin 2x]$$

$$PI = \frac{1}{f(D)} X$$

Step 1 : Use PI formula

$$PI = \frac{1}{(D^2+4)^2} \cos^2 x$$

Step 2 : Separate all the terms

$$PI = \frac{1}{(D^2+4)^2} \frac{1+\cos 2x}{2}$$

Step 3 : Use PI_1, PI_2 and observe f(D)

$$= \frac{1}{(D^2+4)^2}\frac{1}{2} + \frac{1}{(D^2+4)^2}\frac{\cos 2x}{2}$$

Consider

$$PI_1 = \frac{1}{(D^2+4)^2}\frac{1}{2}$$

Step 4 : As X = constant $\therefore$ Replace D by '0'.

$$PI_1 = \frac{1}{(0+4)^2}\frac{1}{2}$$

Step 5 : Simplify.

$$PI_1 = \frac{1}{32}$$

Consider

$$PI_2 = \frac{1}{(D^2+4)^2}\frac{\cos 2x}{2}$$

Step 4 : Use the factors as the zero factor of cos2x is present.

$$PI_2 = \frac{1}{2}\cdot\frac{1}{(D^2+4)^2}\cos 2x$$

Step 5 : Use formula 'C' i.e. $\dfrac{1}{\left(D^2+a^2\right)^2}\cos ax = \dfrac{-x^2}{8a^2}\cos ax$

$$PI_2 = \frac{1}{2}\cdot\frac{-x^2}{8\cdot 4}\cdot\cos 2x$$

Step 6 : Simplify.

$$= \frac{-x^2}{64}\cos 2x$$

$$\therefore \quad P.I. = PI_1 + PI_2$$

$$= \frac{1}{32} - \frac{x^2}{64}\cos 2x$$

Exercise 1.2

1. *Solve* $(D^4+2D^2+1)\,y = \cos x$ **[Ans. : $y = (C_1+C_2x)\cos x + (C_3+C_4x)\sin x - \frac{x^2}{8}\cos x$]**

2. $(D^3+D)\,y = \cos x$ **[Ans. : $y = C_1 + C_2\cos x + C_3\sin x - \frac{x}{2}\cos x$]**

3. $\dfrac{d^2y}{dx^2} + n^2y = h\sin px$

   $y = a,\ \dfrac{dy}{dx} = b$ *at* $x = 0$ **[Ans. : $y = a\cos nx + \left[\frac{b}{n} - \frac{ph}{n(n^2-p^2)}\right]\sin nx + \frac{h\sin px}{(n^2-p^2)}$]**

4. $\dfrac{d^2x}{dt^2} + n^2x = f\cdot\cos(nt+\alpha)$ **[Ans. : $x = C_1\cos nt + C_2\sin nt + \frac{ft}{2n}\sin(nt+\alpha)$]**

1.9 Type 3

$X = x^p$ where p is positive integer then use the binomial series.

1) $$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \frac{n(n-1)(n-2)}{3!}z^3 \ldots$$

2) $$\frac{1}{1+z} = 1 - z + z^2 - z^3 \ldots$$

3) $$\frac{1}{1-z} = 1 + z + z^2 + z^3 \ldots$$

Note : If f (D) involves the single factor like $(D-1)^2$, $(D+2)^3$ then use procedure 3A. For multiple factors $(D-1)\cdot(D-2)^2$ then procedure 3B.

Procedure 3A

Step 1 : Use P.I. formula.

Step 2 : Use the single factor for finding P.I.

Step 3 : Take least power term of D (constant) outside with its sign.

Step 4 : Write f (D) in the form $(1+z)^n$, write z and n.

Step 5 : Use the binomial series $(1+z)^n$ ....

Step 6 : Simplify.

Step 7 : Open the { }

Step 8 : Take the derivatives.

Step 9 : Simplify.

Procedure 3B

Step 1 : Use the P.I. formula.

Step 2 : Keep f (D) as it is (polynomial form).

Step 3 : Take least power term of D outside with its sign.

Step 4 : f (D) will take the form 1 + z or 1 − z write z.

Step 5 : Use binomial series $\frac{1}{1+z}$ or $\frac{1}{1-z}$ ....

Step 6 : Simplify.

Step 7 : Open the { }

Step 8 : Take the derivatives.

Step 9 : Simplify.

1.10 Illustrations on Type 3

Problems using procedure 3A

Example 1.22 : $(D^4 - 2D^3 + D^2)\, y = x^3$

Solution : A.E. $D^2 (D^2 - 2D + 1) = 0$

$$D^2 = 0, \quad (D-1)^2 = 0$$

$$D^2 = 0 \qquad D = 1, 1$$

$$\text{C.F.} = (C_1 + C_2 x)\, e^{0x} + (C_3 + C_4 x)\, e^x$$

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 1 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{(D^4 - 2D^3 + D^2)} x^3$$

Step 2 : Use the single factor.

$$\text{P.I.} = \frac{1}{D^2} \cdot \frac{1}{(D-1)^2} x^3$$

Step 3 : Take the least power term outside.

$$\text{P.I.} = \frac{1}{D^2} \cdot \frac{1}{(-1)^2} \cdot \frac{-1}{(1-D)^2} x^3$$

Step 4 : Write f (D) in the form $(1+z)^n$ $z = -D$, $n = -2$

$$\text{P.I.} = \frac{1}{D^2} [1 + (-D)]^{-2} x^3$$

Step 5 : Use the binomial series $(1+z)^n$.

$$\text{P.I.} = \frac{1}{D^2} \left\{ 1 + (-2)(-D) + \frac{(-2)(-3)}{2!}(-D)^2 + \frac{(-2)(-3)(-4)}{3!}(-D)^3 \right\} x^3$$

Step 6 : Simplify.

$$\text{P.I.} = \frac{1}{D^2} \{1 + 2D + 3D^2 + 4D^3\} x^3$$

Step 7 : Open the { }

$$\text{P.I.} = \frac{1}{D^2} \{x^3 + 2Dx^3 + 3D^2x^3 + 4D^3x^3\}$$

Step 8 : Simplify, take derivatives.

$$\text{P.I.} = \frac{1}{D^2} \{x^3 + 2(3x^2) + 3 \cdot (6x) + 4(6)\}$$

Step 9 : Use $\frac{1}{D} X = \int X \, dx$

$$\text{P.I.} = \frac{1}{D} \left\{ \frac{x^4}{4} + 2 \cdot 3 \cdot \frac{x^3}{3} + 3 \cdot 6 \cdot \frac{x^2}{2} + 4 \cdot 6 \cdot x \right\}$$

Step 10 : Again $\frac{1}{D} X = \int X \, dx$.

$$\text{P.I.} = \left\{ \frac{x^5}{4 \cdot 5} + 2 \cdot \frac{x^4}{4} + 3 \cdot 3 \cdot \frac{x^3}{3} + 4 \cdot 6 \cdot \frac{x^2}{2} \right\}$$

Step 11 : Simplify.

$$\text{P.I.} = \left\{\frac{x^5}{20}+\frac{x^4}{2}+3x^3+12x^2\right\}$$

$$y = \text{C.F.} + \text{P.I.}$$

i.e.
$$y = C_1+C_2x+(C_3+C_4x)\,e^x+\frac{x^5}{20}+\frac{x^4}{2}+3x^3+12x^2$$

Example 1.23 : $(D^3-6D^2+12D-8)\,y = x^2+1$

Solution : A.E.
$$D^3-6D^2+12D-8 = 0$$

$$(D-2)^3 = 0$$

$$D = 2, 2, 2$$

2	1	− 6	12	− 8
		2	− 8	8
	1	− 4	4	$\boxed{0}$

$$(D-2)\;(D^2-4D+4) = 0$$

$$(D-2)\;(D-2)^2 = 0$$

$$(D-2)^3 = 0$$

$$\text{C.F.} = (C_1+C_2x+C_3x^2)\,e^{2x}$$

$$\text{P.I.} = \frac{1}{f(D)}\,X$$

Step 1 : Use P.I formula.

$$\text{P.I.} = \frac{1}{D^3-6D^2+12D-8}\,x^2+1$$

Step 2 : Use the single factor.

$$\text{P.I.} = \frac{1}{(D-2)^3}\,(x^2+1)$$

Step 3 : Take the least power then outside.

$$\text{P.I.} = \frac{1}{(-2)^3}\cdot\frac{1}{\left(-\frac{D}{2}+1\right)^3}\,(x^2+1)$$

Step 4 : Write f(D) in the form $(1+z)^n$, $z = \frac{-D}{2}$, $n = -3$.

$$\text{P.I.} = \frac{1}{-8}\left[1+\left(\frac{-D}{2}\right)\right]^{-3}(x^2+1)$$

Step 5 : Use the binomial series $(1+z)^n = \left\{1+nz+\frac{n(n-1)}{2!}z^2 \ldots\right\}$

$$= \frac{1}{-8}\left\{1+(-3)\left(\frac{-D}{2}\right)+\frac{(-3)(-4)}{2!}\left(\frac{-D}{2}\right)^2\ldots\right\}(x^2+1)$$

Step 6 : Simplify.

$$= \frac{-1}{8}\left\{1+\frac{3}{2}D+\frac{3}{2}D^2\right\}(x^2+1)$$

Step 7 : Take derivatives.

$$= \frac{-1}{8}\left\{(x^2+1)+\frac{3}{2}D(x^2+1)+\frac{3}{2}D^2(x^2+1)\ldots\right\}$$

Step 8 : Simplify.

$$= \frac{-1}{8}\left\{(x^2+1)+\frac{3}{2}(2x)+\frac{3}{2}(2)\right\}$$

Step 9 : Simplify.

$$= \frac{-1}{8}\{x^2+3x+4\}$$

$$y = \text{C.F.} + \text{P.I.}$$

Example 1.24 : $(D^3+2D^2+D)\,y = e^{2x}+x^2+x$

Solution :

$$\text{A.E.} \quad D(D^2+2D+1) = 0$$

$$D(D+1)^2 = 0$$

$$D = 0, \quad D = -1, -1$$

$$\text{C.F.} = C_1 e^{0x} + (C_2 + C_3 x)\, e^{-x}$$

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 1 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{D^3+2D^2+D} e^{2x}+x^2+x$$

Consider $\qquad PI_1 = \dfrac{1}{D(D+1)^2} e^{2x}$

Replace D by 2 only in non zero factor.

$$= \frac{1}{2(2+1)^2} e^{2x}$$

$$PI_1 = \frac{e^{2x}}{18}$$

Also consider $\qquad PI_2 = \dfrac{1}{D^3+2D^2+D} x^2+x$

Step 2 : Use single factor of f(D).

$$PI_2 = \frac{1}{D(D+1)^2}(x^2+x)$$

Step 3 : Take least power term outside.

$$PI_2 = \frac{1}{D}\cdot\frac{1}{(D+1)^2}(x^2+x)$$

Step 4 : Write f(D) in the form of $(1+z)^n$.

$$PI_2 = \frac{1}{D}(1+D)^{-2}(x^2+x)$$

Step 5 : Use the binomial series $(1+z)^n = \left\{1+nz+\dfrac{n(n-1)}{2!}z^2 \ldots\right\}$

$$PI_2 = \frac{1}{D}\left\{1+(-2)(D)+\frac{(-2)(-3)}{2!}(D)^2\ldots\right\}(x^2+x)$$

Step 6 : Simplify.

$$PI_2 = \frac{1}{D}\{1-2D+3D^2\ldots\}(x^2+x)$$

Step 7 : Open the { }

$$PI_2 = \frac{1}{D}\{(x^2+x)-2D(x^2+x)+3D^2(x^2+x)\ldots\}$$

Step 8 : Take the derivatives

$$PI_2 = \frac{1}{D}\{x^2+x-2(2x+1)+3(2)\}$$

Step 9 : Simplify.

$$PI_2 = \frac{1}{D}\{x^2+3x+4\}$$

Step 10 : Use $\frac{1}{D} X = \int X\, dx$

$$PI_2 = \left\{\frac{x^3}{3} + 3\frac{x^2}{2} + 4x\right\}$$

$$y = C.F + P.I \text{ is the complete solution.}$$

Problems using procedure 3B

Example 1.25 : $(D^3 - 2D + 4)\,y = 3x^2 - 5x + 2$

Solution : $$C.F. = C_1 e^{-2x} + e^{-x}(C_2 \cos x + C_3 \sin x)$$

[Refer Problem No. 13 in problems on C.F]

Step 1 : Use P.I. formula.

$$P.I. = \frac{1}{f(D)} X$$

Step 2 : Keep f (D) as it is.

$$P.I. = \frac{1}{D^3 - 2D + 4} 3x^2 - 5x + 2$$

Step 3 : Take the least power term outside.

$$P.I. = \frac{1}{4} \cdot \frac{1}{\left[1 + \frac{D^3 - 2D}{4}\right]} (3x^2 - 5x + 2)$$

Step 4 : f (D) will take the form (1 + z) here $z = \frac{D^3 - 2D}{4}$.

Step 5 : Use binomial series. $\frac{1}{1+z} = \{1 - z + z^2 ....\}$

$$P.I. = \frac{1}{4}\left\{1 - \left(\frac{D^3 - 2D}{4}\right) + \left(\frac{D^3 - 2D}{4}\right)^2 ...\right\} (3x^2 - 5x + 2)$$

Step 6 : Simplify.

$$P.I. = \frac{1}{4}\left\{1 - \frac{D^3}{4} + \frac{D}{2} + \left(\frac{D^6 - 4D^4 + 4D^2}{16}\right)\right\} (3x^2 + 5x + 2)$$

Step 7 : Open { }.

$$P.I. = \frac{1}{4}\left\{(3x^2 - 5x + 2) - \frac{D^3}{4}(3x^2 - 5x + 2) + \frac{D}{2}(3x^2 - 5x + 2)\right.$$
$$\left. + \frac{1}{16}\left[D^6(3x^2 - 5x + 2) - 4D^4(3x^2 - 5x + 2) + 4D^2(3x^2 - 5x + 2)\right]\right\}$$

Step 8 : Take derivatives.

$$\text{P.I.} = \frac{1}{4}\left\{(3x^2-5x+2)-0+\frac{1}{2}(6x-5)+\frac{1}{16}[0-0+4(6)]\right\}$$

Step 9 : Simplify.

$$\text{P.I.} = \frac{1}{4}\{3x^2-2x+1\}$$

$$y = \text{C.F + P.I} \text{ is the complete solution.}$$

Example 1.26 : $(D^3-3D+2)\,y=(x^3-4x^2+2)$

Solution : A.E. $D^3-3D+2 = 0$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$(D-1)(D^2+D-2)$
$D = 1$

$$D = \frac{-1\pm\sqrt{1-8}}{2}$$

$$D = -\frac{1}{2}\pm i\frac{\sqrt{7}}{2}$$

$$\text{C.F.} = C_1e^x+e^{-x/2}\left[C_2\cos\frac{\sqrt{7}x}{2}+C_2\sin\frac{\sqrt{7}x}{2}\right]$$

Step 1 : Use P.I. formula. P.I. $= \frac{1}{f(D)}X$

Step 2 : Keep f (D) as it is

$$\text{P.I.} = \frac{1}{D^3-3D+2}(x^3-4x^2+2)$$

Step 3 : Take least power term outside.

$$\text{P.I.} = \frac{1}{2}\frac{1}{1+\left(\frac{D^3-3D}{2}\right)}(x^3-4x^2+2)$$

Step 4 : Here f (D) will take the form $\frac{1}{1+z}$.

Step 5 : Use binomial series. $\frac{1}{1+z}=\{1-z+z^2-z^3....\}$

$$\text{P.I.} = \frac{1}{2}\left\{1-\left(\frac{D^3-3D}{2}\right)+\left(\frac{D^3-3D}{2}\right)^2-\left(\frac{D^3-3D}{2}\right)^3\right\}(x^3-4x^2+2)$$

Step 6 : As x^3 is involved $\therefore$ every term after D^3 will be zero $\therefore$ Write only significant terms.

$$\text{P.I.} = \frac{1}{2}\left\{1-\left(\frac{D^3-3D}{2}\right)+\left(\frac{D^6-6D^4+9D^2}{4}\right)-\left(\frac{27}{8}D^3\right)\right\}(x^3-4x^2+2)$$

Step 7 : Open the bracket and take the derivatives.

$$\text{P.I.} = \frac{1}{2}\left\{(x^3-4x^2+2)-\left(\frac{6-3(3x^2-8x)}{2}\right)+\frac{0-0+9(6x-8)}{4}-\frac{27}{8}(6)\right\}$$

Step 8 : Simplify.

$$\text{P.I.} = \frac{1}{2}\left\{x^3+\left(-4+\frac{9}{2}\right)x^2+\left(-12+\frac{27}{2}\right)x+\left(2-3-18-\frac{81}{4}\right)\right\}$$

$$\text{P.I.} = \frac{1}{2}\left\{x^3+\frac{x^2}{2}+\frac{3}{2}x+\frac{5}{4}\right\}$$

$$y = \text{C.F.} + \text{P.I.} \text{ is the complete solution.}$$

Exercise 1.3

1. $(D^3+6D^2+12D+8)\,y=e^{-2x}+x^2$ **[Ans. : $y=(C_1+C_2x+C_3x^2)\,e^{-2x}+e^{-2x}\cdot\frac{x^3}{6}+\frac{1}{8}(x^2-3x+3)$]**

2. $(D^2+6D+9)\,y=\cos 3x+3^x+x^2+e^{-3x}$ **[Ans. : $y=(C_1+C_2x)\,e^{-3x}+\frac{\sin 3x}{18}+\frac{3^x}{(\log 3+3)^2}+\frac{1}{9}\left(x^2-\frac{4}{3}x+\frac{2}{3}\right)+\frac{x^2}{2}e^{-3x}$]**

3. $(D^4-a^4)\,y=x^4$ **[Ans. : $y=C_1\cos ax+C_2\sin ax+C_3e^{ax}+C_4e^{-ax}-\frac{1}{a^4}\left(x^4+\frac{24}{a^4}\right)$]**

4. $(D^2+2D+2)\,y=x^2+\cos x$ **[Ans. : $y=e^{-x}(C_1\cos x+C_2\sin x)+\frac{1}{2}(x^2-2x+1)+\frac{1}{5}(2\sin x+\cos x)$]**

5. $(D^4+D^2+1)\,y=53x^2+17$ **[Ans. : P.I. = $53x^2-89$]**

6. $(D^5-D)\,y=12e^x+85mx+2^x$ **[Ans. : P.I. = $3x\,e^x-\frac{85mx^2}{2}+\frac{2^x}{(\log 2)^5-(\log 2)}$]**

7. $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+10y=50x$

   at $x=0$, $y=0$ and $\frac{dy}{dx}=0$ **[Ans. : $y=e^{-3x}[3\cos x+5\sin x]+5x-3$]**

1.11 Type 4

If $X = e^{ax}$ V where

V is any function of x.

We know that

$$D\left(e^{ax}\ V\right) = e^{ax}\ DV + V\,e^{ax}\ a$$

$$= e^{ax}\ (DV + aV)$$

$$D\left(e^{ax}\ V\right) = e^{ax}\ (D+a)\ V$$

i.e. Derivative of e^{ax} V is obtained by taking e^{ax} outside and replacing D by D + a.

Similarly we can show that

$$f(D)\,e^{ax}\ V = e^{ax} f\,(D+a)\ V$$

and $$\boxed{\frac{1}{f\,(D)} e^{ax} V = e^{ax}\left[\frac{1}{f(D+a)} V\right]}$$

Note : To operate $\frac{1}{f\,(D)}$ on a product $(e^{ax} V)$ where V is any function of x.

Take e^{ax} to the left side as a multiple replace D by (D + a) in f (D) and operate $\frac{1}{f\,(D+a)}$ on V.

Note : If V is of the form x^p then the problem reduces to type 3.

If V is sinax or cosax then the problem reduces to type 4.

1.12 Illustrations on Type 4

Example 1.27 : $(D^3 - 7D - 6)\,y = e^{2x}(1+x)$

Solution : A.E. $D^3 - 7D - 6 = 0$

$$\text{C.F.} = C_1 e^{-x} + C_2 e^{3x} + C_3 e^{-2x}$$

[Refer Problem No. 1 in problems on C.F.]

$$\text{P.I.} = \frac{1}{f\,(D)} X$$

Step 1 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{D^3 - 7D - 6} e^{2x}(1+x)$$

Step 2 : Take out e^{ax} and replace D by (D + a) i.e. D + 2

$$\text{P.I.} = e^{2x} \frac{1}{(D+2)^3 - 7(D+2) - 6}(1+x)$$

Step 3 : Simplify f (D + a).

$$= e^{2x} \frac{1}{D^3 + 6D^2 + 5D - 12}(1+x)$$

Step 4 : Problem reduces to Type 3, take least power outside.

$$= \frac{e^{2x}}{-12} \frac{1}{\left[1 - \left(\frac{D^3 + 6D^2 + 5D}{12}\right)\right]}(1+x)$$

Step 5 : f (D) will take the form $\frac{1}{1-z}$

Here $z = \frac{D^3 + 6D^2 + 5D}{12}$ use binomial series. $\frac{1}{1+z} = 1 + z + z^2 \ldots.$

$$= \frac{e^{2x}}{-12}\left\{1 + \left(\frac{D^3 + 6D^2 + 5D}{12}\right) \ldots\right\}(1+x)$$

Step 6 : Open the { }.

$$= \frac{e^{2x}}{-12}\left\{(1+x) + \frac{D^3(1+x) + 6D^2(1+x) + 5D(1+x)}{12}\right\}$$

Step 7 : Take derivatives.

$$= \frac{e^{2x}}{-12}\left\{1 + x + \frac{0+0+5}{12}\right\}$$

Step 8 : Simplify.

$$\text{P.I.} = \frac{e^{2x}}{-12}\left\{x + \frac{17}{12}\right\}$$

Exercise 1.4

1. $(D^3 - 7D - 6)\,y = e^{2x}(1 + x^2)$ **[Ans. : P.I. =** $\frac{e^{2x}}{-12}\left\{x^2 + \frac{5x}{6} + \frac{169}{72}\right\}$**]**

2. $(D^2 + 5D + 6)\,y = 4x^2 e^x$ **[Ans. :** $y = C_1e^{-2x} + C_2e^{-3x} + \frac{e^x}{3}\left(x^2 - \frac{7x}{6} + \frac{37}{72}\right)$**]**

3. $(D^2 - 1)\,y = e^x(1 + x^2)$ **[Ans. :** $y = C_1e^x + C_2e^{-x} + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{2}\right]$**]**

4. $(D^2+6D+9)\,y = \frac{1}{x^3}e^{-3x}$ [Ans. : $y = (C_1 + C_2x)\,e^{-3x} + \frac{e^{3x}}{2x}$]

5. $(D^2-D-1)\,y = e^{-x}x^3$ [Ans. : P.I. $= e^{-x}(x^3+9x^2+48x+126)$]

6. $\left(D^4-2D^3-3D^2+4D+4\right)y = x^2e^x$

[Ans. : $y = \left(C_1+C_2x\right)e^{-x} + \left(C_3+C_4x\right)e^{2x} + \frac{e^x}{4}\left(x^2+2x+\frac{7}{2}\right)$]

7. $\left(D^3-6D^2+11D-6\right)y = xe^x$ [Ans. : $y = C_1e^x + C_2e^{2x} + C_3e^{3x} + \frac{e^x}{2}\left(\frac{x^2}{2}+\frac{3x}{2}\right)$]

8. $\left(D^3-3D+2\right)y = x^2e^x$ [Ans. : $y = \left(C_1+C_2x\right)e^x + C_3e^{-x} + \frac{e^xx^2}{108}\left(3x^2-4x+4\right)$]

Example 1.28 : $(D^4-1)\,y = \cos x\,\cosh x$

Solution :

$$\text{C.F.} = C_1e^x + C_2e^{-x} + C_3\cos x + C_4\sin x$$

[Refer Problem No. 9 in problems on C.F)

$$\text{P.I.} = \frac{1}{f(D)}X$$

Step 1 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{D^4-1}\cos x\ \cosh x$$

Step 2 : Use $\cosh x = \left(\frac{e^x+e^{-x}}{2}\right)$

$$\text{P.I.} = \frac{1}{D^4-1}\cos x\left(\frac{e^x+e^{-x}}{2}\right)$$

Step 3 : Separate the terms.

$$\text{P.I.} = \frac{1}{2}\left\{\frac{1}{D^4-1}e^x\cos x + \frac{1}{D^4-1}e^{-x}\cos x\right\}$$

Step 4 : Take out e^{ax} and replace D by D + a.

$$\text{P.I.} = \frac{1}{2}\left\{e^x\frac{1}{(D+1)^4-1}\cos x + e^{-x}\frac{1}{(D-1)^4-1}\cos x\right\}$$

Step 5 : Simplify. Now the problem is reduced to type 2.

$$\text{P.I.} = \frac{1}{2}\left\{e^x\cdot\frac{1}{D^4+4D^3+6D^2+4D+1-1}\cos x\right.$$

$$\left.+\,e^{-x}\cdot\frac{1}{D^4-4D^3+6D^2-4D+1-1}\cos x\right\}$$

Step 6 : Replace $D^2 = -a^2$, $D^3 = -a^2 D$, $D^4 = a^4$ Here a = 1.

$$\text{P.I.} = \frac{1}{2}\left\{e^x \frac{1}{1-4D-6+4D}\cos x + e^{-x}\frac{1}{1+4D-6-4D}\cos x\right\}$$

Step 7 : Simplify.

$$\text{P.I.} = \frac{1}{2}\left\{e^x \frac{1}{-5}\cos x + e^{-x}\frac{1}{-5}\cos x\right\}$$

Step 8 : Simplify.

$$= \frac{1}{-5}\left(\frac{e^x + e^{-x}}{2}\right)\cos x$$

$$= \frac{-1}{5}\cosh x \cos x$$

Example 1.29 : $(D^4+1)\,y = e^x \cos 2x$

Solution : A.E. $D^4 + 1 = 0$ [Refer Problem No. 23 in problems on C.F.]

$$\text{C.F.} = e^{-x/\sqrt{2}}\left[C_1 \cos\frac{x}{\sqrt{2}} + C_2 \sin\frac{x}{\sqrt{2}}\right] + e^{x/\sqrt{2}}\left[C_3 \cos\frac{x}{\sqrt{2}} + C_4 \sin\frac{x}{\sqrt{2}}\right]$$

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 1 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{D^4+1} e^x \cos 2x$$

Step 2 : Take e^{ax} outside replace D by D + a, here a = 1.

$$\text{P.I.} = e^x \frac{1}{(D+1)^4+1}\cos 2x$$

Step 3 : Simplify.

$$\text{P.I.} = e^x \frac{1}{D^4+4D^3+6D^2+4D+1+1}\cos 2x$$

Step 4 : Replace $D^2 = -4$, $D^3 = -4D$, $D^4 = 16$.

$$\text{P.I.} = e^x \frac{1}{16+4(-4D)+6(-4)+4D+1+1}\cos 2x$$

Step 5 : Simplify.

$$\text{P.I.} = e^x \frac{1}{-12D-6}\cos 2x$$

Step 6 : Rationalise.

$$\text{P.I.} = \frac{e^x}{-6}\cdot\frac{1}{2D+1}\cdot\frac{2D-1}{2D-1}\cos 2x$$

Step 7 : Simplify.

$$\text{P.I.} = \frac{e^x}{-6}\cdot\frac{2D-1}{4D^2-1}\cos 2x$$

Step 8 : Again $D^2 = -4$.

$$\text{P.I.} = \frac{e^x}{-6}\cdot\frac{2D-1}{-16-1}\cos 2x$$

Step 9 : Simplify.

$$\text{P.I.} = \frac{e^x}{102}(2D-1)\cos 2x$$

Step 10 : Open the bracket.

$$\text{P.I.} = \frac{e^x}{102}[2D\cos 2x - \cos 2x]$$

Step 11 : Take the derivatives.

$$\text{P.I.} = \frac{e^x}{102}[-4\sin 2x - \cos 2x]$$

y = C.F + P.I is the complete solution.

Exercise 1.5

1. $(D^4 + D^2 + 1)y = ax^2 + be^{-x}\sin 2x$

[Ans. : $y = e^{-x/2}\left[C_1\cos\frac{x\sqrt{3}}{2} + C_2\sin\frac{x\sqrt{3}}{2}\right] + e^{x/2}\left[C_3\cos\frac{x\sqrt{3}}{2} + C_4\sin\frac{x\sqrt{3}}{2}\right] + a(x^2-2) + \frac{be^{-x}}{-481}(20\cos 2x + 9\sin 2x)$**]**

2. $(D^2 + D - 6)y = e^{-2x}\sin 3x$

[Ans. : $y = C_1e^{-3x} + C_2e^{2x} + \frac{e^{-2x}}{250}[9\cos 3x - 13\sin 3x]$ **]**

3. $(D^3 + 3D)y = \sinh x\sin 3x$

[Ans. : $y = C_1 + C_2\cos\sqrt{3}x + C_2\sin\sqrt{3}x + \frac{1}{610}\{9\cos 3x\sinh x - 23\cosh x\sin 3x\}$ **]**

4. $(D^3 - D^2 + 3D + 5)y = e^x\cos 3x$

[Ans. : $y = C_1e^{-x} + e^x[C_2\cos 2x + C_3\sin 3x] - \frac{e^x}{65}[3\sin 3x + 2\cos 3x]$ **]**

5. $(D^2 - 1)y = \cosh x\cos x$

[Ans. : $y = C_1e^x + C_2e^{-x} + \frac{1}{5}[2\sinh x\sin x - \cosh x\cos x]$**]**

6. $(D^2 - 6D + 13)\,y = 8e^{3x}\sin 4x$ **[Ans. : $y = e^{3x}[C_1\cos 2x + C_2\sin 2x] - \frac{2}{3}e^{3x}\sin 4x$]**

7. $(D^2 + 5D + 6)\,y = e^{-2x}\sin 2x + 4x^2e^x$

[Ans. : $y = C_1e^{-2x} + C_2e^{-3x} + \frac{e^{-2x}}{-10}(\cos 2x + 2\sin 2x) + \frac{e^x}{3}\left(x^2 - \frac{7x}{6} + \frac{37}{72}\right)$]

8. $(D^4 + D^2 + 1)\,y = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$ **[Ans. : P.I. = $\frac{x\,e^{-x/2}}{4\sqrt{3}}\left[\sin\frac{\sqrt{3}x}{2} + \sqrt{3}\cos\frac{\sqrt{3}x}{2}\right]$]**

9. $(D^2 - 2D + 2)\,y = e^{-x}\sin x$ **[Ans. : $y = e^{-x}(C_1\cos x + C_2\sin x) - \frac{x}{2}e^{-x}\cos x$]**

10. $(D^2 - 4D + 8)\,y = e^{2x}\sin(2x + 5)$ **[Ans. : $y = e^{2x}(C_1\cos 2x + C_2\sin 2x) - \frac{x}{4}e^{2x}\cos(2x + 5)$]**

11. $(D^3 - 7D - 6)\,y = \cosh x\cos x$

[Ans. : $y = C_1e^{-x} + C_2e^{-2x} + C_3e^{3x} - \frac{e^x}{100}(\sin x + 3\cos x) - \frac{e^{-x}}{68}(5\sin x - 3\cos x)$]

12. $(D^2 - 1)\,y = \cosh x\sin x\cos x$ **[Ans. : $y = C_1e^x + C_2e^{-x} - \frac{1}{16}\sinh x\cos 2x - \frac{1}{16}\cosh x\sin 2x$]**

1.13 Type 5

Let X = x V where V is any function of x

$$\therefore \quad D(xV) = x\,DV + V$$

$$\therefore \quad D^2(xV) = x\,D^2V + 2DV$$

$$D^3(xV) = x\,D^3V + 3D^2V$$

Similarly $\quad D^n(xV) = x\,D^nV + n\,D^{n-1}V$

$$\therefore \quad f(D)(xV) = x\,f(D)V + f'(D)V$$

Let $\quad f(D)\,V = V_1$ i.e. $V = \frac{1}{f(D)}V_1$

$$\therefore \quad f(D)\left[x\frac{1}{f(D)}V_1\right] = xV_1 + f'(D)\cdot\frac{1}{f(D)}V_1$$

$$\left[x\cdot\frac{1}{f(D)}V_1\right] = \frac{1}{f(D)}xV_1 + \frac{f'(D)}{[f(D)]^2}V_1$$

$$\therefore \quad \frac{1}{f(D)}xV_1 = x\cdot\frac{1}{f(D)}V_1 - \frac{f'(D)}{[f(D)]^2}V_1$$

$$\frac{1}{f(D)}x\,V_1 = \left\{x - \frac{f'(D)}{f(D)}\right\}\frac{1}{f(D)}V_1$$

But here V_1 is a function of x hence

$$\boxed{\frac{1}{f(D)} x V = \left\{x - \frac{f'(D)}{f(D)}\right\} \frac{1}{f(D)} V}$$

Note :

1) Here V is any function of x mostly trigonometric.
2) Power of x must be one.
3) If V is sinax or cosax and f (D) involves the factor $(D^2 + a^2)$ [i.e. the zero factor of sinax or cosax] then don't use type 5 use type 6.

   i.e. for (1) $\frac{1}{D^2 + a^2}$ x sin ax and (2) $\frac{1}{D^2 + a^2}$ x cos ax don't use type 5 use type 6.

1.14 Illustrations on Type 5

Example 1.30 : $(D^2 + 3D + 2)\, y = x e^{-x} \sin x$

Solution : A.E. $(D + 1)(D + 2) = 0$

$$D = -1, \quad D = -2$$

$$C.F. = C_1 e^{-x} + C_2 e^{-2x}$$

$$P.I. = \frac{1}{f(D)} X$$

Step : Use P.I. formula.

$$P.I. = \frac{1}{D^2 + 3D + 2} x e^{-x} \sin x$$

Step : As e^{-x} is present use $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$ i.e. Type 4.

$$P.I. = e^{-x} \frac{1}{(D-1)^2 + 3(D-1) + 2} x \sin x$$

Step : Simplify.

$$P.I. = e^{-x} \left\{\frac{1}{D^2 + D} x \sin x\right\}$$

Step 1 : Apply the formula $\frac{1}{f(D)} xV = \left\{x - \frac{f'(D)}{f(D)}\right\} \frac{1}{f(D)} V$

$$= e^{-x} \left\{x - \frac{2D+1}{D^2 + D}\right\} \frac{1}{D^2 + D} \sin x$$

Step 2 : Replace $D^2 = -a^2$ outside as well as inside the bracket i.e. $D^2 = -1$

$$= e^{-x}\left\{x - \frac{2D+1}{-1+D}\right\}\frac{1}{-1+D}\sin x$$

Step 3 : Rationalise outside and inside the bracket.

$$= e^{-x}\left\{x - \frac{(2D+1)(D+1)}{(D-1)(D+1)}\right\}\frac{D+1}{(D-1)(D+1)}\sin x$$

Step 4 : Simplify.

$$= e^{-x}\left\{x - \frac{2D^2+3D+1}{D^2-1}\right\}\frac{D+1}{D^2-1}\sin x$$

Step 5 : Replace $D^2 = -a^2$ outside as well as inside the { }

$$= e^{-x}\left\{x - \frac{-2+3D+1}{-1-1}\right\}\frac{D+1}{-1-1}\sin x$$

Step 6 : Simplify and take C_2 to the left side of { }

$$= \frac{e^{-x}}{-2}\left\{x + \frac{(3D-1)}{2}\right\}(D+1)\sin x$$

Step 7 : Open the { }

$$= \frac{e^{-x}}{-2}\left\{x(D+1)\sin x + \frac{1}{2}(3D-1)(D+1)\sin x\right\}$$

Step 8 : Simplify 2^{nd} term.

$$= \frac{e^{-x}}{-2}\left\{x(D+1)\sin x + \frac{1}{2}(3D^2+2D-1)\sin x\right\}$$

Step 9 : $D^2 = -a^2$ in 2^{nd} term.

$$= \frac{e^{-x}}{-2}\left\{x(D+1)\sin x + \frac{1}{2}(-3+2D-1)\sin x\right\}$$

Step 10 : Simplify 2^{nd} term.

$$= \frac{e^{-x}}{-2}\left\{x(D+1)\sin x + \frac{1}{2}(2D-4)\sin x\right\}$$

Step 11 : Take the derivatives.

$$= \frac{e^{-x}}{-2}\{x(\cos x + \sin x) + (\cos x - 2\sin x)\}$$

Step 12 : Collect the terms.

$$= \frac{e^{-x}}{-2}\{(x+1)\cos x + (x-2)\sin x\}$$

Exercise 1.6

1. $(D^2-1)\,y = x\,e^x\sin x$ [Ans. : $y = C_1e^x + C_2e^{-x} - \frac{e^x}{25}\{(10x+2)\cos x + (5x-14)\sin x\}$]

2. $(D^2-2D+1)\,y = x\,e^x\sin x$ [Ans. : $y = (C_1+C_2x)\,e^x - e^x\left[x\sin x - 2\cos x\right]$]

3. $(D^2+2D+1)\,y = x\cos x$ [Ans. : $y = (C_1+C_2x)\,e^{-x} + \frac{1}{2}\left[(x-1)\sin x + \cos x\right]$]

4. $(D^3-2D+2)\,y = x\cos x$ [Ans. : P.I. $= \frac{1}{5}\left\{\left(x+\frac{2}{5}\right)\cos x - \left(2x+\frac{14}{5}\right)\sin x\right\}$]

5. $(D^3+3D+2)\,y = x\cos 2x$ [Ans. : P.I. $= \frac{1}{20}\left\{\left(3x-\frac{7}{20}\right)\sin 2x + \left(\frac{12}{5}-x\right)\cos 2x\right\}$]

6. $(D^2+4)\,y = x\sin x$ [Ans. : $y = C_1\cos 2x + C_2\sin 2x + \frac{1}{9}(3x\sin x - 2\cos x)$]

7. $(D^2-1)\,y = x\sin x + e^x(1+x^2)$ [Ans. : P.I. $= \frac{-1}{2}(x\sin x + \cos x) + \frac{x\,e^x}{12}(2x^2-3x+9)$]

8. $(D^2+3D+2)\,y = x\sin 2x$ [Ans. : P.I. $= \frac{-1}{200}\{(10x-24)\sin 2x + (30x-7)\cos 2x\}$]

1.15 Type 6

If $X = x^n \sin ax$ or $x^n \cos ax$

We know that $e^{iax} = \cos ax + i\sin ax$

$$\therefore \quad \frac{1}{f(D)}\, x^n\,[\cos ax + i\sin ax] = \frac{1}{f(D)}\, x^n \cdot e^{iax}$$

Now use type 4

$$= e^{iax}\,\frac{1}{f(D+ai)}\,x^n$$

Now this can be evaluated by type 3 and then equating real and imaginary parts we get the result.

$$\frac{1}{f(D)}\,x^n\sin ax = \text{Img}\ \frac{1}{f(D)}\,x^n e^{iax}$$

$$\frac{1}{f(D)}\,x^n\cos ax = \text{Real}\ \frac{1}{f(D)}\,x^n e^{iax}$$

Illustration on Type 6

Example 1.31 : $\left[(D^2+2D+5)^2 y\right] e^x = x\cos 2x$

Solution :

$$(D^2+2D+5)^2 y = e^{-x}\cdot x\cos 2x$$

A.E. $(D^2+2D+5)^2 = 0$

$$D = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= -1 \pm 2i \quad \text{repeated complex}$$

$$\text{C.F.} = e^{-x}\left[(C_1 + C_2x)\cos 2x + (C_3 + C_4x)\sin 2x\right]$$

Step : Use P.I. formula.

$$\text{P.I.} = \frac{1}{(D^2 + 2D + 5)^2} e^{-x} x \cos 2x$$

Step 1 : Use type 4 i.e. $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$

$$= e^{-x} \frac{1}{\left((D-1)^2 + 2(D-1) + 5\right)^2} x \cos 2x$$

Step 2 : Simplify.

$$= e^{-x} \frac{1}{(D^2 + 4)^2} x \cos 2x$$

Step 3 : Here power of x is one but $D^2 + 4$ is the zero factor for cos 2x $\therefore$ Don't use type 5 use type 6.

$$= e^{-x} \text{ Real } \frac{1}{(D^2 + 4)^2} x\, e^{i2x}$$

Step 4 : Use type 4 $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V.$

$$= e^{-x} \text{ Real } e^{i2x} \frac{1}{\left[(D + 2i)^2 + 4\right]^2} x$$

Step 5 : Simplify.

$$= e^{-x} \text{ Real } e^{i2x} \frac{1}{\left[D^2 + 4iD - 4 + 4\right]^2} x$$

Step 6 : Simplify now the problem is reduced to type 3.

$$= e^{-x} \text{ Real } e^{i2x} \frac{1}{\left[D^2 + 4iD\right]^2} x$$

Step 7 : Take least power term outside.

$$= e^{-x} \text{ Real } e^{i2x} \frac{1}{(4iD)^2} \cdot \frac{1}{\left[1 + \frac{D}{4i}\right]^2} x$$

Step 8 : Write f (D) in the form $(1+z)^n$ {Note : $i^2 = -1$}

$$= e^{-x} \text{ Real } e^{i2x} \frac{1}{-16 D^2}\left[1+\frac{D}{4i}\right]^{-2} x$$

Step 9 : Use the binomial series $(1+z)^n = 1+nz+\frac{n(n-1)}{2!}z^2....$

$$= e^{-x} \text{ Real } e^{i2x} \frac{1}{-16 D^2}\left[1+(-2)\left(\frac{D}{4i}\right)+...\right] x$$

Step 10 : Simplify and open the bracket.

$$= e^{-x} \text{Real } e^{i2x} \frac{1}{-16 D^2}\left[x-\frac{1}{2i}\right]$$

Step 11 : Use $\frac{1}{i} = -i$.

$$= e^{-x} \text{ Real } e^{i2x} \frac{1}{-16 D^2}\left\{x+\frac{i}{2}\right\}$$

Step 12 : $\frac{1}{D}X = \int X\,dx$ $\therefore$ Integrals twice.

$$= e^{-x} \text{ Real } e^{-i2x} \frac{1}{-16}\left\{\frac{x^3}{2\cdot 3}+\frac{i}{2}\cdot\frac{x^2}{2}\right\}$$

Step 13 : Use $e^{iax} = (\cos ax + i \sin ax)$

$$= \frac{e^{-x}}{-16} \text{Real } (\cos 2x + i \sin 2x)\left(\frac{x^3}{6}+\frac{ix^2}{4}\right)$$

Step 14 : Use (a + ib) (c + id) = (ac − bd) + i (ad + bc).

$$= \text{Real} \frac{e^{-x}}{-16}\left\{\left(\frac{x^3}{6}\cos 2x - \frac{x^2}{4}\sin 2x\right)+i\left(\frac{x^3}{6}\sin 2x+\frac{x^2}{4}\cos 2x\right)\right\}$$

Step 15 : Write the real term as the P.I.

$$= \frac{e^{-x}}{-16}\left(\frac{x^3}{6}\cos 2x - \frac{x^2}{4}\sin 2x\right)$$

$$y = \text{C.F. + P.I. is the complete solution.}$$

Example 1.32 : $(D^2-2D+4)^2 = x e^x \cos(\sqrt{3}x+\alpha)$

Solution : A.E. is $(D^2-2D+4)^2 = 0$

$$D = \frac{2\pm\sqrt{4-16}}{2}$$

$$D = \frac{2 \pm \sqrt{-12}}{2}$$

$$D = \frac{2 \pm i2\sqrt{3}}{2}$$

$$D = 1 \pm i\sqrt{3} \quad \text{repeated complex}$$

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step : Use P.I. formula.

$$\text{P.I.} = \frac{1}{(D^2 - 2D + 4)^2} x\ e^x \cos(\sqrt{3}x + \alpha)$$

Step 1 : Use type 4.

$$= e^x \frac{1}{\left[(D+1)^2 - 2(D+1) + 4\right]^2} x \cos(\sqrt{3}x + \alpha)$$

Step 2 : Simplify.

$$= e^x \frac{1}{(D^2 + 3)^2} x \cos(\sqrt{3}x + \alpha)$$

Step 3 : Here power of x is one but $D^2 + 3$ is the zero factor for $\cos(\sqrt{3}x + \alpha)$ $\therefore$ use type 6.

$$= e^x \text{ Real } \frac{1}{(D^2 + 3)^2} x\ e^{i(\sqrt{3}x + \alpha)}$$

Step 4 : Use type 4.

$$= e^x \text{ Real } e^{i(\sqrt{3}x + \alpha)} \frac{1}{\left[(D + i\sqrt{3})^2 + 3\right]^2} x$$

Step 5 : Simplify.

$$= e^x \text{ Real } e^{i(\sqrt{3}x + \alpha)} \frac{1}{\left[D^2 + 2i\sqrt{3}D - 3 + 3\right]^2} x$$

Step 6 : Simplify now the problem is reduced to type 3.

$$= e^x \text{ Real } e^{i(\sqrt{3}x + \alpha)} \frac{1}{\left[D^2 + 2i\sqrt{3}D\right]^2} x$$

Step 7 : Take the least power term outside.

$$= e^{-x} \text{ Real } e^{i(\sqrt{3}x + \alpha)} \frac{1}{(i\sqrt{3}D)^2} \frac{1}{\left[1 + \frac{D}{2i\sqrt{3}}\right]^2} x$$

Step 8 : Write f (D) in the form $(1+z)^n$.

$$= e^{-x} \text{ Real } e^{i(\sqrt{3}x+\alpha)} \frac{1}{(-12D^2)}\left[1+\frac{D}{i\sqrt{3}}\right]^{-2} x$$

Step 9 : Use binomial series $(1+z)^n = 1+nz+\frac{n(n-1)}{2!}z^2 \ldots.$

$$= e^{-x} \text{ Real } e^{i(\sqrt{3}x+\alpha)} \frac{1}{(-12D^2)}\left[1+(-2)\left(\frac{D}{2i\sqrt{3}}\right)\ldots\right] x$$

Step 10 : Simplify and open the bracket.

$$= e^{-x} \text{ Real } e^{i(\sqrt{3}x+\alpha)} \frac{1}{-12D^2}\left[x-\frac{2}{2i\sqrt{3}}\right]$$

Step 11 : Use $\frac{1}{i} = -i$

$$= e^{-x} \text{ Real } e^{i(\sqrt{3}x+\alpha)} \frac{1}{-12D^2}\left[x+\frac{i\,2}{2\sqrt{3}}\right]$$

Step 12 : Use $\frac{1}{D}X = \int X\,dx$

$$= e^{-x} \text{ Real } e^{i(\sqrt{3}x+\alpha)} \frac{1}{-12}\left[\frac{x^3}{2\cdot 3}+\frac{i\,2}{2\sqrt{3}}\cdot\frac{x^2}{2}\right]$$

Step 13 : Use $e^{iax} = (\cos ax + i \sin ax)$

$$= \frac{e^{-x}}{-12} \text{Real}\left[\cos(\sqrt{3}x+\alpha)+i\sin(\sqrt{3}x+\alpha)\right]\left[\frac{x^3}{6}+\frac{i\,x^2}{2\sqrt{3}}\right]$$

Step 14 : Use (a + ib) (c + id) = (ac – bd) + i (ad + bc)

$$= \frac{e^{-x}}{-12} \text{Real}\left[\frac{x^3}{6}\cos(\sqrt{3}x+\alpha)-\frac{x^2}{2\sqrt{3}}\sin(\sqrt{3}x+\alpha)\right]$$

$$+ i\left[\frac{x^2}{2\sqrt{3}}\cos(\sqrt{3}x+\alpha)+\frac{x^3}{6}\sin(\sqrt{3}x+\alpha)\right]$$

Step 15 : Write the real term as the P.I.

$$= \frac{e^{-x}}{-12}\left[\frac{x^3}{6}\cos(\sqrt{3}x+\alpha)-\frac{x^3}{2\sqrt{3}}\sin(\sqrt{3}x+\alpha)\right]$$

y = C.F. + P.I. is the complete solution.

Example 1.33 : $(D^2-1)y=x^2 \sin 3x$

Solution : A.E. $D^2-1 = 0$

$$(D-1)(D+1) = 0$$

$$D = 1, D = -1$$

$$\text{C.F.} = C_1e^{-x}+C_2e^{x}$$

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 1 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{D^2-1} x^2 \sin 3x$$

Step 2 : Use type 6, As sin ax is present ∴ Imaginary part of e^{iax}.

$$\text{P.I.} = \text{Img} \frac{1}{D^2-1} x^2e^{i3x}$$

Step 3 : Use type 4 $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$

$$= \text{Img } e^{i3x} \frac{1}{(D+i3)^2-1} x^2$$

Step 4 : Simplify.

$$= \text{Img } e^{i3x} \frac{1}{D^2+6iD-9-1} x^2$$

Step 5 : Simplify.

$$= \text{Img } e^{i3x} \frac{1}{D^2+6iD-10} x^2$$

Step 6 : Take the least power term outside with its sign.

$$= \text{Img } e^{i3x} \frac{1}{-10} \frac{1}{1-\left(\frac{D^2+6iD}{10}\right)} x^2$$

Step 7 : Use binomial series $\frac{1}{1-z} = 1+z+z^2\ldots.$

$$= \text{Img } e^{i3x} \frac{1}{-10}\left[1+\left(\frac{D^2+6iD}{10}\right)+\left(\frac{D^2+6iD}{10}\right)^2\ldots\right] x^2$$

$$= \text{Img } \frac{e^{i3x}}{-10}\left[-1+\frac{D^2+6iD}{10}+\frac{D^4+12iD^3-36D^2}{100}\right] x^2$$

Step 8 : Simplify.

$$= \text{Img } \frac{e^{i3x}}{-10}\left[x^2 + \frac{2+6i(2x)}{10} + \frac{0+0-36(2)}{100}\right]$$

Step 9 : Open the bracket and take derivatives of x^2.

$$= \text{Img } \frac{e^{i3x}}{-10}\left[x^2 + \frac{2}{10} + \frac{12xi}{10} - \frac{72}{100}\right]$$

Step 10 : Collect real and Img terms.

$$= \text{Img } \frac{e^{i3x}}{-10}\left[\left(x^2 - \frac{13}{25}\right) + \frac{6x}{5}\, i\right]$$

Step 11 : Use $e^{iax} = (\cos ax + i \sin ax)$

$$= \text{Img}\left(\frac{\cos 3x + i \sin 3x}{-10}\right)\left[\left(x^2 - \frac{13}{25}\right) + \frac{6x}{5}\, i\right]$$

Step 12 : Use (a + ib) (c + id) = (ac − bd) + i (ad + bc)

$$= \frac{1}{-10} \text{ Img}\left[\left(x^2 - \frac{13}{25}\right)\cos 3x - \frac{6x}{5}\sin 3x\right]$$

$$+ \; i\left[\frac{6x}{5}\cos 3x + \left(x^2 - \frac{13}{25}\right)\sin 3x\right]$$

Step 13 : Write the imaginary part as P.I.

$$= \frac{1}{-10}\left[\frac{6x}{5}\cos 3x + \left(x^2 - \frac{13}{25}\right)\sin 3x\right]$$

y = C.F. + P.I. is the complete solution.

Exercise 1.7

1. $(D^2 + 4)y = x \sin^2 x$ **[Ans. : $y = C_1 \cos 2x + C_2 \sin 2x + \frac{x}{8} - \frac{1}{32}(x \cos 2x + 2x^2 \sin 2x)$]**

2. $(D^2 + 1)y = x^2 \sin 2x$ **[Ans. : $y = C_1 \cos x + C_2 \sin x - \frac{8x}{9}\cos 2x + \frac{1}{27}(26 - 9x^2)\sin 2x$]**

3. $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$ **[Ans. : $y = e^{2x}\left[(C_1 + C_2 x) + (3 - 2x^2)\sin 2x - 4x \cos 2x\right]$]**

4. $(D^2 - 1)y = x^2 \cos x$ **[Ans. : $y = C_1 e^{x} + C_2 e^{-x} - \frac{1}{2}\left[(x^2 - 1)\cos x - 2x \sin x\right]$]**

5. $(D^4 + 2D^2 + 1)y = x^2 \cos x$ **[Ans. : P.I. = $\frac{x^3}{12}\sin x - \left(\frac{x^4 - 9x^2}{48}\right)\cos x$]**

6. $(D^4 + 8D^2 + 16)y = x \cos 2x$ **[Ans. : P.I. = $\frac{-x^2}{192}[2x \cos 2x - 3 \sin 2x]$]**

1.16 Type 7

If it is not possible to apply any of above types then use type7 i.e. if X involves tan ax sec ax, cosec ax, cot ax, $\sin e^x$, $\cos e^x$, e^{e^x}, $\frac{1}{1+e^x}$, $\frac{1}{1+e^{-x}}$, log x then use type 7 i.e. the general method for finding P.I.

1) $$\frac{1}{D-a}X = e^{ax}\int X\, e^{-ax}\, dx$$

2) $$\frac{1}{D+a}X = e^{-ax}\int X\, e^{+ax}\, dx$$

Example 1.34 : $(D^2+3D+2)y = e^{e^x}$

Solution : A.E. $(D+2)(D+1) = 0$

$$D = -2, D = -1$$

$$\text{C.F.} = C_1e^{-2x}+C_2e^{-x}$$

$$\text{P.I.} = \frac{1}{f(D)}X$$

Step 1 : Use P.I. formula.

$$= \frac{1}{(D+2)(D+1)}e^{e^x}$$

Step 2 : Use $\frac{1}{(D-a)(D-b)} = \frac{1}{a-b}\left[\frac{1}{D-a}-\frac{1}{D-b}\right]$

$$= \left[\frac{1}{D+1}-\frac{1}{D+2}\right]e^{e^x}$$

Step 3 : Separate the terms.

$$= \left[\frac{1}{D+1}e^{e^x}-\frac{1}{D+2}e^{e^x}\right]$$

Step 4 : Use type 7 formula.

$$= \left\{e^{-x}\int e^x e^{e^x}dx - e^{-2x}\int e^{2x}e^{e^x}dx\right\}$$

Step 5 : Write e^{2x} as $e^x \cdot e^x$ in 2nd term.

$$= \left\{e^{-x}\int e^{e^x}e^x dx - e^{-2x}\int e^x e^{e^x}e^x dx\right\}$$

Step 6 : Put $e^x = t$

$\therefore e^x dx = dt$

$$= \left\{e^{-x}\int e^t dt - e^{-2x}\int t\, e^t dt\right\}$$

Step 7 : Integrate 1st term and for 2nd term use generalised rule of integration by parts i.e. $\int uv dx = uv_1 - u'v_2 \ldots$

$$= \left\{e^{-x}(e^t) - e^{-2x}\left[\underbrace{t\,(e^t)}_{u\,v_1} - \underbrace{(1)\,(e^t)}_{u'v_2}\right]\right\}$$

Step 8 : Simplify.

$$= e^t\left\{e^{-x} - e^{-2x}\cdot t + e^{-2x}\right\}$$

Step 9 : Put $t = e^x$

$$= e^{e^x}\left\{e^{-x} - e^{-2x}\cdot e^x + e^{-2x}\right\}$$

Step 10 : Simplify.

$$= e^{e^x}\left\{e^{-x} - e^{-x} + e^{-2x}\right\}$$

$$= e^{e^x}\cdot e^{-2x}$$

Example 1.35 : $(D^2-1)y = e^{-x}\sin e^{-x} + \cos e^{-x}$

Solution : A.E. $(D-1)(D+1) = 0$

$$D = 1, -1$$

$$\text{C.F.} = C_1 e^{-x} + C_2 e^x$$

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 1 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{(D-1)(D+1)} e^{-x}\sin e^{-x} + \cos e^{-x}$$

Step 2 : Use $\dfrac{1}{(D-a)(D-b)} = \dfrac{1}{a-b}\left[\dfrac{1}{D-a} - \dfrac{1}{D-b}\right]$

$$\text{P.I.} = \frac{1}{2}\left[\frac{1}{D-1} - \frac{1}{D+1}\right]\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)$$

Step 3 : Separate the two terms.

$$\text{P.I.} = \frac{1}{2}\left\{\frac{1}{D-1}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right) - \frac{1}{D+1}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)\right\}$$

Consider

$$PI_1 = \frac{1}{2}\left\{\frac{1}{D-1} e^{-x} \sin e^{-x} + \cos e^{-x}\right\}$$

Step 4 : Use $\frac{1}{D-a} X = e^{ax} \int X e^{-ax} dx$

$$= \frac{1}{2} e^{x} \int \left(e^{-x} \sin e^{-x} + \cos e^{-x}\right) dx$$

$$= \frac{1}{2} e^{x} \int \left(e^{-x} \sin e^{-x} + \cos e^{-x}\right) e^{-x} dx$$

Step 5 : Put $e^{-x} = t \quad \therefore \; - e^{-x} dx = dt$ i.e. $e^{-x} dx = -dt$

$$= \frac{1}{2} e^{x} \int (t \sin t + \cos t)(-dt)$$

Step 6 : Separate two integrals.

$$= \frac{-1}{2} e^{x} \left[\int t \sin t \; dt + \int \cos t \; dt\right]$$

Step 7 : For $\int t \sin t \; dt$ use general integration by parts $\int uv dx = uv_1 - u'v_2 \ldots$

$$\therefore \int t \sin t \; dt = \underset{u}{(t)} \; \underset{v_1}{(-\cos t)} - \underset{u'}{(1)} \; \underset{v_2}{(-\sin t)}$$

$$= \frac{-e^{x}}{2} \left\{[(t)\;(-\cos t) - (1)\;(-\sin t)] + \sin t\right\}$$

Step 8 : Simplify.

$$= \frac{-e^{x}}{2} \{-t \cos t + 2 \sin t\}$$

Step 9 : Put $t = e^{-x}$

$$= \frac{-e^{x}}{2} \left\{-e^{-x} \cos e^{-x} + 2 \sin e^{-x}\right\}$$

Step 10 : Simplify.

$$PI_1 = \frac{1}{2} \cos e^{-x} - e^{x} \sin e^{-x}$$

Consider

$$PI_2 = \frac{1}{2} \frac{1}{D+1} \left(e^{-x} \sin e^{-x} + \cos e^{-x}\right)$$

Step 11 : Use type 7 formula $\frac{1}{D+a} X = e^{-ax} \int e^{ax} \; X \; dx$

$$= \frac{1}{2} e^{-x} \int -e^{x} \left(\cos e^{-x} + e^{-x} \sin e^{-x}\right) dx$$

Step 12 : Use $\int e^x \, [f(x)+f'(x)] \, dx = e^x \, f(x)$

Here $f(x) = \cos e^{-x}$ $\therefore f'(x) = -\sin e^{-x} \; (-e^{-x}) = e^{-x} \sin e^{-x}$

$$= \frac{1}{2} e^{-x} \cdot e^x \; \cos \; e^{-x}$$

Step 13 : Simplify.

$$PI_2 = \frac{1}{2} \cos \; e^{-x}$$

Thus

Step 14 : $PI = PI_1 - PI_2$

$$= -e^x \; \sin e^{-x}$$

$$\therefore \qquad y = C.F. + P.I.$$

$$= C_1 e^{-x} + C_2 e^x - e^x \sin e^{-x}$$

Example 1.36 : $(D^2 - D - 2)y = 2 \log x + \frac{1}{x} + \frac{1}{x^2}$

Solution :

$$D^2 - D - 2 = 0$$

$$(D-2)(D+1) = 0$$

$$D = 2, -1$$

$$C.F. = C_1 e^{2x} + C_2 e^{-x}$$

$$P.I. = \frac{1}{f(D)} \; X$$

Step 1 : Use P.I. formula.

$$= \frac{1}{(D-2)(D+1)} \left(2 \log x + \frac{1}{x} + \frac{1}{x^2} \right)$$

Step 2 : Use $\frac{1}{(D-a)(D-b)} = \frac{1}{a-b} \left[\frac{1}{D-a} - \frac{1}{D-b} \right]$

$$= \frac{1}{-3} \left[\frac{1}{D+1} - \frac{1}{D-2} \right] \left(2 \log x + \frac{1}{x} + \frac{1}{x^2} \right)$$

Consider

$$PI_1 = \frac{1}{D+1} \left(2 \log x + \frac{1}{x} + \frac{1}{x^2} \right)$$

Step 3 : Use P.I. $= \frac{1}{D+1} X = e^{-x} \int e^{x} \; X\,dx$

$$= e^{-x} \int e^{x}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right) dx$$

Step 4 : Write $\frac{1}{x} = \frac{-1}{x} + \frac{2}{x}$

$$= e^{-x} \int e^{x}\left[\left(2\log x - \frac{1}{x}\right) + \left(\frac{2}{x} + \frac{1}{x^2}\right)\right] dx$$

Step 5 : Use $\int e^{x} \left[f(x) + f'(x)\right] dx = e^{x} \; f(x)$

$$= e^{-x} \cdot e^{x} \cdot \left(2\log x - \frac{1}{x}\right)$$

Step 6 : Simplify.

$$= 2\log x - \frac{1}{x}$$

Consider

$$PI_2 = \frac{1}{D-2}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

Step 7 : Use $\frac{1}{D-a} X = e^{ax} \int e^{-ax} \; X\,dx$

$$= e^{2x} \int e^{-2x}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right) dx$$

Step 8 : Put $-2x = t \;\therefore\; x = \frac{-t}{2} \;\therefore dx = \frac{-dt}{2}$

$$= e^{2x} \int e^{t}\left[2\log\left(\frac{t}{-2}\right) - \frac{2}{t} + \frac{4}{t^2}\right] \frac{dt}{-2}$$

Step 9 : Write $\frac{-2}{t} = \frac{-4}{t} + \frac{2}{t}$

$$= \frac{e^{2x}}{-2} \int e^{t}\left[\left(2\log\left(\frac{t}{-2}\right) - \frac{4}{t}\right) + \left(\frac{2}{t} + \frac{4}{t^2}\right)\right] dt$$

Step 10 : Use $\int e^{t} \; [f(t) + f'(t)] \; dt = e^{t} \; f(t)$

$$= \frac{e^{2x}}{-2} \cdot e^{t}\left[2\log\left(\frac{t}{-2}\right) - \frac{4}{t}\right]$$

Step 11 : Put $t = -2x$

$$= \frac{e^{2x}}{-2} \cdot e^{-2x}\left[2\log\left(\frac{-2x}{2}\right) - \left(\frac{4}{-2x}\right)\right]$$

Step 12 : Simplify.

$$= \frac{1}{-2}\left[2\log x + \frac{2}{x}\right]$$

Step 13 : Simplify.

$$= -\log x - \frac{1}{x}$$

Thus

$$\text{P.I.} = -\frac{1}{3}[PI_1 - PI_2]$$

$$= -\frac{1}{3}[3\log x]$$

$$= -\log x$$

$$\therefore \quad y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^{2x} + C_2 e^{-x} - \log x$$

Example 1.37: $(D^2 + 5D + 6)y = e^{-2x} \sec^2 x(1 + 2\tan x)$

Solution :

$$D^2 + 5D + 6 = 0$$

$$(D + 3)(D + 2) = 0$$

$$D = -3, -2$$

$$\text{C.F.} = C_1 e^{-3x} + C_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 1 : Use P.I. formula.

$$= \frac{1}{D^2 + 5D + 6} e^{-2x} \sec^2 x(1 + 2\tan x)$$

Step 2 : $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+2)} V$

$$= e^{-2x} \frac{1}{(D-2)^2 + 5(D-2) + 6} \sec^2 x(1 + 2\tan x)$$

Step 3 : Simplify.

$$= e^{-2x} \frac{1}{D^2 + D} \sec^2 x(1 + 2\tan x)$$

Step 4 : Factorise f (D).

$$= e^{-2x} \frac{1}{D(D+1)} \sec^2 x(1+2\tan x)$$

Step 5 : Use $\frac{1}{(D-a)(D-b)} = \frac{1}{a-b}\left[\frac{1}{D-a} - \frac{1}{D-b}\right]$

$$= e^{-2x}\left[\frac{1}{D} - \frac{1}{D+1}\right] \sec^2 x\,(1+2\tan x)$$

Consider

$$PI_1 = \frac{1}{D}\sec^2 x\,(1+2\tan x)$$

Step 6 : Use $\frac{1}{D}X = \int X\,dx$

$$= \int \sec^2 x\,(1+2\tan x)\,dx$$

Step 7 : Put $\tan x = t$, $\sec^2 x\,dx = dt$

$$= \int (1+2t)\,dt$$

Step 8 : Integrate w.r.t t.

$$= t + 2\cdot\frac{t^2}{2}$$

$$PI_1 = \tan x + \tan^2 x$$

Also consider

$$PI_2 = \frac{1}{D+1}\sec^2 x\,(1+2\tan x)$$

Step 9 : Use P.I. formula $\frac{1}{D+a} X = e^{ax}\int X e^{-ax}\,dx$

$$= e^{-x}\int e^{x}\left(\sec^2 x + 2\tan x \sec^2 x\right)\,dx$$

Step 10 : Use $\int e^x[f(x) + f'(x)]\,dx = e^x f(x)$

$$= e^{-x}\cdot(e^x\cdot\sec^2 x)$$

Step 11 : Simplify.

$$= \sec^2 x$$

Step 12 : $\therefore$ P.I. $= e^{-2x}\,[PI_1 - PI_2]$

$$= e^{-2x}\left[\tan x + \tan^2 x - \sec^2 x\right]$$

Step 13 : Use $\sec^2 x - \tan^2 x = 1$

$$\therefore \quad \text{P.I.} = e^{-2x}\,[\tan x - 1]$$

Thus

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^{-3x} + C_2 e^{-2x} + e^{-2x}(\tan x - 1)$$

Example 1.38 : $(D^2 + D)\, y = \dfrac{1}{1+e^x}$

Solution : AE $D(D+1) = 0$

$$D = 0, \; D = -1 \text{ real roots.}$$

$$\text{C.F.} = C_1 e^{0x} + C_2 e^{-x}$$

Step 1 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{D(D+1)}\,\frac{1}{1+e^x}$$

Step 2 : Use $\dfrac{1}{(D-a)(D-b)} = \dfrac{1}{a-b}\left[\dfrac{1}{D-a} - \dfrac{1}{D-b}\right]$

$$\text{P.I.} = \left[\frac{1}{D} - \frac{1}{D+1}\right]\frac{1}{1+e^x}$$

Step 3 : Separate PI_1 and PI_2.

Consider

$$PI_1 = \frac{1}{D}\,\frac{1}{1+e^x}$$

Step 4 : Use $\dfrac{1}{D} X = \int X \, dx$

$$= \int \frac{1}{1+e^x}\, dx$$

Step 5 : As the derivative of denominator term is not present in the numerator write $e^x = \dfrac{1}{e^{-x}}$.

$$= \int \frac{1}{1 + \dfrac{1}{e^{-x}}}\, dx$$

Step 6 : Simplify.

$$= \int \frac{e^{-x}}{e^{-x}+1}\, dx$$

Step 7 : Put $e^{-x} + 1 = t \quad -e^{-x}dx = dt$

$$= \int \frac{-dt}{t}$$

Step 8 : Use $\int \frac{1}{t} dt = \log t$

$$= -\log t$$

Step 9 : Put $t = e^{-x} + 1$

$$= -\log(e^{-x} + 1)$$

Step 10 : Simplify.

$$= -\log\left(\frac{1}{e^x} + 1\right)$$

Step 11 : Take L.C.M.

$$= -\log\left(\frac{1+e^x}{e^x}\right)$$

Step 12 : Use $\log(a/b) = \log a - \log b$

$$= -\left[\log(1+e^x) - \log e^x\right]$$

$$= \log e^x - \log(e^x + 1)$$

Step 13 : Use $\log e^x = x$

$$= x - \log(1+e^x)$$

Step 14 :

Consider

$$PI_2 = \frac{1}{D+1} \frac{1}{1+e^x}$$

Step 15 : Use $\frac{1}{D+a} X = e^{-ax} \int e^{ax} X\, dx$, type 7.

$$= e^{-x} \int e^x \cdot \frac{1}{1+e^x} dx$$

Step 16 : As the derivative of denominator is present in numerator put $1 + e^x = t$ $\therefore e^x dx = dt$.

$$= e^{-x} \int \frac{dt}{t}$$

Step 17 : Use $\int \frac{dt}{t} = \log t$

$$= e^{-x} \log t$$

Step 18 : Put $t = 1 + e^x$

$$= e^{-x} \log(1+e^x)$$

Step 19 : Now.

$$\therefore \quad \text{P.I.} = PI_1 - PI_2$$

Step 20 : Substitute the values of PI_1 and PI_2.

$$= x - \log(1+e^x) - e^{-x}\log(1+e^x)$$

Step 21 : Take $\log(1+e^x)$ common from 2nd and 3rd term.

$$= x - (1+e^{-x})\log(1+e^x)$$

$$y = \text{C.F.} + \text{P.I. is the complete solution.}$$

Example 1.39 : $(D^2-1)y = \dfrac{1}{(1+e^{-x})^2}$

Solution : A.E. $(D^2-1) = 0$

$$(D-1)(D+1) = 0$$

$$D = 1, -1 \quad \text{real roots}$$

$$\text{C.F.} = C_1e^x + C_2e^{-x}$$

$$\text{P.I.} = \frac{1}{f(D)}X$$

Step 1 : Use P.I. formula.

$$= \frac{1}{(D-1)(D+1)}\frac{1}{(1+e^{-x})^2}$$

Step 2 : Use $\dfrac{1}{(D-a)(D-b)} = \dfrac{1}{a-b}\left[\dfrac{1}{D-a} - \dfrac{1}{D-b}\right]$

$$= \frac{1}{2}\left[\frac{1}{D-1} - \frac{1}{D+1}\right]\frac{1}{(1+e^{-x})^2}$$

Step 3 : Separate PI_1 and PI_2.

Consider $\quad PI_1 = \dfrac{1}{D-1}\dfrac{1}{(1+e^{-x})^2}$

Step 4 : Use $\dfrac{1}{D-a}X = e^{ax}\int X e^{-ax}dx$ type 7 formula.

$$= e^x\int e^{-x}\frac{1}{(1+e^{-x})^2}dx$$

Step 5 : Derivative of denominator term is present in numerator.

Put $e^{-x}+1 = t \quad -e^{-x}dx = dt$

$$= e^x\int\frac{-dt}{t^2}$$

Step 6 : Use $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$= e^x \left[\frac{1}{t}\right]$$

Step 7 : Put $t = 1 + e^{-x}$

$$= e^x \frac{1}{1+e^{-x}}$$

Step 8 : Write $e^{-x} = \frac{1}{e^x}$

$$= e^x \frac{1}{1+\frac{1}{e^x}}$$

Step 9 : Simplify.

$$= \frac{e^{2x}}{e^x+1}$$

Step 10 :

Consider $\quad PI_2 = \frac{1}{D+1} \frac{1}{(1+e^{-x})^2}$

Step 11 : Use type 7 formula $\frac{1}{D+a} X = e^{-ax} \int X\ e^{-ax}\ dx$

$$= e^{-x} \int e^x \frac{1}{(1+e^{-x})^2} dx$$

Step 12 : Write $e^{-x} = \frac{1}{e^x}$

$$= e^{-x} \int e^x \frac{1}{\left(1+\frac{1}{e^x}\right)^2} dx$$

Step 13 : Simplify.

$$= e^{-x} \int \frac{e^x \cdot e^{2x}}{(e^x+1)^2} dx$$

Step 14 : Put $e^x + 1 = t$, $e^x = t-1$, $e^x dx = dt$

$$= e^{-x} \int \frac{(t-1)^2\ dt}{t^2}$$

Step 15 : Use $(a+b)^2 = a^2 - 2ab + b^2$.

$$= e^{-x} \int \frac{t^2-2t+1}{t^2} dt$$

Step 16 : Simplify and separate the terms.

$$= e^{-x} \int \left(1 - \frac{2}{t} + \frac{1}{t^2}\right) dt$$

Step 17 : Integrate.

$$= e^{-x}\left[t - 2\log t - \frac{1}{t}\right]$$

Step 18 : Put $t = 1 + e^x$

$$= e^{-x}\left[(1+e^x) - 2\log(1+e^x) - \frac{1}{1+e^x}\right]$$

Step 19 : Multiply e^{-x} inside the bracket.

$$= -2e^{-x}\log(1+e^x) - \frac{e^{-x}}{1+e^x} + e^{-x}(1+e^x)$$

Now $\qquad$ P.I. $= \frac{1}{2}[PI_1 - PI_2]$

Substitute PI_1 and PI_2.

$$\text{P.I.} = \frac{1}{2}\left[\frac{e^{2x}}{e^x+1} + 2e^{-x}\log(1+e^x) + \frac{e^{-x}}{1+e^x} - e^{-x}(1+e^x)\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x) + \frac{e^{2x} + e^{-x} - e^{-x}(1+e^x)^2}{1+e^x}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x) + \frac{e^{2x} + e^{-x} - e^{-x}(1+2e^x+e^{2x})}{1+e^x}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x) + \frac{e^{2x} + e^{-x} - e^{-x} - 2 - e^x}{1+e^x}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x) + \frac{e^{2x} - 2 - e^x}{1+e^x}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x) + \frac{e^{2x} - 1 - 1 - e^x}{1+e^x}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x) + \frac{(e^x-1)(e^x+1) - (1+e^x)}{1+e^x}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x) + \frac{[e^x - 1 - 1](e^x+1)}{1+e^x}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^{x})+\frac{(e^{x}-2)(1+e^{x})}{(1+e^{x})}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^{x})+e^{x}-2\right]$$

$$= e^{-x}\log(1+e^{x})+\frac{e^{x}}{2}-1$$

y = C.F. + P.I. is the complete solution.

Example 1.40 : $(D^3+D)\,y = \operatorname{cosec} x$

Solution : A.E. $D(D^2+1) = 0$

$$D(D+i)(D-i) = 0$$

$$D = 0,\ D = \pm i \quad \text{real complex}$$

$$\text{C.F.} = C_1 e^{0x} + e^{0x}[C_2\cos x + C_3 \sin x]$$

$$\text{P.I.} = \frac{1}{f(D)}\,X$$

Step 1 : Use P.I. formula.

$$= \frac{1}{D(D+i)(D-i)}\operatorname{cosec} x$$

Step 2 : Consider $\frac{1}{D(D+i)(D-i)}$ ∴ use partial fractions.

$$= \frac{A}{D}+\frac{B}{D+i}+\frac{C}{D-i}$$

$$A = \lim_{D\to 0} D\cdot\left[\frac{1}{D(D+i)(D-i)}\right] = \frac{1}{1}$$

$$B = \lim_{D\to -i}(D+i)\left[\frac{1}{D(D+i)(D-i)}\right] = \frac{1}{(-i)(-2i)} = \frac{-1}{2}$$

$$C = \lim_{D\to i}(D-i)\left[\frac{1}{D(D+i)(D-i)}\right] = \frac{1}{(i)(2i)} = \frac{-1}{2}$$

Step 3 : Substitute A, B, C.

$$\therefore \quad = \frac{1}{D}+\frac{-1/2}{D+i}+\frac{-1/2}{D-i}$$

Step 4 : Consider P.I.

$$\text{P.I.} = \left[\frac{1}{D}+\frac{-1/2}{D+i}+\frac{-1/2}{D-i}\right]\operatorname{cosec} x$$

Step 5 : Separate PI_1, PI_2, PI_3 and consider PI_1.

$$PI_1 = \frac{1}{D}\operatorname{cosec} x = \int \operatorname{cosec} x \, dx$$

$$= \log(\operatorname{cosec} x - \cot x)$$

Step 6 : Consider PI_2

$$PI_2 = \frac{-1/2}{D+i}\operatorname{cosec} x$$

Step 7 : Use type 7 formula $\frac{1}{D+a} X = e^{-ax}\int X e^{ax} dx$

$$= \frac{-1}{2} e^{-ix}\int e^{ix}\operatorname{cosec} x \, dx$$

Step 8 : Use $e^{ia} = \cos\theta + i\sin\theta$

$$= \frac{-1}{2} e^{-ix}\int (\cos x + i\sin x)\operatorname{cosec} x \, dx$$

Step 9 : Multiply by cosec x inside the bracket $\operatorname{cosec} x = \frac{1}{\sin x}$.

$$= \frac{-1}{2} e^{-ix}\int (\cot x + i)\, dx$$

Step 10 : Integrate.

$$= \frac{-1}{2} e^{-ix}[\log\sin x + ix]$$

Step 11 : Replacing i by – i we get.

$$PI_3 = \frac{-1/2}{D-i}\operatorname{cosec} x = \frac{-i}{2} e^{ix}[\log\sin x - ix]$$

Step 12 : Consider $PI_2 + PI_3$.

$$= \frac{-1}{2}\left\{e^{-ix}(\log\sin x) + e^{-ix}(ix) + e^{ix}(\log\sin x) - e^{ix}(ix)\right\}$$

Step 13 : Take log sin x common from 1[st] and 3[rd] term and (– ix) common from 2[nd] and 4[th] term.

$$= \frac{-1}{2}\left\{\log\sin x\,(e^{ix} + e^{-ix}) - ix\,(e^{ix} - e^{ix})\right\}$$

Step 14 : Use $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ and $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$$= \frac{-1}{2}\{\log\sin x\,(2\cos x) - ix\,(2i\sin x)\}$$

Step 15 : Simplify $i^2 = -1$.

$$= \{-\cos x \log \sin x - x \sin x\}$$

Step 16 :

$$\therefore \quad \text{P.I.} = PI_1 + PI_2 + PI_3 = \log(\operatorname{cosec} x - \cot x) - x \sin x - \cos x \log \sin x$$

Example 1.41 : $(D^2 + a^2)y = \tan ax$

Solution : A.E. $D^2 + a^2 = 0$

$$D^2 = -a^2$$

$$D = \pm ai$$

$$\text{C.F.} = C_1 \cos ax + C_2 \sin ax$$

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 1 : Use P.I. formula and use $D^2 + a^2 = (D+ai)(D-ai)$.

$$= \frac{1}{(D+ai)(D-ai)} \tan ax$$

Step 2 : Use $\frac{1}{(D-a)(D-b)} = \frac{1}{a-b}\left[\frac{1}{D-a} - \frac{1}{D-b}\right]$

$$PI_1 = \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right] \tan ax$$

Step 3 : Separate PI_1, PI_2 and consider

$$PI_1 = \frac{1}{D-ai} \tan ax$$

Step 4 : Use type 7 formula.

$$= e^{iax} \int e^{-aix} \tan ax \, dx$$

Step 5 : Use $e^{-i\theta} = \cos\theta - i \sin\theta$

$$= e^{aix} \int (\cos ax - i \sin ax) \tan ax \, dx$$

Step 6 : Multiply tan ax inside the bracket $\tan ax = \frac{\sin ax}{\cos ax}$.

$$= e^{aix} \int \sin ax - i \frac{\sin^2 ax}{\cos ax} \, dx$$

Step 7 : Write $\sin^2 \theta = 1 - \cos^2 \theta$.

$$= e^{aix} \int \sin ax - i \frac{1 - \cos^2 ax}{\cos ax} dx$$

Step 8 : Separate all the terms.

$$= e^{aix} \int \sin ax - i\left(\frac{1}{\cos ax} - \cos ax\right) dx$$

Step 9 : Simplify.

$$= e^{aix} \int \sin ax - i \sec ax + i \cos ax \;\; dx$$

Step 10 : Collect sin ax and cos ax together.

$$= e^{aix} \int \sin ax + i \cos ax - i \sec ax \;\; dx$$

Step 11 : Integrate.

$$= e^{aix} \left[-\frac{\cos ax}{a} + \frac{i}{a} \sin ax - \frac{i}{a} \log(\sec ax + \tan ax) \right]$$

Step 12 : Take $\frac{1}{a}$ outside.

$$= \frac{e^{aix}}{a} [-(\cos ax - i \sin ax) - i \log(\sec ax + \tan ax)]$$

Step 13 : Use $e^{-iax} = \cos ax - i \sin ax$.

$$= \frac{e^{aix}}{a} [-e^{-aix} - i \log(\sec ax + \tan ax)]$$

Step 14 : Multiply e^{aix} outside.

$$PI_1 = \frac{1}{a} [-1 - i e^{aix} \log(\sec ax + \tan ax)]$$

Step 15 : Replacing i by – i we get.

$$PI_2 = \frac{1}{D + ai} \tan ax$$

$$= \frac{1}{a} [-1 + i e^{-aix} \log(\sec ax + \tan ax)]$$

Step 16 :

Now $\quad$ $P.I. = \frac{1}{2ai} [PI_1 - PI_2]$

Step 17 : Substitute PI_1 and PI_2.

$$\text{P.I.} = \frac{1}{2ai}\cdot\frac{1}{a}\{[-1 - i\, e^{aix}\log(\sec ax + \tan ax)]$$

$$-[-1 + i\, e^{-aix}\log(\sec ax + \tan ax)]\}$$

Step 18 : Take log (sec ax + tan ax) common.

$$= \frac{1}{2a^2 i}\{(-i)\log(\sec ax + \tan ax)\,(e^{aix} + e^{-aix})\}$$

Step 19 : Use $e^{i\theta} + e^{-i\theta} = 2\cos\theta$.

$$= \frac{1}{2a^2 i}\{-i\, 2\cos ax \cdot \log(\sec ax + \tan ax)\}$$

Step 20 : Simplify.

$$= \frac{-1}{a^2}\cos ax \log(\sec ax + \tan ax)$$

$$y = \text{C.F.} + \text{P.I. is the complete solution.}$$

Example 1.42 : $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = e^{-x}\sec^3 x$

Solution : Let $D = \dfrac{d}{dx}$

∴ the equation becomes

$$D^2y + 2Dy + 2y = e^{-x}\sec^3 x$$

$$(D^2 + 2D + 2)\,y = e^{-x}\sec^3 x$$

A.E. $D^2 + 2D + 2 = 0$ Use $D = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ a = 1, b = 2, c = 2 complex roots.

$$D = \frac{-2 \pm \sqrt{4 - 18}}{2}$$

$$D = -1 \pm i$$

$$\text{C.F.} = e^{-x}\,[C_1\cos x + C_2 \sin x]$$

$$\text{P.I.} = \frac{1}{f(D)}\,X$$

Step 1 : Use P.I. formula.

$$= \frac{1}{D^2 + 2D + 2}\,e^{-x}\sec^3 x$$

Step 2 : Use type 4 formula $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$ i.e. replace D by D – 1 and take e^{-x} outside.

$$= e^{-x} \frac{1}{(D-1)^2 + 2(D-1) + 2} \sec^3 x$$

Step 3 : Simplify.

$$= e^{-x} \frac{1}{D^2 + 1} \sec^3 x$$

Step 4 : Use $D^2 + 1 = (D-i)(D+i)$.

$$= e^{-x} \frac{1}{(D-i)(D+i)} \sec^3 x$$

Step 5 : Use $\frac{1}{(D-a)(D-b)} = \frac{1}{a-b}\left[\frac{1}{D-a} - \frac{1}{D-b}\right]$

$$= e^{-x} \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right] \sec^3 x$$

Step 6 : Separate PI_1 and PI_2.

Consider $\quad PI_1 = \frac{1}{D-i} \sec^3 x$

Step 7 : Use type 7 formula.

$$= e^{ix} \int e^{-ix} \sec^3 x \; dx$$

Step 8 : Use $e^{-ix} = \cos x - i \sin x$

$$= e^{ix} \int (\cos x - i \sin x) \sec^3 x \; dx$$

Step 9 : Multiply $\sec^3 x$ inside the bracket.

$$= e^{ix} \int \sec^2 x - i \tan x \sec^2 x \; dx$$

Step 10 : Integrate use $\int [f(x)] \cdot f'(x) \, dx = \frac{f(x)^2}{2}$ for 2nd term.

$$PI_1 = e^{ix}\left(\tan x - i \frac{\tan^2 x}{2}\right)$$

Step 11 : Replacing i by – i we get PI_2.

$$PI_2 = \frac{1}{D+i} \sec^3 x = e^{-ix}\left(\tan x + i \frac{\tan^2 x}{2}\right)$$

Step 12 : P.I. $= \frac{e^{-x}}{2i}[PI_1 - PI_2]$

Now $$\text{P.I.} = \frac{e^{-x}}{2i}\left[e^{ix}\left(\tan x - i\frac{\tan^2 x}{2}\right) - e^{-ix}\left(\tan x + i\frac{\tan^2 x}{2}\right)\right]$$

Step 13 : Take tanx and $\frac{\tan^2 x}{2}$ common.

$$= \frac{e^{-x}}{2i}\left\{\tan x\,(e^{ix} - e^{-ix}) - \frac{i}{2}\tan^2 x\,(e^{ix} + e^{-ix})\right\}$$

Step 14 : Use $e^{ix} - e^{-ix} = 2i\sin x$, $e^{ix} + e^{-ix} = 2\cos x$.

$$= \frac{e^{-x}}{2i}\left\{\tan x\,(2i\sin x) - \frac{i}{2}\tan^2 x\,(2\cos x)\right\}$$

Step 15 : Simplify write $\tan x = \frac{\sin x}{\cos x}$ in 2^{nd} term.

$$= \frac{e^{-x}}{2i}\left\{2i\sin x\tan x - i\tan x\cdot\frac{\sin x}{\cos x}\cdot\cos x\right\}$$

Step 16 : Simplify.

$$= \frac{e^{-x}}{2i}\{2i\sin x\tan x - i\sin x\tan x\}$$

$$= \frac{e^{-x}}{2}\sin x\tan x$$

$\therefore$ $$y = \text{C.F.} + \text{P.I.}$$

Example 1.43 : $(D^2 + 2D + 1)\,y = e^{-x}\log x$

Solution : A.E. $(D+1)^2 = 0$

$$D = -1, -1$$

$$\text{C.F.} = (C_1 + C_2 x)\,e^{-x} \quad \text{repeated roots.}$$

Step 1 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{(D+1)^2}e^{-x}\log x$$

Step 2 : Use type 4 take out e^{-x} and replace D by D – 1.

$$= e^{-x}\frac{1}{(D-1+1)^2}\log x$$

Step 3 : Simplify.

$$= e^{-x}\frac{1}{D^2}\log x$$

Step 4 : Use $\frac{1}{D}X = \int X\,dx$, $\int \log x\,dx = x\log x - x$

$$= e^{-x}\frac{1}{D}[x\log x - x]$$

Step 5 : Again $\frac{1}{D}X = \int X\,dx$

$$= e^{-x}\left[\int x\log x\,dx - \frac{x^2}{2}\right]$$

Step 6 : Use by parts $\int uv dx = u\int v dx - \int\left(\frac{dy}{dx}\int v dx\right)dx$

$$= e^{-x}\left[\log x\left(\frac{x^2}{2}\right) - \int \frac{1}{x}\cdot\frac{x^2}{2}dx - \frac{x^2}{2}\right]$$

Step 7 : Simplify.

$$= e^{-x}\left[\frac{x^2}{2}\log x - \frac{1}{2}\cdot\frac{x^2}{2} - \frac{x^2}{2}\right]$$

Step 8 : Simplify.

$$= e^{-x}\left[\frac{x^2}{2}\log x - \frac{3}{4}x^2\right]$$

$$y = \text{C.F.} + \text{P.I. is the complete solution.}$$

Exercise 1.8

1. $(D^2+3D+2)y = \sin e^x$ **[Ans. : $y = C_1e^{-2x} + C_2e^{-x} - e^{-2x}\sin e^x$]**
2. $(D^2-3D+2)y = \frac{1}{e^{e^{-x}}} + \cos\left(\frac{1}{e^x}\right)$ **[Ans. : $y = C_1e^{2x} + C_2e^x + e^{2x}\left[\frac{1}{e^{e^{-x}}} - \cos\left(\frac{1}{e^x}\right)\right]$]**
3. $(D^2-9D+18)y = e^{e^{-3x}}$ **[Ans. : $y = C_1e^{3x} + C_2e^{6x} + \frac{1}{9}e^{6x}e^{e^{-3x}}$]**
4. $(D^2-2D-3)y = 3e^{-3x}\sin(e^{-3x}) + \cos(e^{-3x})$ **[Ans. : $y = C_1e^{3x} + C_2e^{-x} - \frac{e^{3x}}{3}\sin e^{-3x}$]**
5. $(D^2+5D+6)y = e^{e^x}$ **[Ans. : $y = C_1e^{-2x} + C_2e^{-3x} + (e^{-2x} - 2e^{-3x})e^{e^x}$]**
6. $(D^2+2D+1)y = \frac{e^{-x}}{x+2}$ **[Ans. : $y = (C_1 + C_2x)e^{-x} + e^{-x}\{x\log(x+2) - 2\log(x+2) - x\}$]**
7. $(D^2+9)y = \operatorname{cosec} 3x$ **[Ans. : $y = C_1\cos 3x + C_2\sin 3x + \frac{1}{9}\sin 3x\log\sin 3x - \frac{1}{3}x\cos 3x$]**
8. $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ **[Ans. : $y = C_1\cos x + C_2\sin x + \sin\log\sin x - x\cos x$]**

1.17 Legendre's Differential Equations

General form for 3rd order differential equation is

$$a_0\,(ax+b)^3\frac{d^3y}{dx^3}+a_1(ax+b)^2\frac{d^2y}{dx^2}+a_2(ax+b)\frac{dy}{dx}+a_3y = X$$

Here the coefficients are not constants so we can't solve such example by using previous methods.

∴ We must reduce the above equation to the linear differential equation with constant coefficients.

∴ Put $(ax + b) = e^z$ i.e. $z = \log(ax + b)$

We know that

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{a}{ax+b}$$

$$\therefore \quad (ax+b)\frac{dy}{dx} = a\cdot\frac{dy}{dz}$$

Let $$D = \frac{d}{dz}$$

$$\therefore \quad \underbrace{(ax+b)\frac{dy}{dx}}_{\text{Variable coefficient}} = \underbrace{a\,Dy}_{\text{constant coefficient}}$$

Thus the substitution $ax+b=e^z$ reduces the variable coefficient to the constant coefficient similarly we can show that

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2D\,(D-1)\,y$$

$$(ax+b)^3\frac{d^3y}{dx^3} = a^3D\,(D-1)\,(D-2)\,y \text{ and so on.}$$

1.18 Cauchys Differential Equation or Homogeneous Differential Equaions

It is the particular case of Legendres differential equation if we put a = 1 and b = 0 in Legendre's then we get Cauchy's D.E. Thus the general form of 3rd order differential equation is

$$a_0 x^3 \frac{d^3y}{dx^3} + a_1 x^2 \frac{d^2y}{dx^2} + a_2 x \frac{dy}{dx} + a_3 y = X$$

Put $z = e^z$ i.e. $z = \log x$

$$\therefore \qquad x\frac{dy}{dx} = D\,y$$

$$x^2 \frac{d^2y}{dx^2} = D\,(D-1)\,y$$

$$x^3 \frac{d^3y}{dx^3} = D\,(D-1)\,(D-2)\,y$$

where $\qquad D = \frac{d}{dz}$

1.19 Procedure to Solve Legendre's or Cauchy's Differential Equation

Step 1 : Put $(ax + b) = e^z$ or $z = \log\,(ax + b)$

$$\therefore \qquad (ax+b)\frac{dy}{dx} = a\,Dy$$

$$(ax+b)^2 \frac{d^2y}{dx^2} = a^2 D\,(D-1)\,y$$

$$(ax+b)^3 \frac{d^3y}{dx^3} = a^3 D\,(D-1)\,(D-2)\,y$$

and so on where $D = \frac{d}{dz}$

Step 2 : Substituting in the given equation we get a linear differential equation with constant coefficients in y and z.

Step 3 : Solve by previous methods.

$$\therefore \qquad y = C.F + P.I$$

As $D = \frac{d}{dz}$ here C.F and P.I will be in terms of z.

Step 4 : Resubstitute for $z = \log\,(ax + b)$ or $e^z = (ax + b)$ we get y in terms of x.

Example 1.44 : $(2x+1)^2 \frac{d^2y}{dx^2} - 2\,(2x+1)\frac{dy}{dx} - 12y = 6x$

Solution : Step 1 : Legendre's D.E. with $a = 2$, $b = 1$

$\therefore$ Put $2x+1 = e^z$ i.e. $z = \log\,(2x + 1)$

$$\therefore \quad x = \frac{e^z - 1}{2}$$

Step 2 : Thus $(2x+1)\dfrac{dy}{dx} = 2\ Dy$

$$(2x+1)^2 \frac{d^2y}{dx^2} = 4\ D\ (D-1)\ y \qquad \text{where } D = \frac{d}{dz}$$

Step 3 : The equation becomes

$$4D\ (D-1)\ y - 2\cdot 2\cdot D\ y - 12y = 6\left(\frac{e^z - 1}{2}\right)$$

Step 4 : Take 4 common to the left side and simplify RHS.

$$4\ [D^2 - D - D - 3]\ y = 3\ (e^z - 1)$$

Step 5 : Divide by 4.

$$(D^2 - 2D - 3)\ y = \frac{3}{4}(e^z - 1)$$

Step 6 : The above equation is linear differential equation with constant coefficient, which we can solve by previous methods. Note that $D = \dfrac{d}{dz}$.

Step 7 :

$$\begin{aligned} \text{A.E.} \quad D^2 - 2D - 3 &= 0 \\ (D-3)\ (D+1) &= 0 \\ D &= 3 \quad D = -1 \\ \text{C.F.} &= C_1 e^{3z} + C_2 e^{-1z} \end{aligned}$$

Step 8 : Use PI formula.

$$\text{P.I.} = \frac{1}{f(D)} Z$$

Step 9 : This PI is of type 1. Take factors of f (D) for finding P.I.

$$\text{P.I.} = \frac{1}{(D-3)\ (D+1)} \frac{3}{4} (e^z - 1)$$

Step 10 : Separate PI_1 and PI_2.

$$= \frac{3}{4}\left\{\frac{1}{(1-3)\ (1+1)} e^z - \frac{1}{-3} 1\right\}$$

Step 11 : Replace D by a only in non-zero factor, for constant replace D by zero.

$$= \frac{3}{4}\left\{\frac{1}{(D-3)\ (D+1)} e^z - \frac{1}{(D-3)\ (D+1)} 1\right\}$$

Step 12 : Simplify.

$$= \frac{3}{4}\left\{\frac{e^z}{-4}+\frac{1}{3}\right\}$$

Step 13 : Simplify.

$$= \frac{-3e^z}{16}+\frac{1}{4}$$

Step 14 : y = C.F. + P.I.

$$\therefore \qquad y = C_1 e^{3z} + C_2 e^{-z} - \frac{3}{16}e^z + \frac{1}{4}$$

Step 15 : Put $e^z = 2x+1$

$$y = C_1(2x+1)^3 + C_2(2x+1)^{-1} - \frac{3}{16}(2x+1) + \frac{1}{4}$$

Example 1.45 : $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Solution : Step 1 : Legendre's differential equation with a = 3, b = 2.

Put $3x + 2 = e^z$ i.e. $z = \log(3x+2)$

Step 2 : $(3x+2)\dfrac{dy}{dx} = 3\,D\,y$

$$(3x+2)^2 \frac{d^2y}{dx^2} = 9\,D\,(D-1)\,y$$

Step 3 : Thus the equation becomes

$$9\,D\,(D-1)\,y + 3\cdot 3\,Dy - 36y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

Step 4 : Take 9 common and simplify RHS.

$$9\,[D^2 - D + D - 4]\,y = 3\frac{(e^{2z}-4e^z+4)}{9} + \frac{4\,(e^z-2)}{3} + \frac{3}{3}$$

$$= \frac{e^{2z}-4e^z+4+4e^z-8+3}{3}$$

$$9\,(D^2-4)\,y = \frac{e^{2z}-1}{3}$$

Step 5 : Divide by 9.

$$(D^2-4)\,y = \frac{1}{27}(e^{2z}-1)$$

Step 6 : Which is linear differential equation with constant coefficient.

Step 7 :

A.E. $D^2 - 4 = 0$

$$(D-2)(D+2) = 0$$

$$D = 2, -2$$

$$\text{C.F.} = C_1 e^{2z} + C_2 e^{-2z}$$

Step 8 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{f(D)} Z$$

Step 9 : This P.I. is of type 1 $\therefore$ Take factors of f (D) for finding P.I.

$$\text{P.I.} = \frac{1}{(D-2)(D+2)} \frac{1}{27}(e^{2z} - 1)$$

Step 10 : Separate PI_1 and PI_2.

$$= \frac{1}{27}\left\{\frac{1}{(D-2)(D+2)} e^{2z} - \frac{1}{(D-2)(D+2)} 1\right\}$$

Step 11 : Replace D by a only in non zero factor. For constant D = 0.

$$= \frac{1}{27}\left\{\frac{1}{(D-2)} \cdot \frac{1}{(2+2)} e^{2z} - \frac{1}{(0-2)(0+2)}\right\}$$

Step 12 : Simplify.

$$= \frac{1}{27}\left\{\frac{1}{4} \cdot \frac{1}{D-2} e^{2z} + \frac{1}{4}\right\}$$

Step 13 : For zero factor use formula $F_2 \; \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$.

$$= \frac{1}{108}\left\{\frac{z}{1} e^{2z} + 1\right\}$$

Step 14 : y = C.F. + P.I.

$$= C_1 e^{2z} + C_2 e^{-2z} + \frac{1}{108}\{z\, e^{2z} + 1\}$$

Step 15 : Now $e^z = 3x + 2$

$$= C_1 (3x+2)^2 + C_2 (3x+2)^{-2} + \frac{1}{108}\{(3x+2)^2 \log(3x+2) + 1\}$$

Example 1.46 : $(5+2x)^2 \dfrac{d^2y}{dx^2} - 6(5+2x)\dfrac{dy}{dx} + 8y = 5\log(5+2x)$

Solution : Step 1 : Legendre's with a = 2, b = 5

Put $5 + 2x = e^z$, $z = \log(5+2x)$

Step 2 : $(5+2x)\dfrac{dy}{dx} = 2\,Dy$

$$(5+2x)^2 \frac{d^2y}{dx^2} = 4\,D\,(D-1)\,y$$

Step 3 : The equation becomes.

$$4\,D\,(D-1)\,y - 6\cdot 2\cdot Dy + 8y = 5z$$

Step 4 : Take 4 common.

$$4\,[D^2 - D - 3D + 2]\,y = 5z$$

Step 5 : Divide by 4.

$$(D^2 - 4D + 2)\,y = \frac{5}{4}z$$

Step 6 : Which is linear differential equation with constant coefficient.

Step 7 :

$$D^2 - 4D + 2 = 0$$

$$D = \frac{+\,4 \pm \sqrt{16-8}}{2}$$

$$D = \frac{+\,4 \pm \sqrt{8}}{2}$$

$$D = +\,2 \pm \sqrt{2}$$

Which are real roots.

$$\text{C.F.} = C_1 e^{(+2+\sqrt{2})\,z} + C_2 e^{(2-\sqrt{2})\,z}$$

Step 8 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{D^2 - 4D + 2}\,\frac{5}{4}z$$

Step 9 : This P.I. is of type 3 $\therefore$ take the least power term outside.

$$= \frac{5}{2}\left[\frac{1}{1+\left(\dfrac{D^2-4D}{2}\right)}\right]\frac{z}{4}$$

Step 10 : Here f (D) takes the form $\frac{1}{1+z} = 1 - z + z^2 \ldots$

$$= \frac{5}{2}\left\{1-\left(\frac{D^2-4D}{2}\right)\ldots\right\}\frac{1}{4}z$$

Step 11 : Multiply z inside and take the derivatives.

$$= \frac{5}{2}\cdot\frac{1}{4}\left\{z-\frac{0-4}{2}\right\}$$

Step 12 : Simplify.

$$= \frac{5}{8}\{z+2\}$$

Step 13 : y = C.F. + P.I.

$$y = C_1 e^{(2+\sqrt{2})z} + C_2 e^{(2-\sqrt{2})z} + \frac{5}{8}(z+2)$$

Step 14 : Put $e^z = (5+2x)$ i.e. z = log (5 + 2x).

$$\therefore \quad y = C_1 (5+2x)^{2+\sqrt{2}} + C_2 (5+2x)^{2-\sqrt{2}} + \frac{5}{8}[\log(5+2x)+2]$$

Example 1.47 : *The radial displacement u in a rotating disc at a distance 'r' from axis is given by* $\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}+kr=0$. *Find the displacement if u = 0 for r = 0, r = a.*

Solution :

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}+kr = 0$$

Step 1 : Multiple by r^2.

$$r^2\frac{d^2u}{dr^2}+r\frac{du}{dr}-u+kr^3 = 0$$

Step 2 : Take kr^3 to RHS.

$$r^2\frac{d^2u}{dr^2}+r\frac{du}{dr}-u = -kr^3$$

Step 3 : Which is homogeneous in u and r.

$\therefore$ Put $r = e^z$

$\therefore r\frac{du}{dr} = D\,u$

and $r^2\frac{d^2u}{dr^2} = D(D-1)\,u$ where $D = \frac{d}{dz}$

Step 4 : The equation becomes.

$$D(D-1)u + Du - u = -k e^{3z}$$

Step 5 : Simplify.

$$(D^2 - D + D - 1)\,u = -k\,e^{3z}$$

$$(D^2 - 1)\,u = -k\,e^{3z}$$

Step 6 :

A.E. $$D^2 - 1 = 0$$

$$(D-1)(D+1) = 0$$

$$D = 1, -1$$

$$\text{C.F.} = C_1 e^{z} + C_2 e^{-z}$$

Step 7 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{f(D)}\,Z$$

Step 8 :

$$\text{P.I.} = \frac{1}{D^2 - 1}\; -k\,e^{3z}$$

Step 9 : This P.I. is of type 1 $\therefore$ Replace D by a only in non zero factor.

$$\text{P.I.} = \frac{1}{(3)^2 - 1}\,-k\,e^{3z}$$

Step 10 : Simplify.

$$\text{P.I.} = \frac{-k}{8}\,e^{3z}$$

Step 11 : u = C.F. + P.I. is the complete solution.

$$u = C_1 e^{z} + C_2 e^{-z} - \frac{k}{8}\,e^{3z}$$

Step 12 : Put $e^{z} = r$

$$u = C_1 r + \frac{C_2}{r} - \frac{k}{8}\,r^3 \qquad \text{... (1)}$$

Step 13 : Here the conditions are given to find the values of C_1 and C_2.

Given u = 0 for r = 0 substitute in (1).

$$0 = C_1(0) + \frac{C_2}{(0)} - \frac{k}{8}(0)^3$$

If C_2 is non zero then RHS becomes infinity which is not possible $\therefore C_2$ must be zero.

Thus (1) becomes

$$u = C_1 r + 0 - \frac{k}{8} r^3 \qquad \text{... (2)}$$

Also given u = 0 for r = a substitute in (2).

$$0 = C_1 a - \frac{k}{8} a^3$$

$$C_1 = \frac{k}{8} a^2$$

Thus (2) becomes

$$u = \frac{k}{8} a^2 r - \frac{k}{8} r^3$$

$$\therefore \quad u = \frac{k}{8} r (a^2 - r^2)$$

Example 1.48 : $u = r\dfrac{d}{dr}\left(r\dfrac{du}{dr}\right) + ar^3$

Solution : Hint : $u = r\left[\dfrac{d}{dr}\left(r\dfrac{du}{dr}\right)\right] + ar^3$ differentiate wrt r using product rule

$$u = r\left[r\frac{d^2u}{dr^2} + 1\frac{du}{dr}\right] + ar^3$$

$$\therefore \quad r^2 \frac{d^2u}{dr^2} + r\frac{du}{dr} - u = -ar^3$$

which is same as that of previous problem.

Example 1.49 : $\left(\dfrac{d^2}{dx^2} - \dfrac{2}{x^2}\right)^2 y = x^2$

Solution : Step 1 : Separate the operator.

$$\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right) y = x^2 \qquad \text{... (1)}$$

Step 2 :

Let $\left(\dfrac{d^2}{dx^2} - \dfrac{2}{x^2}\right) y = V \qquad \text{... (2)}$

$\therefore$ The equation (1) becomes.

$$\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right) V = x^2$$

Step 3 : Multiply by x^2

$$\left(x^2\frac{d^2}{dx^2}-2\right)V = x^4$$

Step 4 : Open the bracket.

$$x^2\frac{d^2v}{dx^2}-2V = x^4$$

Step 5 : Which is homogeneous in V and x.

$\therefore$ Put $x = e^z$

$$\therefore x^2\frac{d^2v}{dx^2} = D(D-1)V \quad \text{where } D = \frac{d}{dz}$$

Thus the equation becomes.

$$D(D-1)V - 2V = e^{4z}$$

$$(D^2-D-2)V = e^{4z}$$

which is the differential equation in V and x $\therefore$ Solve for V.

Step 6 :

A.E. $\quad D^2-D-2 = 0$

$$(D-2)(D+1) = 0$$

$$D = 2,\ D = -1$$

$$\text{C.F.} = C_1e^{2z}+C_2e^{-z}$$

Step 7 : Use PI formula for V.

$$\text{P.I.} = \frac{1}{D^2-D-2}e^{4z}$$

Step 8 : This P.I. is from type 1 $\therefore$ Replace D by a $D = 4$.

$$\text{P.I.} = \frac{1}{16-4-2}e^{4z}$$

$$\text{P.I.} = \frac{e^{4z}}{10}$$

Step 9 : As the C.F. and P.I. is for V thus V = C.F. + P.I.

$$\therefore \quad V = C_1e^{2z}+C_2e^{-z}+\frac{e^{4z}}{10}$$

Step 10 : Put $e^z = x$

$$V = C_1x^2+\frac{C_2}{x}+\frac{x^4}{10}$$

Step 11 : Substitute the value of V in equation (2).

$$\therefore \quad \left(\frac{d^2}{dx^2}-\frac{2}{x^2}\right)y = C_1x^2+\frac{C_2}{x}+\frac{x^4}{10}$$

Step 12 : Multiply x^2 and open bracket on LHS.

$$x^2\frac{d^2y}{dx^2}-2y = C_1x^4+C_2x+\frac{x^6}{10}$$

Step 13 : Which is the homogeneous differential equation in y and x $\therefore$ Solve for y.

Put $x = e^z$

$$\therefore\ x^2\frac{d^2y}{dx^2} = D\,(D-1)\,y$$

$\therefore$ The equation becomes

$$D\,(D-1)\,y-2y = C_1e^{4z}+C_2e^{z}+\frac{e^{6z}}{10}$$

$$(D^2-D-2)\,y = C_1e^{4z}+C_2e^{-z}+\frac{e^{6z}}{10}$$

Step 14 :

$$\text{A.E.}\qquad D^2-D-2 = 0$$

$$(D-2)\,(D+1) = 0$$

$$D = 2,\ D = -1$$

$$\text{C.F.} = C_3e^{2z}+C_4e^{-z}$$

Step 15 : Use PI formula for y.

$$\text{P.I.} = \frac{1}{D^2-D-2}C_1e^{4z}+C_2e^{z}+\frac{e^{6z}}{10}$$

Step 16 : Take factors of f (D) and separate all the terms as PI_1, PI_2, PI_3 ...

$$\text{P.I.} = \frac{1}{(D-2)\,(D+1)}C_1e^{4z}+\frac{1}{(D-2)\,(D+1)}C_2e^{z}+\frac{1}{(D-2)\,(D+1)}\frac{e^{6z}}{10}$$

Put $D = 4$ $\qquad D = 1$ $\qquad D = 6$

Step 17 : Replace D by a only in non zero factor.

$$\text{P.I.} = \frac{C_1e^{4z}}{10}+\frac{C_2e^{z}}{-2}+\frac{e^{6z}}{280}$$

Step 18 : y = C.F. + P.I.

$$y = C_3e^{2z}+C_4e^{-z}+\frac{C_1e^{4z}}{10}-\frac{C_2}{2}e^{z}+\frac{e^{6z}}{280}$$

Step 19 : Put $x = e^z$

$$y = C_3x^2 + \frac{C_4}{x} + \frac{C_1}{10}x^4 - \frac{C_2}{2}x + \frac{x^6}{280}$$

Example 1.50 : *Find the equation of curve which satisfies the Differential Equation* $4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + y = 0$ *and crosses x axis at an angle of 60° at x = 1.*

Solution :

Step 1 :

$$4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + y = 0$$

Which is homogenous in y and x

$\therefore$ Put $x = e^z$

$\therefore$ $x\frac{dy}{dx} = D\,y$

$x^2\frac{d^2y}{dx^2} = D\,(D-1)\,y$ where $D = \frac{d}{dz}$

Step 2 : $\therefore$ The equation becomes

$$4\,D\,(D-1)\,y - 4\,Dy + y = 0$$

Step 3 : Simplify.

$$(4D^2 - 8D + 1)\,y = 0$$

Step 4 :

A.E. $4D^2 - 8D + 1 = 0$

$$D = \frac{8 \pm \sqrt{64-16}}{2\cdot 4}$$

$$= \frac{8 \pm \sqrt{48}}{8}$$

$$= \frac{8 \pm 4\sqrt{3}}{8}$$

$$= 1 \pm \frac{\sqrt{3}}{2}$$

Both the roots are real.

$$\therefore \quad C.F. = C_1e^{\left(1+\frac{\sqrt{3}}{2}\right)z} + C_2e^{\left(1-\frac{\sqrt{3}}{2}\right)z}$$

Step 5 : As RHS = 0 $\therefore$ PI = 0.

$$\therefore \quad y = \text{C.F.} + \text{P.I.}$$

$$y = C_1 e^{\left(1+\frac{\sqrt{3}}{2}\right)z} + C_2 e^{\left(1-\frac{\sqrt{3}}{2}\right)z}$$

Step 6 : Put $e^z = x$

$$y = C_1\, x^{\left(1+\frac{\sqrt{3}}{2}\right)} + C_2\, x^{\left(1-\frac{\sqrt{3}}{2}\right)} \qquad \text{... (1)}$$

Step 7 : Given that curve crosses x axis at x = 1 means y = 0 at x = 1

Substitute in (1)

$$0 = C_1 + C_2 \qquad \text{... (2)}$$

Step 8 : Also given that curve crosses x axis at an angle of 60° at x = 1.

means at x = 1, $\frac{dy}{dx} = \tan 60 = \sqrt{3}$

$\therefore$ Find $\frac{dy}{dx}$ from equation (1).

$$\frac{dy}{dx} = C_1\left(1+\frac{\sqrt{3}}{2}\right)x^{\sqrt{3}/2} + C_2\left(1-\frac{\sqrt{3}}{2}\right)x^{-\sqrt{3}/2}$$

Substitute x = 1, $\frac{dy}{dx} = \sqrt{3}$

$$\sqrt{3} = C_1\left(1+\frac{\sqrt{3}}{2}\right) + C_2\left(1-\frac{\sqrt{3}}{2}\right)$$

Simplify.

$$\sqrt{3} = C_1 + C_2 + \frac{\sqrt{3}}{2}(C_1 - C_2)$$

$$\sqrt{3} = 0 + \frac{\sqrt{3}}{2}(C_1 - C_2) \qquad \{\text{As } C_1 + C_2 = 0 \text{ from 2}\}$$

$$2 = C_1 - C_2 \qquad \text{... (3)}$$

Step 9 : Solve equation (2) and (3) for C_1 and C_2.

$$0 = C_1 + C_2$$

$$2 = C_1 - C_2$$

$$\Rightarrow \quad C_1 = 1 \quad C_2 = -1$$

Step 10 : Substitute C_1 and C_2 in equation (1)

$$y = x^{\left(1+\frac{\sqrt{3}}{2}\right)} - x^{\left(1-\frac{\sqrt{3}}{2}\right)}$$

Example 1.51 : $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = \dfrac{x^3}{1+x^2}$

Solution : Step 1 : Which is homogeneous in y and x

Put $x = e^z$

$$\therefore x\frac{dy}{dx} = D\,y$$

$$x^2 \frac{d^2y}{dx^2} = D\,(D-1)\,y$$

Step 2 : $\therefore$ The equation becomes

$$D\,(D-1)\,y + Dy - y = \frac{e^{3z}}{1+e^{2z}}$$

$$(D^2 - D + D - 1)\,y = \frac{e^{3z}}{1+e^{2z}}$$

$$(D^2 - 1)\,y = \frac{e^{3z}}{1+e^{2z}}$$

Step 3 : Consider A.E.

$$D^2 - 1 = 0$$

$$(D-1)\,(D+1) = 0$$

$$D = 1,\ D = -1 \quad \text{real roots.}$$

$$\therefore \quad \text{C.F.} = C_1 e^z + C_2 e^{-z}$$

Step 4 : Use P.I. formula.

$$= \frac{1}{D^2 - 1}\,\frac{e^{3z}}{1+e^{2z}}$$

Step 5 : Use factor of f (D).

$$= \frac{1}{(D-1)\,(D+1)}\,\frac{e^{3z}}{1+e^{2z}}$$

Step 6 : Use $\dfrac{1}{(D-a)\,(D-b)} = \dfrac{1}{a-b}\left[\dfrac{1}{D-a} - \dfrac{1}{D-b}\right]$

$$= \frac{1}{2}\left[\frac{1}{D-1} - \frac{1}{D+1}\right]\frac{e^{3z}}{1+e^{2z}}$$

Step 7 : Use type 7 formula for both.

$$= \frac{1}{2}\left\{e^z \int e^{-z}\,\frac{e^{3z}}{1+e^{2z}}\,dz - e^{-z}\int e^{z}\,\frac{e^{3z}}{1+e^{2z}}\,dz\right\}$$

Step 8 : Simplify.

$$= \frac{1}{2}\left\{e^{z}\int \frac{e^{2z}}{1+e^{2z}}dz - e^{-z}\int \frac{e^{2z}\cdot e^{2z}}{1+e^{2z}}dz\right.$$

Step 9 : Put $1+e^{2z}=t$, $2e^{2z}dz=dt$, $e^{2z}dz=\frac{dt}{2}$, $e^{2z}=t-1$.

$$\therefore \quad = \frac{1}{2}\left\{e^{z}\int \frac{dt/2}{t} - e^{-z}\int \frac{(t-1)\,dt/2}{t}\right.$$

Step 10 : Simplify (i.e. divide by t in 2nd term).

$$= \frac{1}{4}\left\{e^{z}\int \frac{dt}{t} - e^{-z}\int\left(1-\frac{1}{t}\right)dt\right\}$$

Step 11 : Integrate.

$$= \frac{1}{4}\{e^{z}\log t - e^{-z}(t-\log t)\}$$

Step 12 : Put the value of t.

$$= \frac{1}{4}\left\{e^{z}\log(1+e^{2z}) - e^{-z}\left[(1+e^{2z})-\log(1+e^{2z})\right]\right\}$$

Step 13 : Take $\log(1+e^{2z})$ common and multiply e^{-z} inside.

$$= \frac{1}{4}\{(e^{z}+e^{-z})\log(1+e^{2z})-(e^{z}+e^{-z})\}$$

Step 14 : Take $e^{z}+e^{-z}$ common.

$$= \frac{1}{4}\{\log(1+e^{2z})-1\}\ (e^{z}+e^{-z})$$

Step 15 : y = C.F. + P.I.

$$y = C_1e^{z}+C_2e^{-z}+\frac{1}{4}(e^{z}+e^{-z})\{\log(1+e^{2z})-1\}$$

Step 16 : Put $e^{z}=x$

$$y = C_1x+\frac{C_2}{x}+\frac{1}{4}\left(x+\frac{1}{x}\right)\{\log(1+x^2)-1\}$$

Example 1.52 : $(x^2D^2+3xD+1)\,y=(1-x)^{-2}$

Solution :

Step 1 : Here $D=\frac{d}{dx}$ $\therefore$ Substituting we get

$$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y = \frac{1}{(1-x)^2}$$

Step 2 : Which is homogeneous in y and x

$\therefore$ Put $x = e^z$

$$\therefore\ x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)\,y$$

Step 3 : The equation becomes

$$D\,(D-1)\,y + 3\,Dy + y = \frac{1}{(1-e^z)^2}$$

$$(D^2+2D+1)\,y = \frac{1}{(1-e^z)^2}$$

Step 4 : Consider A.E.

$$D^2+2D+1 = 0$$

$$(D+1)^2 = 0$$

$$D = -1 \text{ twice}$$

$$\therefore\quad \text{C.F.} = (C_1+C_2z)\,e^{-z}$$

Step 5 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{(D+1)^2}\,\frac{1}{(1-e^z)^2}$$

Step 6 : As D + 1 is the repeated factor $\therefore$ Operate $\frac{1}{D+1}$ twice.

$\therefore$ Consider $\frac{1}{D+1}\,\frac{1}{(1-e^z)^2}$

Step 7 : Use P.I. formula from type 7.

$$= e^{-z}\int e^z\,\frac{1}{(1-e^z)^2}\,dz$$

Step 8 : Put $1-e^z = t$

$\therefore\ -e^z\,dz = dt$

$$\therefore\quad = e^{-z}\int\frac{-dt}{t^2}$$

Step 9 : Integrate using $\int\frac{1}{t^2}\,dt = -\frac{1}{t}$

$$= e^{-z}\left[\frac{1}{t}\right]$$

Step 10 : Put the value of t.

$$\frac{1}{D+1}\frac{1}{(1-e^{z})^2} = e^{-z}\frac{1}{(1-e^{z})}$$

Step 11 : Consider

$$\text{P.I.} = \frac{1}{(D+1)^2}\frac{1}{(1-e^{z})^2}$$

Step 12 : Write $(D+1)^2$ as $(D+1)(D+1)$.

$$= \frac{1}{D+1}\left[\frac{1}{D+1}\frac{1}{(1-e^{z})^2}\right]$$

Step 13 : We know $\frac{1}{D+1}\frac{1}{(1-e^{z})^2} = \frac{e^{-z}}{1-e^{z}}$ from step 10 $\therefore$ Substitute

$\therefore$
$$\text{P.I.} = \frac{1}{D+1}\left[\frac{e^{-z}}{1-e^{z}}\right]$$

Step 14 : Use type 7 formula.

$$\text{P.I.} = e^{-z}\int e^{z}\cdot\frac{e^{-z}}{1-e^{z}}\,dz$$

Step 15 : Simplify.

$$\text{P.I.} = e^{-z}\int \frac{1}{1-e^{z}}\,dz$$

Step 16 : Write $e^{z} = \frac{1}{e^{-z}}$

$$\text{P.I.} = e^{-z}\int \frac{1}{1-\frac{1}{e^{-z}}}\,dz$$

Step 17 : Simplify.

$$\text{P.I.} = e^{-z}\int \frac{e^{-z}}{e^{-z}-1}\,dz$$

Step 18 : Put $e^{-z}-1 = t$

$\therefore\ -e^{-z}dz = dt$

$\therefore\ e^{-z}dz = -dt$

$$\text{P.I.} = e^{-z}\int \frac{-dt}{t}$$

Step 19 : Integrate

$$P.I. = e^z(-\log t)$$

Step 20 : Put the value of t.

$$P.I. = -e^{-z}\log(e^{-z}-1)$$

Step 21 : y = C.F. + P.I.

$$y = (C_1+C_2z)e^{-z} - e^{-z}\log(e^{-z}-1)$$

Step 22 : Take e^{-z} common and write $e^{-z} = \frac{1}{e^z}$.

$$y = \frac{1}{e^z}\left\{(C_1+C_2z)-\log\left(\frac{1}{e^z}-1\right)\right\}$$

Step 23 : Put $e^z = x$ i.e. Z = log x.

$$y = \frac{1}{x}\left\{1+C_2\log x-\log\left(\frac{1}{x}-1\right)\right\}$$

$$y = \frac{1}{x}\left\{C_1+C_2\log x-\log\left(\frac{1-x}{x}\right)\right\}$$

$$y = \frac{1}{x}\{C_1+C_2\log x-[\log(1-x)-\log x]\}$$

$$y = \frac{1}{x}\{C_1+C_2\log x-\log(1-x)+\log x\}$$

Example 1.53 : $x^3\frac{d^2y}{dx^2}+3x^2\frac{dy}{dx}+xy = \sin(\log x)$

Solution :

Step 1 : Here the coefficient of $\frac{d^2y}{dx^2}$ is x^3.

We need x^2 ∴ Divide by x

$$\therefore x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y = \frac{1}{x}\sin(\log x)$$

Step 2 : Which is homogeneous in y and x.

∴ Put $x = e^z$ or $z = \log x$

$$\therefore x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Step 3 : $\therefore$ The equation becomes

$$D(D-1)y + 3Dy + y = \frac{1}{e^z}\sin(z)$$

$$(D^2+2D+1)y = e^{-z}\sin z$$

Step 4 : Consider A.E.

$$D^2+2D+1 = 0$$

$$(D+1)^2 = 0$$

$$D = -1, -1 \quad \text{real repeated}$$

$$\text{C.F.} = (C_1+C_2z)\,e^{-z}$$

Step 5 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{(D+1)^2}\,e^{-z}\sin z$$

Step 6 : Use $\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$ type 4 formula.

$$\text{P.I.} = e^{-z}\frac{1}{(D-1+1)^2}\sin z$$

$$= e^{-z}\frac{1}{D^2}\sin z$$

Step 7 : The problem is reduces to type 2.

$\therefore D^2 = -a^2$

$$\text{P.I.} = e^{-z}\frac{1}{-1}\sin z$$

$$= -e^{-z}\sin z$$

Step 8 : y = C.F. + P.I.

$$y = (C_1+C_2z)\,e^{-z} - e^{-z}\sin z$$

Step 9 : Take e^{-z} common write $e^{-z} = \frac{1}{e^z}$.

$$y = \frac{1}{e^z}[C_1+C_2z-\sin z]$$

Step 10 : Put $e^z = x$, $Z = \log x$.

$$y = \frac{1}{x}\,[C_1+C_2\log x-\sin(\log x)]$$

Exercise 1.9 (Problems Reducible to Type 1)

1. $(x+2)^2 \frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x+4$

[Ans. : $y = (x+2)\{C_1 + C_2 \log(x+2)\} + \frac{3}{2}(x+2)[\log(x+2)]^2 - 2]$

2. $(x+2)^2 \frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 5x+6$ [Ans. : P.I. $= \frac{5}{2}[\log(x+2)]^2 \cdot (x+2) - 4]$

3. $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

[Ans. : $y = \frac{C_1}{x} + x\,[C_2 \cos(\log x) + C_3 \sin(\log x)] + 5x + \frac{2}{x}\log x]$

4. $(x^3D^3 + x^2D^2 - 2)y = x + \frac{1}{x^3}$ [Ans. : $y = C_1x^2 + C_2\cos(\log x) + C_3\sin(\log x) - \frac{x}{2} - \frac{1}{50}x^3]$

Hint : *Here given D is* $\frac{d}{dx}$ *thus the equation is* $x^3\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} - 2y = x + \frac{1}{x^3}$

5. $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 3x - 7$

[Ans. : $y = C_1x + x^{-1/2}\left[C_2\cos\left(\frac{\sqrt{3}}{2}\log x\right) + C_3\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] + x\log x + 7]$

6. $\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)^2 y = 0$ [Ans. : $y = C_1x^4 + C_2x^2 + C_3x + \frac{C_4}{x}]$

7. $\left(\frac{d}{dx} + \frac{1}{x}\right)^2 y = \frac{1}{x^4}$ [Ans. : $y = C_1 + \frac{C_2}{x} + \frac{1}{2x^2}]$

Problems Reducible to Type 2 and 3

8. $(x+2)^2\frac{d^2y}{dx^2} + (x+2)\frac{dy}{dx} + y = 2\sin\log(x+2)$

[Ans. : $y = C_1\cos z + C_2\sin z - z\cos z$ where $z = \log(x+2)]$

9. $(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 4\cos[\log(x+1)]$

[Ans. : $y = C_1\cos z + C_2\sin z + 2z\sin z$ where $z = \log(x+1)]$

10. $(x^2D^2 - xD + 4)y = \cos(\log x) + x\sin(\log x)$ [Ans. : $y = e^z[C_1\cos\sqrt{3}z + C_2\sin\sqrt{3}z] + \frac{e^z}{2}\sin z$

$+ \frac{1}{13}(3\cos z - 2\sin z)$ where $z = \log x]$

11. $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = A + B\log x$ [Ans. : $y = C_1 + C_2\log x + \frac{Ax^2}{4} + \frac{Bx^2}{4}(\log x - 1)]$

12. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2\log x$ [Ans. : $y = x^2\{C_1\cos(\log x) + C_2\sin(\log x) + \log x\}$]

13. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x\log x$ [Ans. : $y = x\left[C_1 + C_2\log x + \frac{1}{6}(\log x)^3\right]$]

14. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x$ [Ans. : $y = C_1x^4 + \frac{C_2}{x} - \frac{x^2}{6} - \frac{1}{2}\log x + \frac{3}{8}]$

15. $(2x+1)^2 \dfrac{d^2y}{dx^2} + 2(2x+1)\dfrac{dy}{dx} + 4y = 4\sin[2\log(2x+1)]$

[Ans. : $C_1\cos\log(2x+1) + C_2\sin\log(2x+1) - \frac{1}{3}\sin[2\log(2x+1)]$]

16. $(x+1)^2 \dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} - y = 2\log(x+1) + (x-1)$

[Ans. : $y = C_1(x+1) + \frac{C_2}{x+1} + \frac{1}{3}(x-3)\log(x+1) + 2$]

17. $(4x+1)^2 \dfrac{d^2y}{dx^2} + 2(4x+1)\dfrac{dy}{dx} + y = 2x+1$ [Ans. : $y = [C_1 + C_2\log(4x+1)](4x+1)^{1/4} + \left(\frac{2x+5}{9}\right)$]

18. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = x^2\sin(\log x)$

[Ans. : $y = x^2[C_1\cos\log x + C_2\sin\log x] - \frac{x^2}{2}\log x\cos\log x$]

19. $(x+1)^2 \dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} = (2x+3)(2x+4)$

[Ans. : $y = C_1 + C_2\log(x+1) + (x+1)^2 + 6(x+1) + [\log(x+1)]^2$]

20. $(x^3D^3 + x^2D^2 - 2)y = x^2 + x^{-3}$ [Ans. : $y = C_1x^2 + C_2\cos\log x + C_3\sin\log x - \frac{1}{50x^3} + \frac{x^2}{5}\log x$]

21. $(x^2D^2 + xD - 4)y = \left(x + \frac{1}{x}\right)^2$ [Ans. : $y = C_1x^2 + \frac{C_2}{x^2} + \frac{1}{2}[\log x\sinh(2\log x) - 1]$]

22. *Find the equation of the curve that satisfies the differential equation* $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + x = 0$ *and crosses x axis at an angle of 45° at (1, 0).* [Ans. : $y = \frac{3}{2}\log x + \frac{1}{4x^2} - \frac{1}{4}$]

1.20 Lagrange's Method of Variation of Parameters

Consider the linear second order differential equation with constant coefficients.

$$f(D)\,y = X \qquad \text{... (i)}$$

Let its C.F be $\quad y = C_1y_1 + C_2y_2 \qquad$ ... (ii)

Assume $\quad$ P.I $= uy_1 + vy_2$

As P.I satisfies equation (i)

$$\therefore \quad y = uy_1 + vy_2$$

Differentiate w.r.t. x

$$y' = u'y_1 + v'y_2 + uy_1' + vy_2'$$

Assume $u'y_1 + v'y_2 = 0 \qquad$ ... (iii)

Thus $\quad y' = uy_1' + vy_2' \qquad$ ... (iv)

Again $$y'' = uy_1'' + vy_2'' + u'y_1' + v'y_2' \quad \text{... (v)}$$

Substituting y, y', y" in equation (i) we get

$$u'y_1' + v'y_2' = X \quad \text{... (vi)}$$

Thus we have two equations in u' and v' i.e. equation (iii) and (vi).

$$u'y_1 + v'y_2 = 0$$

$$u'y_1' + v'y_2' = X$$

Solving these two equations by Cramer's rule we get

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\Delta u}{\Delta}$$

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\Delta v}{\Delta}$$

Thus $$u = \int \frac{\Delta u}{\Delta}\, dx \quad \text{and } v = \int \frac{\Delta v}{\Delta}\, dx$$

Method of variation of parameters to find particular integral

Procedure 1 :

Step 1 : If the differential equation is of order two then the complementary function will be of the form C.F. = $C_1 y_1 + C_2 y_2$

Step 2 : Assume

$$\text{P.I.} = uy_1 + vy_2$$

Step 3 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad \Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} \quad \Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix}$$

where y_1' and y_2' are derivatives of y_1 and y_2 also X is the RHS of equation [i.e. f (D) y = X]

Step 4 : Then

$$u = \int \frac{\Delta u}{\Delta} dx$$

$$v = \int \frac{\Delta v}{\Delta} dx$$

Step 5 : Substitute u and v in P.I. = $uy_1 + vy_2$

Procedure 2 :

Step 1 : If the differential equation is of order three then C.F. will be

$C.F. = C_1 y_1 + C_2 y_2 + C_3 y_3$

Step 2 : $\therefore$ Assume

$$P.I. = u y_1 + v y_2 + w y_3$$

$$\Delta = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$\Delta u = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ X & y_2'' & y_3'' \end{vmatrix}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & X & y_3'' \end{vmatrix}$$

$$\Delta w = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & X \end{vmatrix}$$

Step 3 : Find $u = \int \frac{\Delta u}{\Delta} dx,\ v = \int \frac{\Delta v}{\Delta} dx,\ w = \int \frac{\Delta w}{\Delta} dx$

Step 4 : $P.I = u y_1 + v y_2 + w y_3$

1.21 Solve by Method of Variation of Parameters

Example 1.54 : $(D^3 + D)\, y = cosec\, x$

Solution : Step 1 : Consider A.E. $D^3 + D = 0$

$$D(D^2 + 1) = 0$$

$$D = 0, \quad D^2 = -1$$

$$D = 0, \quad D = \pm i,\ \alpha = 0,\ \beta = 1 \text{ complex}$$

$$\therefore \quad C.F. = C_1 e^{0x} + e^{0x}(C_2 \cos x + C_3 \sin x)$$

$$C.F. = C_1 + C_2 \cos x + C_3 \sin x$$

Step 2 : Comparing with $C.F. = C_1 y_1 + C_2 y_2 + C_3 y_3$ we get.

$$y_1 = 1,\ y_2 = \cos x,\ y_3 = \sin x$$

Step 3 : Assume P.I. = $uy_1 + vy_2 + wy_3$. To find u, v, w find derivatives of y_1 y_2 y_3.

$$y_1 = 1 \quad y_2 = \cos x \quad y_3 = \sin x$$

$$y_1' = 0 \quad y_2' = -\sin x \quad y_3' = +\cos x$$

$$y_1'' = 0 \quad y_2'' = -\cos x \quad y_3'' = -\sin x$$

Step 4 : Find Δ.

$$\Delta = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$\Delta = 1(\sin^2 x + \cos^2 x) - \cos x(0-0) + \sin x(0-0)$$

$$= 1$$

Step 5 : Find Δu.

$$\Delta u = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ X & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \operatorname{cosec} x & -\cos x & -\sin x \end{vmatrix}$$

$$= 0\,(\sin^2 x + \cos^2 x) - \cos x(0 - \cos x \operatorname{cosec} x) + \sin x(0 + \sin x \cdot \operatorname{cosec} x)$$

$$= 0 + \frac{\cos^2 x}{\sin x} + \sin x(1)$$

$$= \frac{\cos^2 x + \sin^2 x}{\sin x}$$

$$= \frac{1}{\sin x} = \operatorname{cosec} x$$

Step 6 : Find Δv.

$$\Delta v = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & X & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \operatorname{cosec} x & -\sin x \end{vmatrix}$$

$$= 1\,(0 - \operatorname{cosec} x \ \cos x) - (0-0) + \sin x(0-0)$$

$$= -\cot x$$

Step 7 : Find Δw.

$$\Delta w = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & X \end{vmatrix} = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \operatorname{cosec} x \end{vmatrix}$$

$$= 1(-\sin x \operatorname{cosec} x - 0) - \cos x(0-0) + 0(0-0)$$

$$= -1$$

Step 8 : Find u, v, w.

$$u = \int \frac{\Delta u}{\Delta} dx = \int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x - \cot x)$$

$$v = \int \frac{\Delta v}{\Delta} dx = \int -\cot x \, dx = -\log \sin x$$

$$w = \int \frac{\Delta w}{\Delta} dx = \int -1 \, dx = -x$$

Step 9 : Substitute u, v, w y_1, y_2, y_3 in P.I. = $uy_1 + vy_2 + wy_3$

$\therefore$ $$\text{P.I.} = \log(\operatorname{cosec} x - \cot x) = \cos x \log \sin x - x \sin x$$

Step 10 : y = C.F. + P.I. is the complete solution of D.E.

Example 1.55 : $(D^2 - 4)y = 2\,sech\,2x$

Solution : Step 1 : Consider A.E. $D^2 - 4 = 0$

$$(D-2)(D+2) = 0$$

$$D = 2, -2$$

$$\text{C.F.} = C_1' e^{2x} + C_2' e^{-2x}$$

Note : In this problem as the R.H.S is hyperbolic. $\therefore$ If the C.F. is hyperbolic then the integration becomes easy.

$\therefore$ Consider $$\text{C.F.} = C_1 \cosh 2x + C_2 \sinh 2x$$

Step 2 : Comparing with C.F. = $C_1 y_1 + C_2 y_2$

$$y_1 = \cosh 2x \quad y_2 = \sinh 2x$$

Step 3 : Assume P.I. = $uy_1 + vy_2$

Find derivatives of y_1, y_2.

$$y_1 = \cosh 2x \qquad y_2 = \sinh 2x$$

$$y_1' = 2 \sinh 2x \quad y_2' = 2 \cosh 2x$$

Step 4 : Find Δ

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cosh 2x & \sinh 2x \\ 2\sinh 2x & 2\cosh 2x \end{vmatrix}$$

$$= 2(\cosh^2 2x - \sinh^2 2x)$$

$$= 2\,(1) \qquad \because \cosh^2\theta - \sinh^2\theta = 1$$

$$= 2$$

Step 5 : Find Δu.

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sinh 2x \\ 2\operatorname{sech} 2x & 2\cosh 2x \end{vmatrix}$$

$$= 2 \cdot \frac{\sinh 2x}{\cosh 2x}$$

Step 6 : Find Δv.

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} = \begin{vmatrix} \cosh 2x & 0 \\ 2\sinh 2x & 2\operatorname{sech} 2x \end{vmatrix}$$

$$= 2 \cdot \cosh 2x \cdot \operatorname{sech} 2x = 2$$

Step 7 :

$$u = \int \frac{\Delta u}{\Delta}\,dx \qquad v = \int \frac{\Delta v}{\Delta}\,dx$$

$$u = \int \frac{2\sinh 2x}{\cosh 2x}\,dx, \quad v = \int \frac{2}{2}\,dx$$

$$u = \log(\cosh 2x), \qquad v = x$$

$$\because \int \frac{f'(x)}{f(x)}\,dx = \log f(x)$$

Step 8 : $\text{P.I} = uy_1 + vy_2$

$$\text{P.I.} = [\log \cosh 2x]\cosh 2x + x \sinh 2x$$

Step 9 : y = C.F. + P.I. is the complete solution.

Example 1.56 : $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = e^x \tan x$

Solution : $D^2 - 2D + 2 = 0$

Step 1 : $\text{C.F} = e^x (C_1 \cos x + C_2 \sin x)$

Step 2 : Comparing with $\text{C.F} = C_1 y_1 + C_2 y_2$

$$y_1 = e^x \cos x, \quad y_2 = e^x \sin x$$

Step 3 : Assume $\text{P.I} = u\,y_1 + v\,y_2$

Find derivatives of y_1, y_2

$$y_1' = e^x(\cos x - \sin x), \; y_2' = e^x(\sin x + \cos x)$$

Step 4 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$= e^x \cos e^x (\sin x + \cos x) - e^x \sin x\, e^x (\cos x - \sin x)$$

$$= e^{2x}\left(\cos^2 x + \sin^2 x\right) = e^{2x}$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} = -X\, y_2$$

$$= -e^x \tan x \cdot e^x \sin x = -e^{2x} \cdot \frac{\sin^2 x}{\cos x}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} = X\, y_1$$

$$= e^x \tan x \cdot e^x \cos x$$

$$= e^{2x} \sin x$$

Step 5 :

$$u = \int \frac{\Delta u}{\Delta}\, dx$$

$$= \int \frac{-\sin^2 x}{\cos x}\, dx$$

$$= \int -\frac{\left(1 - \cos^2 x\right)}{\cos x}\, dx$$

$$= \int -\sec x + \cos x\, dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

$$v = \int \frac{\Delta v}{\Delta}\, dx$$

$$= \int \sin x\, dx$$

$$= -\cos x$$

Step 6 :

$$\text{P.I} = u\, y_1 + v\, y_2$$

$$= e^x \cos x\left[-\log(\sec x + \tan x) + \sin x\right] + e^x \sin x\, [-\cos x]$$

$$= -e^x \cos x \log(\sec x + \tan x)$$

Step 7 : y = C.F + P.I is the complete solution.

$$y = e^x (C_1 \cos x + C_2 \sin x) - e^x \cos x - \log(\sec x + \tan x)$$

Example 1.57 : $(D^2+1)\,y = \dfrac{1}{1+\sin x}$

Solution :

Step 1 :

$$D^2+1 = 0$$
$$D^2 = -1$$
$$D = \pm i$$
$$\therefore \quad C.F = C_1 \cos x + C_2 \sin x$$

Step 2 : Compare C.F with C.F = $C_1 y_1 + C_2 y_2$

Step 3 : Assume P.I = $u\,y_1 + v\,y_2$

Step 4 : Find derivatives of y_1, y_2

$$\therefore \quad y_1 = \cos x \quad y_2 = \sin x$$
$$y_1' = -\sin x \quad y_2' = \cos x$$

Step 5 : Find Δ, Δu, Δv.

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$
$$= \cos^2 x + \sin^2 x$$
$$= 1$$
$$\Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} = -X\,y_2$$
$$= -\frac{1}{1+\sin x}\sin x$$
$$= -\frac{\sin x}{1+\sin x}$$
$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} = X\,y_1$$
$$= \frac{1}{1+\sin x}\cdot\cos x$$
$$= \frac{\cos x}{1+\sin x}$$

Step 6 : $u = \int \dfrac{\Delta u}{\Delta}\,dx$, $v = \int \dfrac{\Delta v}{\Delta}\,dx$

$$u = \int -\frac{\sin x}{1+\sin x}\,dx$$

$$= \int -\frac{\sin x\,(1-\sin x)}{1-\sin^2 x}\,dx \quad \text{(Rationalise)}$$

$$= \int \frac{-\sin x+\sin^2 x}{\cos^2 x}\,dx$$

$$= \int \frac{-\sin x}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x}\,dx$$

$$= \int -\sec x\tan x+\tan^2 x\,dx$$

$$= \int -\sec x\tan x+\sec^2 x-1\,dx$$

$$u = -\sec x + \tan x - x$$

$$v = \int \frac{\Delta v}{\Delta}\,dx$$

$$= \int \frac{\cos x}{1+\sin x}\,dx$$

$$= \log(1+\sin x)$$

Step 7 : $\quad P.I = u\,y_1 + v\,y_2$

$$P.I = \cos x\,(-\sec x+\tan x-\mathbf{x})+\sin x\,(\log(1+\sin x))$$

$$= -1+\sin x - x\cos x+\sin x\log(1+\sin x)$$

Step 8 : $\quad y = C.F + P.I$ is the complete solution.

$$y = C_1\cos x+C_2\sin x-1+\sin x-x\cos x+\sin x\log(1+\sin x)$$

Example 1.58 : $\left(D^2-4D+4\right)y=e^{2x}\sec^2 x$

Solution :

Step 1 : $D^2-4D+4 = 0$

$$(D-2)^2 = 0$$

$$D = 2, 2$$

$$C.F = (C_1+C_2x)\,e^{2x}$$

Step 2 : Compare with $C.F = C_1y_1+C_2y_2$

Step 3 : Assume $P.I = u\,y_1+v\,y_2$

Step 4 : Find derivatives of y_1, y_2.

$$y_1 = e^{2x} \qquad y_2 = x\,e^{2x}$$

$$y_1' = 2e^{2x} \qquad y_2' = (2x+1)\,e^{2x}$$

Step 5 : Find Δ, Δu, Δv

$$\Delta = y_1 y_2' - y_1' y_2 = e^{2x}\,(2x+1)\,e^{2x} - 2e^{2x}\cdot xe^{2x}$$

$$= e^{4x}$$

$$\Delta u = -X\,y_2 = -\,e^{2x}\sec^2 x\cdot x\,e^{2x}$$

$$= -x\sec^2 x\,e^{4x}$$

$$\Delta v = X\,y_1 = e^{2x}\,\sec^2 x\cdot e^{2x}$$

$$= e^{4x}\,\sec^2 x$$

Step 6 :

$$u = \int \frac{\Delta u}{\Delta} = \int \frac{-x\sec^2 x\,e^{4x}}{e^{4x}}\,dx$$

$$= -\int x\sec^2 x\,dx$$

$$= -\{x\cdot\tan x - (1)\,(\log\sec x)\}$$

$$= -x\tan x + \log\sec x$$

$$v = \int \frac{\Delta v}{\Delta}\,dx = \int \frac{e^{4x}\,\sec^2 x}{e^{4x}}\,dx$$

$$= \int \sec^2 x\,dx$$

$$= \tan x$$

Step 6 :

$$P.I = u\,y_1 + v\,y_2$$

$$= e^{2x}\,(-x\tan x + \log\sec x) + e^{2x}\cdot\tan x\cdot x$$

$$= e^{2x}\,\log\sec x$$

Step 7 :

$$y = C.F + P.I$$

$$y = (C_1 + C_2 x)\,e^{2x} + e^{2x}\,\log\sec x$$

is the complete solution.

Example 1.59 : $(D^2+1)\,y = x\sin x$

Solution :

$$D^2 + 1 = 0$$

$$D^2 = -1$$

$$D = \pm i$$

$$C.F = C_1 \cos x + C_2 \sin x$$

Step 2 : Compare with $C.F = C_1 y_1 + C_2 y_2$

Step 3 : Assume $P.I = u\, y_1 + v\, y_2$

Step 4 : Find derivatives of y_1, y_2

$$y_1 = \cos x \qquad y_2 = \sin x$$

$$y_1' = -\sin x \quad y_2' = \cos x$$

Step 5 : Find Δ, Δu, Δv.

$$\Delta = y_1 y_2' - y_1' y_2$$

$$= \cos x \cdot \cos x + \sin x \sin x$$

$$= 1$$

$$\Delta u = -X y_2$$

$$= -x \sin x \cdot \sin x$$

$$= -x \sin^2 x = \frac{-x}{2}(1 - \cos 2x)$$

$$\Delta v = X y_1$$

$$= x \sin x \cos x$$

$$= \frac{x}{2} \sin 2x$$

Step 6 : $$u = \int \frac{\Delta u}{\Delta} dx, \; v = \int \frac{\Delta v}{\Delta} dx$$

$$u = \int -\frac{x}{2} + \frac{x}{2} \cos 2x\, dx$$

$$= -\frac{x^2}{4} + \frac{1}{2} \int x \cos 2x\, dx$$

$$= -\frac{x^2}{4} + \frac{1}{2}\left\{x\left(\frac{\sin 2x}{2}\right) - (1)\left(\frac{-\cos 2x}{4}\right)\right\}$$

$$= -\frac{x^2}{4} + \frac{x}{4} \sin 2x + \frac{1}{8} \cos 2x$$

$$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{x}{2} \sin 2x\, dx$$

$$= \frac{1}{2}\left\{x\left(\frac{-\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right\}$$

$$= -\frac{x}{4}\cos 2x+\frac{1}{8}\sin 2x$$

Step 7 :

$$\text{P.I} = u\,y_1+v\,y_2$$

$$\text{P.I} = \left(-\frac{x^2}{4}+\frac{x}{4}\sin\ 2x+\frac{1}{8}\cos 2x\right)\cos x+\left(-\frac{x}{4}\cos 2x+\frac{1}{8}\sin 2x\right)\sin x$$

$$= \frac{-x^2}{4}\cos x+\frac{x}{4}(\sin 2x\cos x-\cos 2x\sin x)$$

$$+\frac{1}{8}(\cos 2x\cos x+\sin 2x\sin x)$$

$$= -\frac{x^2}{4}\cos x+\frac{x}{4}\sin x+\frac{1}{8}\cos x$$

Step 8 :

$$y = \text{C.F} + \text{P.I}$$

$$= C_1\ \cos x+C_2\ \sin x-\frac{x^2}{4}\cos x+\frac{x}{4}\sin x+\frac{1}{8}\cos x$$

is the complete solution of the equation.

Example 1.60 : $(D^2+1)y=3x-8\cot x$

Solution : Step 1 : C.F = $C_1\ \cos x+C_2\ \sin x$

Step 2 : Assume P.I = $u\,y_1+v\,y_2$

Step 3 : Find y_1', y_2'

$$y_1 = \cos x \qquad y_2 = \sin x$$

$$y_1' = -\sin x \qquad y_2' = \cos x$$

Step 4 : Find Δ, Δu, Δv

$$\Delta = y_1 y_2'-y_1' y_2$$

$$= \cos^2 x+\sin^2 x=1$$

$$\Delta u = -X\,y_2=-\ \sin x\,(3x-8\cot x)$$

$$= -\ 3x\ \sin\ x\ +\ 8\ \cos\ x$$

$$\Delta v = X\,y_1=\cos x\,(3x-8\cot x)$$

$$= 3x\cos x\ -\ 8\ \frac{\cos^2 x}{\sin x}$$

$$= 3x\cos x-8\ \frac{(1-\sin^2 x)}{\sin x}$$

$$= 3x\ \cos\ x-8\ \text{cosec}\ x+8\ \sin\ x$$

Step 5 : Find u, v.

$$u = \int \frac{\Delta u}{\Delta} = \int (-3x \sin x + 8 \cos x)\, dx$$

$$= -3\{x(-\cos x) - (1)(-\sin x)\} + 8 \sin x$$

$$= 3x \cos x - 3 \sin x + 8 \sin x$$

$$= 3x \cos x + 5 \sin x$$

$$v = \int \frac{\Delta v}{\Delta} = \int 3x \cos x - 8 \operatorname{cosec} x + 8 \sin x$$

$$= 3[x \sin x + \cos x] - 8 \log(\operatorname{cosec} x - \cot x) - 8 \cos x$$

$$= 3x \sin x - 5 \cos x - 8 \log(\operatorname{cosec} x - \cot x)$$

Step 6 : $\therefore$

$$\text{P.I} = u y_1 + v y_2$$

$$= \cos x\,[3x \cos x + 5 \sin x]$$

$$+ \sin x\,[3x \sin x - 5 \cos x - 8 \log(\operatorname{cosec} x - \cot x)]$$

$$= 3x(\cos^2 x + \sin^2 x) - 8 \sin x \log(\operatorname{cosec} x - \cot x)$$

$$= 3x - 8 \sin x \log(\operatorname{cosec} x - \cot x)$$

Step 7 :

$$y = \text{C.F} + \text{P.I}$$

$$= C_1 \cos x + C_2 \sin x + 3x - 8 \sin x \log(\operatorname{cosec} x - \cot x)$$

is the complete solution.

Example 1.61 : $(D^2 + 1)y = \operatorname{cosec} x$

Solution : Step 1 : C.F $= C_1 \cos x + C_2 \sin x$

Step 2 : Let P.I $= u y_1 + v y_2$

Step 3 : Find derivatives of y_1, y_2

$$y_1 = \cos x \qquad y_2 = \sin x$$

$$y_1' = -\sin x \qquad y_2' = \cos x$$

Step 4 : Find Δ, Δu, Δv

$$\Delta = y_1 y_2' - y_1' y_2 = \cos^2 x + \sin^2 x = 1$$

$$\Delta u = -x\, y_2 = -\operatorname{cosec} x \sin x = -1$$

$$\Delta v = x\, y_1 = \operatorname{cosec} x \cos x = \cot x$$

Step 5 : Find u, v

$$u = \int \frac{\Delta u}{\Delta}\,dx \qquad v = \int \frac{\Delta v}{\Delta}\,dx$$

$$= \int -1\,dx \qquad = \int \cot x\,dx$$

$$= -x \qquad = \log \sin x$$

Step 6 :

$$P.I = u y_1 + v y_2$$

$$= -x \cos x + \sin x \log \sin x$$

Step 7 :

$$y = C.F + P.I$$

$$y = C_1 \cos x + C_2 \sin x - x \cos x + \sin x \log \sin x$$

is the complete solution.

Example 1.62 : $(D^2 + 2D + 5)y = 4e^{-x} \tan 2x + 5e^x$

Solution :

Step 1 : $D^2 + 2D + 5 = 0$

$$D = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= -1 \pm 2i$$

$$C.F = e^{-x}(C_1 \cos 2x + C_2 \sin 2x)$$

Step 2 : Compare with $C.F = C_1 y + C_2 y_2$

Step 3 : Assume $P.I = u y_1 + v y_2$

Step 4 : Write y_1 y_2 and find y_1', y_2'

$$y_1 = e^{-x} \cos 2x, \quad y_2 = e^{-x} \sin 2x$$

$$y_1' = e^{-x}(-\cos 2x - 2 \sin 2x)$$

$$y_2' = e^{-x}(-\sin 2x + 2 \cos 2x)$$

Step 5 : Find Δ, Δu, Δv

$$\Delta = y_1 y_2' - y_1' y_2$$

$$= e^{-2x}\left(-\sin 2x \cos 2x + 2\cos^2 2x + \sin 2x \cos 2x + 2 \sin^2 2x\right)$$

$$= e^{-2x} \cdot 2$$

$$\Delta u = -X y_2$$

$$= -e^{-x} \sin 2x \left(4e^{-x} \tan 2x + 5e^x\right)$$

$$= -4\,e^{-2x} \frac{\sin^2 2x}{\cos 2x} - 5 \sin 2x$$

$$\Delta v = X y_1$$
$$= e^{-x} \cos 2x\left(4e^{-x} \tan 2x+5e^{x}\right)$$
$$= 4e^{-2x} \sin 2x+5 \cos 2x$$

Step 6 : Find u, v

$$u = \int \frac{\Delta u}{\Delta} dx$$
$$= \int \frac{-2\left(1-\cos^2 2x\right)}{\cos 2x}+\frac{5 \sin 2x}{2e^{-2x}} dx$$
$$= -2\int \sec 2x \, dx+2\int \cos 2x \, dx+\frac{5}{2}\int e^{2x} \sin 2x \, dx$$
$$= -\log (\sec 2x+\tan 2x)+\sin 2x+\frac{5}{2}\cdot\frac{e^{2x}}{4+4} [2 \sin 2x-2 \cos 2x]$$
$$= -\log (\sec 2x+\tan 2x)+\sin 2x+\frac{5e^{2x}}{8} (\sin 2x-\cos 2x)$$

$$v = \int \frac{\Delta v}{\Delta} dx$$
$$= \int 2 \sin 2x + \frac{5}{2} \frac{\cos 2x}{2e^{-2x}} dx$$
$$= -\cos 2x+\int \frac{5}{2} e^{2x} \cos 2x \, dx$$
$$= -\cos 2x+\frac{5}{2}\frac{e^{2x}}{4+4}(2 \cos 2x+2 \sin 2x)$$
$$= -\cos 2x+\frac{e^{2x}}{8} (\cos 2x+\sin 2x)$$

Step 7 :

$$P.I = u y_1 +v y_2$$
$$P.I = e^{-x} \cos 2x\left[-\log (\sec 2x+\tan 2x)+\sin 2x+\frac{5}{8} e^{2x} (\sin 2x-\cos 2x)\right]$$
$$+ e^{-x} \sin 2x\left[-\cos 2x+\frac{e^{2x}}{8} (\cos 2x+\sin 2x)\right]$$
$$= -e^{-x} \cos 2x[\log (\text{sex } 2x+\tan 2x)]+\frac{5}{8} e^{x}$$

Step 8 :

$$y = \text{C.F.} + \text{P.I.}$$
$$y = e^{-x} \left(c_1 \cos 2x+c_2 \sin 2x\right)-e^{-x} \cos 2x[\log(\sec 2x+\tan 2x)]+\frac{5}{8} e^{x}$$

is the complete solution

Exercise 1.10

Solve by method of variation of parameters.

1. $(D^3 + D)y = \cot x$

   [Ans. : $y = C_1 + C_2 \cos x + C_3 \sin x + \log \sin x - \cos x \log (\operatorname{cosec} x - \cot x)$]

2. $(D^3 + D)y = \sec x$ [Ans. : P.I $= \log(\sec x + \tan x) + x \cos x + \sin x \log \cos x$]

3. $(D^2 + 9)y = \dfrac{1}{1 + \sin 3x}$

   [Ans. : $y = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{9}[\sin 3x - 1 - 3x \cos 3x + \sin 3x \log(1 + \sin 3x)]$]

4. $(D^3 + D)y = \sec x \tan x$ [Ans. : P.I $= \cos \log \cos x + x \sin x + \cos x$]

5. $(D^3 + D)y = \operatorname{cosec}^2 x$ [Ans. : P.I $= -2 \cot x - \tan x \sin x$]

6. $(D^3 + D)y = \tan x$ [Ans. : P.I $= \log \sec x - 1 - \sin x \log(\sec x + \tan x)$]

7. $(D^2 + 4)y = 4\sec^2 2x$ [Ans. : $y = C_1 \cos 2x + C_2 \sin 2x - 1 + \sin 2x \log(\sec 2x + \tan 2x)$]

8. $(D^2 - 1)y = (1 + e^{-x})^2$ [Ans. : $y = C_1 e^x + C_2 e^{-x} - 1 + e^{-x} \log(1 + e^x)$]

9. $(D^2 - 1)y = \dfrac{2}{1 + e^x}$ [Ans. : $y = C_1 e^x + C_2 e^{-x} + e^x \log(1 + e^x) - e^{-x} \log(1 + e^x)$]

10. $(D^2 + 3D + 2)y = e^{e^x}$ [Ans. : $C_1 e^{-x} + C_2 e^{-2x} + e^{-2x} e^{e^x}$]

11. $(D^2 - 1)y = e^{-x} \sin e^{-x} + \cos e^{-x}$ [Ans. : $y = C_1 e^x + C_2 e^{-x} - e^{-x} \sin e^{-x}$]

12. $(D^2 + a^2)y = \tan ax$ [Ans. : $y = C_1 \cos ax + C_2 \sin ax - \frac{1}{a^2} \log(\sec ax + \tan ax) \cos ax$]

13. $(D^2 + D)y = \dfrac{1}{1 + e^x}$ [Ans. : $y = C_1 e^{0x} + C_2 e^{-x} + x - (1 + e^{-x}) \log(1 + e^x)$]

14. $(D^2 + 4)y = \sec 2x$ [Ans. : $y = C_1 \cos 2x + C_2 \sin 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x \log \cos 2x$]

15. $y'' - 6y' + 9y = \dfrac{e^{3x}}{x^2}$ [Ans. : $y = (C_1 + C_2 x) e^{3x} - e^{3x}[1 + \log x]$]

16. $(D^2 - 4D + 4)y = e^{2x} \sec^2 x$ [Ans. : $y = [C_1 + C_2 x + \log \sec x] e^{2x}$]

17. $(D^2 - 2D)y = e^x \sin x$ [Ans. : $y = C_1 + C_2 e^{2x} - \frac{e^x}{2} \sin x$]

18. $(D^2 + 9)y = \operatorname{cosec} 3x$ [Ans. : $y = C_1 \cos 3x + C_2 \sin 3x - \frac{x}{3} \cos 3x + \frac{1}{9} \sin 3x \log \sin 3x$]

19. $(D^2 + 1)y = x + \tan x$ [Ans. : $y = C_1 \cos x + C_2 \sin x + x - \cos x \log(\sec x + \tan x)$]

University Questions

Dec. - 98

1. *Solve any two :*

 i) $\dfrac{d^3 y}{dx^3} - y = x e^x$

 ii) $\dfrac{d^2 y}{dx^2} - 2m \dfrac{dy}{dx} + m^2 y = e^{mx} + 2^x$

 iii) $\left(D^2 + 1\right)^2 y = \sin^2(x / 2)$ *where* $D = \dfrac{d}{dx}$ **[10 Marks]**

2. *Solve* $x^2 \dfrac{d^3 y}{dx^3} + 3x \dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} - \dfrac{1}{x} y = 3x^2 - \dfrac{\sin(\log x)}{x}$ **[5 Marks]**

3. *Use method of variation of parameters to solve*

 $\dfrac{d^2 y}{dx^2} - 8 \dfrac{dy}{dx} + 16y = \dfrac{e^{4x}}{x^4}.$ **[5 Marks]**

May - 99

1. *Solve (any two) :*

 i) $\operatorname{cosec} x \dfrac{d^4 y}{dx^4} + y \operatorname{cosec} x = \sin 2x$

 ii) $\left(D^2 - 2D + 1\right) y = x e^x \sin x$

 iii) $\left(D^2 + 5D + 6\right) y = e^{-2x} \sec^2 x \left(1 + 2\tan x\right)$ **[8 Marks]**

2. *The radial displacement 'u' in a rotating disc at a distance r from axis is given by :*

 $\dfrac{d^2 u}{dr^2} + \dfrac{1}{r} \dfrac{du}{dr} - \dfrac{u}{r^2} + kr = 0$

 Find the displacement if u = 0 for r = 0, r = a. **[5 Marks]**

3. *Use method of variation of parameters to solve :*

 $\left(D^3 + D\right) y = \operatorname{cosec} x$ **[6 Marks]**

Dec. - 99

1. *Solve the following :*

 i) $\left(D^2 + 2D + 1\right) y = 3x + 2 + 3e^x$

 ii) $\left(D^2 - 1\right) y = e^{-x} \sin\left(e^{-x}\right) + \cos\left(e^{-x}\right)$

 iii) $\left(D^4 - 2D^3 - 3D^2 + 4D + 4\right) y = x^2 e^x$ **[12 Marks]**

2. *Solve* $\left(D^2 + 4\right) y = \sec 2x$ *by by the method of variation of parameters.* **[5 Marks]**

May - 2000

1. *Solve any three :*

 i) $\left(D^4 - 7D^2 + 12\right) y = \sin 3x \cdot \cos 2x \cdot \cos x;$

 ii) $\left(D^2 - 6D + 13\right) y = e^{3x} \cdot \sin 4x + 3^x;$

iii) $(D^2 - 1)y = x \cdot \sin 3x;$

iv) $(D^4 + 5D^3 + 6D^2 - 4D - 8)y = 4e^{-2x} + 3x$ **[12 Marks]**

2. Solve : $\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)^2 y = x^2$ **[5 Marks]**

3. Solve by the method of variation of parameters :
$(D^2 + 4)y = 2\sec^2(2x)$ **[5 Marks]**

Dec. - 2000

1. Solve any three of the following :

i) $(D^2 - 4D + 4)y = 8(e^{2x} + \sin 2x + x^2)$

ii) $(D^2 + a^2)y = \sec ax$

iii) $(D^2 - 1)y = x \sin 3x + \cos x$

iv) $(D^2 - 2D + 4)y = e^x \cos^2 x \left(\text{where } D \equiv \frac{d}{dx}\right)$ **[12 Marks]**

2. Solve by the method of variation of parameters :
$\frac{d^3 y}{dx^3} + \frac{dy}{dx} = \operatorname{cosec} x$ **[7 Marks]**

May - 2001

1. Solve any three :

i) $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x$

ii) $(D^2 - 4D + 1)y = e^{2x} \cdot \sin 2x$

iii) $(D^2 + 1)^2 y = 24 \cdot x \cdot \cos x$

iv) $(D^2 + 3D + 2)y = \sin(e^x)$

where $\left(D \equiv \frac{d}{dx}\right)$ **[12 Marks]**

2. Solve : $(3x + 2)^2 \frac{d^2 y}{dx^2} + 3(3x + 2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$ **[5 Marks]**

3. Solve by method of variation of parameters :
$(D^2 - 1)y = e^{-x} \cdot \sin(e^{-x}) + \cos(e^{-x})$ **[6 Marks]**

Dec. - 2001

1. Solve any three :

i) $(D^2 + 2D + 2)y = \frac{e^{-x}}{\cos^3 x}$

ii) $(D^3 - 3D^2 + 3D - 1)y = x^{1/2} e^x$

iii) $(D^4 - 1)y = \sinh x \sin x$

iv) $\left(D^2+1\right)y = x^2 \sin 2x$ **[12 Marks]**

2. Solve by method of variation of parameter
$\left(x^3 D^3 - 3x^2 D^2 + 6xD - 6\right)y = x^4 \log x.$ **[6 Marks]**

May - 2002

1. Solve any three :

i) $\dfrac{d^2 y}{dx^2} + 4y = x \sin x$

ii) $\dfrac{d^4 y}{dx^4} - 16y = \cos 2x$

iii) $\dfrac{d^2 y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{e^x}$

iv) $\dfrac{d^2 y}{dx^2} + 9y = 9\left(4 - x^2\right)$ **[12 Marks]**

2. Solve :
$x^2 \dfrac{d^2 y}{dx^2} - 4x \dfrac{dy}{dx} + 6y = x^5$ **[4 Marks]**

3. Solve the following differential equation by using the method of variation of parameters :
$\dfrac{d^2 y}{dx^2} + y = \operatorname{cosec} x.$ **[5 Marks]**

Dec - 2002

1. Solve any three :

i) $\left(\dfrac{d^2 y}{dx^2} + 2\dfrac{dy}{dx} + y\right) = x \cos x$

ii) $\left(D^2 + D\right)y = \dfrac{1}{1+e^x}$

iii) $\dfrac{d^3 y}{dx^3} - \dfrac{d^2 y}{dx^2} = 3x + x e^x$

iv) $\dfrac{d^2 y}{dx^2} - \dfrac{dy}{dx} - 6y = e^x x^2 + 4^{-x}$ **[12 Marks]**

2. The radial displacement 'u' in a rotating disc at a distance 'r' from axis is given by
$\dfrac{d^2 u}{dr^2} + \dfrac{1}{r}\dfrac{du}{dr} - \dfrac{u}{r^2} + kr = 0$
Find the displacement if $u = 0$, for $r = 0$, $r = a$. **[5 Marks]**

3. Solve the differential equation by using the method of variation of parameters
$\left(D^2+1\right)y = 3x - 8\cot x$ **[5 Marks]**

May - 2003

1. Solve any three :

i) $x^2 \dfrac{d^3 y}{dx^3} + 3x \dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} = x^2 \log x$

ii) $\frac{d^2 y}{dx^2} - \frac{dy}{dx} - y = e^{-x} \cdot x^3 + 4^x$

iii) $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin\left(e^{-x}\right)$

iv) $\frac{d^2 y}{dx^2} + 4y = x \sin^2 x$ **[12 Marks]**

2. *Solve by the method of variation of parameters :*

$\left(D^2 + 1\right)y = cosec\ x \left(D \equiv \frac{d}{dx}\right).$ **[5 Marks]**

Dec. - 2003

1. *Solve any three :*

i) $\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

ii) $\frac{d^2 y}{dx^2} + y = \sin x \sin 2x$

iii) $\frac{d^2 y}{dx^2} - \frac{dy}{dx} + y = x^3 - 3x^2 + 1$

iv) $\left(x^3 D^3 + 2x^2 D^2 + 2\right)y = 10\ x$ **[12 Marks]**

2. *Solve by method of variation of parameter :*

$\left(D^2 + 4\right) y = \sec 2x$ **[5 Marks]**

May - 2004

1. *Solve (any three) :*

i) $\left(D^2 - 4D + 4\right) y = \cos x \cosh x$

ii) $\left(D^4 + 1\right) y = 8\left(e^{2x} + \sin 2x + x^2\right)$

iii) $\left(D^2 - 2D + 1\right) y = x e^x \sin x$

iv) $\left(D^2 + 1\right) y = cosec\ x$

where $D = \frac{d}{dx}$ **[12 Marks]**

2. *Solve :*

i) $x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = \log x \cos(\log x)$

ii) $(1+2x)^2 \frac{d^2 y}{dx^2} - 6(1+2x)\frac{dy}{dx} + 16y - 8(1+2x)^2$ **[9 Marks]**

3. *Use method of variation of parameter to solve :*

$\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \tan x.$ **[6 Marks]**

Dec. - 2004

1. *Solve any three :*

   i) $\frac{d^3 y}{dx^3} + \frac{dy}{dx} = e^{2x} + \cos x + 5^x$

   ii) $(D^2 + 3D + 4)y = \sin 2x + x^2 + 1$

   iii) $(D^2 + 3D + 2)y = \cos(e^x)$

   iv) $\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x} + x^3 e^x$

   v) $x^2 \frac{d^2 y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^5$ **[12 Marks]**

2. *Solve by the method of variation of parameter*

   $(D^2 + 16)y = \operatorname{cosec} 4x$ **[5 Marks]**

May - 2005

1. *Solve any three :*

   i) $\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \sin x$

   ii) $\frac{d^3 y}{dx^3} - 7\frac{dy}{dx} - 6y = e^{2x}(1 + x^2)$

   iii) $\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x} \sec^2 x$ *[using variation of parameters]*

   iv) $x^2 \frac{d^2 y}{dx^2} - 4x\frac{dy}{dx} + 6y = -x^4 \sin x$

   v) $\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin(e^x)$ **[4 Marks]**

2. *Solve any three :*

   i) $\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 4y = 8(e^{2x} + \sin 2x)$

   ii) $\frac{d^3 y}{dx^3} + 3\frac{dy}{dx} = \cosh 2x \sinh 3x$

   iii) $\operatorname{cosec} x \frac{d^4 y}{dx^4} + y \operatorname{cosec} x = \sin 2x$

   iv) $(3x + 2)^2 \frac{d^2 y}{dx^2} + 3(3x + 2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

   v) $\frac{d^2 y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x} \cdot x^{-2}$ *[by variation of parameters]* **[12 Marks]**

Dec. - 2005

1. *Solve any three of the following equations :*

   i) $\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + 2y = e^{-x} \sec^3 x$

   ii) $\frac{d^2 y}{dx^2} + 4y = \cos x \cdot \cos 2x \cdot \cos 3x$

iii) $\frac{d^2 y}{dx^2} + y = x^2 \sin 2x$

iv) $\frac{d^2 y}{dx^2} + y = \operatorname{cosec} x$ *[By variation of parameters]*

v) $(1+x)^2 \frac{d^2 y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$ **[12 Marks]**

2. *Solve any three of the following equations :*

i) $(D^2 - 4D + 3)\, y = x^3 e^{2x}$

ii) $\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + 5y = 10 \sin x$

iii) $(D^2 - 3D + 2)y = \cos\left(\frac{1}{e^x}\right)$

iv) $\frac{d^2 y}{dx^2} + 4y = \tan 2x$ *[By variation of parameters]*

v) $(2x+1)^2 \frac{d^2 y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 8x.$ **[12 Marks]**

May - 2006

1. *Solve any three :*

i) $(D^2 + 3D + 2)y = x \sin 2x$

ii) $(D^2 + 2D + 1)y = e^{-x} \log x$

iii) $(D^2 + a^2)y = \sec ax$, *solve by method of variation of parameter.*

iv) $(x^2 D^2 + xD + 1)y = \cos(\log x) + x\sin(\log x).$ **[12 Marks]**

2. *Solve any three :*

i) $(D^2 - 1)\, y = \frac{2}{1+e^x}$

ii) $(D^3 - 2D + 4)y = 3x^2 - 5x + 2$

iii) $(D^2 + 3D + 2)y = e^{e^x}$, *by method of variation of parameter.*

iv) $\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)^2 y = x.$ **[12 Marks]**

❑❑❑

2 Simultaneous Linear Differential Equations

2.1 Introduction

In this chapter we solve differential equations in which there is one independent variable and two or more dependent variables. for example,

$$\frac{dx}{dt}+y = t$$

$$\frac{dy}{dt}-x = e^t$$

Here t is independent variable and x, y are dependent variables. Such equations are known as simultaneous equations. Methods of solving these equations are similar to those of solving algebraic equations.

Illustrations on Simultaneous Differential Equation

Example 2.1 : *The currents x and y in coupled circuits are given by*

$$L\frac{dx}{dt}+Rx-R(x-y)=E$$

$$L\frac{dy}{dt}+Ry-R(x-y)=0$$

Where L, R, E are constants. Find x and y in terms of t given x = 0, y = 0, when t = 0.

Solution :

Step 1 : Use $D = \frac{d}{dt}$

$$L\,Dx+Rx+Rx-Ry = E$$

$$L\,Dy+Ry-Rx+Ry = 0$$

Step 2 : Collect the terms of x and y.

$$(LD + 2R)\,x - Ry = E \qquad \ldots (1)$$

$$-Rx + (LD + 2R)\,y = 0 \qquad \ldots (2)$$

Step 3 : Solving for x using cramers rule.

$$\begin{vmatrix} LD+2R & -R \\ -R & LD+2R \end{vmatrix} x = \begin{vmatrix} E & -R \\ 0 & LD+2R \end{vmatrix}$$

Step 4 : Simplify.

$$\left(L^2D^2 + 4RLD + 4R^2 - R^2\right) x = (LD+2R)\,E$$

$$(LD+R)(LD+3R)\,x = 2RE \quad \text{As } DE = \frac{d}{dt}E$$

Step 5 : Find C.F for x.

$$(LD+R)(LD+3R) = 0$$

$$D = \frac{-R}{L} \quad D = \frac{-3R}{L}$$

$$C.F = C_1 e^{-Rt/L} + C_2 e^{-3Rt/L}$$

Step 6 : Find P.I for x.

$$P.I = \frac{1}{(LD+R)(LD+3R)}\, 2RE$$

As 2RE is a constant $\therefore$ put D = 0

$$P.I = \frac{2RE}{3R^2} = \frac{2E}{3R}$$

Step 7 : Write x = C.F + P.I

$$x = C_1 e^{-Rt/L} + C_2 e^{-3Rt/L} + \frac{2E}{3R} \qquad \text{... (3)}$$

Step 8 : Use the equation where the coefficient of y is simple i.e. equation (1).

$$\therefore \quad Ry = (LD + 2R)\,x - E$$

$$Ry = L\frac{dx}{dt} + 2Rx - E$$

Step 9 : Substitute x and $\frac{dx}{dt}$ to find y.

$$Ry = L\left[\frac{-R}{L} C_1 e^{-Rt/L} - \frac{3RC_2}{L} e^{-3Rt/L}\right]$$

$$+ 2R\left[C_1 e^{-Rt/L} + C_2 e^{-3Rt/L} + \frac{2E}{3R}\right] - E$$

$$\therefore \quad Ry = C_1 R e^{-Rt/L} - RC_2 e^{-3Rt/L} + \frac{1}{3}E$$

$$\therefore \quad y = C_1 e^{-Rt/L} - C_2 e^{-3Rt/L} + \frac{E}{3R} \qquad \text{... (4)}$$

Step 10 : Given at t = 0, x = 0 and y = 0 $\therefore$ to find C_1 and C_2 put t = 0, x = 0 in (3) and t = 0, y = 0 (4).

$$0 = C_1 + C_2 + \frac{2E}{3R}$$

$$0 = C_1 - C_2 + \frac{E}{3R}$$

Step 11 : Find C_1 and C_2. Adding we get

$$0 = 2C_1 + \frac{E}{R} \Rightarrow C_1 = \frac{-E}{2R}$$

Substituting we get $C_2 = -\frac{E}{6R}$

Step 12 : Substitute C_1 and C_2 in (3) and (4).

$$\therefore \quad x = \frac{-E}{2R} e^{-Rt/L} - \frac{E}{6R} e^{-3Rt/L} + \frac{2E}{3R}$$

$$y = \frac{-E}{2R} e^{-Rt/L} + \frac{E}{6R} e^{-3Rt/L} + \frac{E}{3R}$$

Example 2.2 : *Solve the system of equations*

$$\frac{dx}{dt} + 4x + 3y = t \; ; \; \frac{dy}{dt} + 2x + 5y = e^t$$

Solution :

Step 1 : Use $D = \frac{d}{dt}$. Then the given equations are

$$Dx + 4x + 3y = t$$

$$Dy + 2x + 5y = e^t$$

Step 2 : Collect the terms of x and y

$$(D+4)\,x + 3y = t \quad \text{... (1)}$$

$$(D+5)\,y + 2x = e^t \quad \text{... (2)}$$

Step 3 : Solving for x using cramers rule.

$$\begin{vmatrix} D+4 & 3 \\ 2 & D+5 \end{vmatrix} x = \begin{vmatrix} t & 3 \\ e^t & D+5 \end{vmatrix}$$

Step 4 : Simplify.

$$\left(D^2 + 9D + 14\right) x = Dt + 5t - 3e^t$$

$$\left(D^2 + 9D + 14\right) x = 1 + 5t - 3e^t$$

Step 5 : Find C.F for x.

$$D^2 + 9D + 14 = 0$$

$$D^2 + 7D + 2D + 14 = 0$$

$D = -2,\ D = -7$

$$\therefore \quad C.F = C_1 e^{-2t} + C_2 e^{-7t}$$

Step 6 : Find P.I for x

$$P.I = \frac{1}{D^2 + 9D + 14}\left[1 + 5t - 3e^t\right]$$

$$= \frac{1}{D^2 + 9D + 14} 1 + 5\frac{1}{D^2 + 9D + 14}(t) - 3\frac{1}{D^2 + 9D + 14} e^t$$

As 1 is constant we put D = 0

$$= \frac{1}{0 + 9(0) + 14}(1) + \frac{5}{14}\left[1 + \frac{1}{14}\left(9D + D^2\right)\right]^{-1}(t) - \frac{3}{1 + 9 + 14}(e^t)$$

$$= \frac{1}{14} + \frac{5}{14}\left[1 + (-1)\frac{1}{14}\left(9D + D^2\right) + \ldots\right](t) - \frac{3}{24} e^t$$

$$= \frac{1}{14} + \frac{5}{14}\left[t - \frac{9}{14}(1)\right] - \frac{1}{18} e^t$$

$$= \frac{5}{14} t - \frac{31}{196} - \frac{1}{8} e^t$$

Step 7 : $\therefore$ G.S. for x is x = C.F + P.I

$$x = C_1 e^{-2t} + C_2 e^{-7t} + \frac{5}{14} t - \frac{31}{196} - \frac{1}{8} e^t \qquad \ldots (3)$$

Step 8 : Use the equation where coefficient of y is simple.

$$(D + 4)\, x + 3y = t$$

$$3y = t - (D + 4)\, x$$

Step 9 : Substitute x, $\frac{dx}{dt}$ and find y.

$$y = \frac{1}{3}\left[t - (D + 4)\left(C_1 e^{-2t} + C_2 e^{-7t} + \frac{5}{14} t - \frac{31}{196} - \frac{1}{8} e^t\right)\right]$$

$$= \frac{1}{3}\left[t - \left(-2C_1 e^{-2t} - 7C_2 e^{-7t} + \frac{5}{14} - \frac{1}{8} e^t\right.\right.$$

$$\left.\left. + 4C_1 e^{-2t} + 4C_2 e^{-7t} + \frac{10}{7} t - \frac{31}{49} - \frac{1}{2} e^t\right)\right]$$

$$y = -\frac{2}{3}C_1 e^{-2t} + C_2 e^{-7t} - \frac{1}{7}t + \frac{5}{24}t + \frac{9}{98} \quad \text{... (4)}$$

Step 10 : Hence, from (3) and (4), the required solution of the system of equations are

$$x = C_1 e^{-2t} + C_2 e^{-7t} + \frac{5}{14}t - \frac{31}{196} - \frac{1}{8}e^t$$

$$y = -\frac{2}{3}C_1 e^{-2t} + C_2 e^{-7t} - \frac{1}{7}t + \frac{5}{24}t + \frac{9}{98}$$

Example 2.3 : *The velocity components of a particle moving in a plane are given by* $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$. *If the particle is initially at the point (2, 0); find the parametric equations of the curve.*

Solution :

Step 1 : Use $D = \frac{d}{dt}$; then given equations are

$$Dx + y = \sin t \quad \text{... (1)}$$

$$x + Dy = \cos t \quad \text{... (2)}$$

Step 2 : Solving for x using cramers rule.

$$\begin{vmatrix} D & 1 \\ 1 & D \end{vmatrix} x = \begin{vmatrix} \sin t & 1 \\ \cos t & D \end{vmatrix}$$

Step 3 : Simplify.

$$\left(D^2 - 1\right) x = D \sin t - \cos t$$

$$\left(D^2 - 1\right) x = 0$$

Step 4 : Find C.F for x.

$D^2 - 1 = 0$ giving $D = \pm 1$

we can write the above C.F. as

$$\text{C.F.} = A e^t + B e^{-t}$$

$$\therefore \quad \text{C.F} = C_1 \cosh t + C_2 \sinh t$$

Step 5 : Find P.I for x.

Here P.I = 0

Step 6 : Hence G.S. for x is

$$X = \text{C.F} + \text{P.I}$$

$$x = C_1 \cosh t + C_2 \sinh t \quad \text{... (3)}$$

Step 7 : Use the equation where coefficient of y is simple.

i.e. $$Dx + y = \sin t$$

$$y = \sin t - \frac{dx}{dt}$$

Step 8 : Substitute x and $\frac{dx}{dt}$ to find y.

$$y = \sin t - D(C_1 \cosh t + C_2 \sinh t)$$

$$y = \sin t - (C_1 \sinh t + C_2 \cosh t) \quad \text{... (4)}$$

Step 9 : Given that initially at t = 0, x = 2, y = 0.

To find C_1 and C_2 substituting the above values in (3) and (4)

$$C_1 = 2 \qquad C_2 = 0$$

Step 10 : Substituting these in (3) and (4), the required solution is

$$x = 2 \cosh t, \quad y = \sin t - 2 \sinh t$$

Example 2.4 : *Solve* $(D+2)x+(D+1)y=t,\ 5x+(D+3)y=t^2$

Solution : Step 1 : We are given that

$$(D+2)x+(D+1)y = t \quad \text{... (1)}$$

$$5x+(D+3)y = t^2 \quad \text{... (2)}$$

Step 2 : Solving for y using cramers rule.

$$\begin{vmatrix} D+2 & D+1 \\ 5 & D+3 \end{vmatrix} y = \begin{vmatrix} D+2 & t \\ 5 & t^2 \end{vmatrix}$$

Step 3 : Simplifying, we get

$$(D^2+1)y = 2t^2 - 3t$$

Step 4 : Find C.F for y.

$D^2 + 1 = 0$ giving $D = \pm i$

$$\therefore \quad C.F = C_1 \cos t + C_2 \sin t$$

Step 5 :

$$P.I = \frac{1}{D^2+1}(2t^2-3t) = [1+D^2]^{-1}(2t^2-3t)$$

$$= \{1-D^2+...\}(2t^2-3t) = \{(2t^2-3t)-D^2(2t^2-3t)\}$$

$$= 2t^2 - 3t - 4$$

Step 6 : $\therefore$ G.S. for y is

$$y = C_1\cos t + C_2\sin t + 2t^2 - 3t - 4$$

Step 7 : Use the equation where coefficient of x is simple.

i.e. $$5x + (D+3)y = t^2$$

$$5x = t^2 - (D+3)y$$

Step 8 : Substitute y, $\frac{dy}{dt}$ to find x.

$$x = \frac{1}{5}\left[t^2 - (D+3)\left(C_1\cos t + C_2\sin t + 2t^2 - 3t - 4\right)\right]$$

$$= \frac{1}{5}\left[t^2 - \left(-C_1\sin t + C_2\cos t + 4t - 3\right.\right.$$

$$\left.\left.+ 3C_1\cos t + 3C_2\sin t + 6t^2 - 9t - 12\right)\right]$$

$$x = \frac{(C_1 - 3C_2)}{5}\sin t - \frac{(3C_1 + C_2)}{5}\cos t - t^2 + t + 3 \quad \text{... (4)}$$

Step 9 : Hence (3) and (4) gives the G.S. of the given equations.

$$x = \frac{(C_1 - 3C_2)}{5}\sin t - \frac{(3C_1 + C_2)}{5}\cos t - t^2 + t + 3$$

$$y = C_1\cos t + C_2\sin t + 2t^2 - 3t - 4$$

Example 2.5 : *For the equations* $\frac{d^2x}{dt^2} = n^2(4y - 5x)$ *and* $\frac{d^2y}{dt^2} = n^2(4x - 5y)$ *with conditions* $x(0) = a$ $y(0) = x'(0) = y'(0) = 0$ *show that* $y = x \tan nt \tan (2n\ t)$

Solution :

Step 1 : Use $\frac{d}{dt} = D$; then above equations are

$$\left(D^2 + 5n^2\right)x - 4n^2 y = 0 \quad \text{... (1)}$$

$$4n^2 x - \left(D^2 + 5n^2\right)y = 0 \quad \text{... (2)}$$

Step 2 : Solving for y using cramer's rule.

$$\begin{vmatrix} D^2 + 5n^2 & -4n^2 \\ 4n^2 & -\left(D^2 + 5n^2\right) \end{vmatrix} y = \begin{vmatrix} D^2 + 5n^2 & 0 \\ 4n^2 & 0 \end{vmatrix}$$

Step 3 : Simplify.

$$\left[-\left(D^2 + 5n^2\right)^2 + 16n^4\right]y = 0$$

$$\left[\left(D^2 + 5n^2\right)^2 - 16n^4\right]y = 0$$

$$[D^2+5n^2+4n^2][D^2+5n^2-4n^2]y = 0$$
$$(D^2+9n^2)(D^2+n^2)y = 0$$

Step 4 : Find C.F for y.

$$(D^2+9n^2)(D^2+n^2)y = 0$$

Solving above equation we get D = $\pm$ ni, $\pm$ 3 in

$$\therefore \quad C.F = C_1\cos nt + C_2\sin nt + C_3\cos 3nt + C_4\sin 3nt$$

Here P.I = 0

$$\therefore \quad y = C.F + P.I$$

i.e.
$$y = C_1\cos nt + C_2 \sin nt + C_3\cos 3nt + C_4\sin 3nt \quad \text{... (3)}$$

Step 6 : Use the equation where coefficient of x is simple.

i.e.
$$4n^2x-(D^2+5n^2)y = 0$$

Step 7 : Substituting for y to find x.

$$x = \frac{1}{4n^2}\left[(D^2+5n^2)(C_1\cos nt + C_2\sin nt + C_3\cos 3nt + C_4\sin 3nt)\right]$$

Simplify and find the derivative

$$= \frac{1}{4n^2}\left[-n^2C_1\cos nt - n^2C_2\sin nt - 9n^2C_3\cos 3nt - 9n^2C_4\sin 3nt\right.$$
$$\left. + 5n^2C_1\cos nt + 5n^2C_2\sin nt + 5n^2C_3\cos 3nt + 5n^2C_4\sin 3nt\right]$$
$$x = C_1\cos nt + C_2\sin nt - C_3\cos 3nt - C_4\sin 3nt \quad \text{... (4)}$$

Step 8 : Now to find C_1 and C_2 use initial conditions in (3) and (4), to get x (0) = a (i.e. x = a at t = 0) giving

$$C_1 - C_3 = a \quad \text{... (A)}$$

y (0) = 0 (i.e. y = 0 at t = 0), giving

$$C_1 + C_3 = 0 \quad \text{... (B)}$$

$\therefore$ From (A) and (B)

$$C_1 = \frac{a}{2}, \quad C_3 = -\frac{a}{2}$$

Step 9 : Now differentiating (3) and (4), w.r.t. 't'.

$$\frac{dx}{dt} = -nC_1\sin nt + nC_2\cos nt + 3nC_3\sin 3nt - 3nC_4\cos 3nt$$

$$\frac{dy}{dt} = -nC_1\sin nt + nC_2\cos nt - 3nC_3\sin 3nt + 3nC_4\cos 3nt$$

Step 10 : Again using conditions.

$x'(0) = 0\left(\text{i.e.} \frac{dx}{dt} = 0, \text{ at } t = 0\right)$, giving

$$nC_2 - 3nC_4 = 0 \Rightarrow C_2 - 3C_4 = 0 \quad \text{... (C)}$$

$$y'(0) = 0 \text{ giving, } nC_2 + 3nC_4 = 0 \Rightarrow C_2 + 3C_4 = 0 \quad \text{... (D)}$$

$\therefore$ From (C) and (D), $C_2 = 0$, $C_4 = 0$

Step 11 : Substituting the values of C_1, C_2, C_3, C_4 in (3) and (4) we get

$$y = \frac{a}{2}\cos nt - \frac{a}{2}\cos 3nt$$

$$x = \frac{a}{2}\cos nt + \frac{a}{2}\cos 3nt$$

$$\therefore \quad \frac{y}{x} = \frac{\cos nt - \cos 3nt}{\cos nt + \cos 3nt} = \frac{2\sin 2nt \sin nt}{2\cos 2nt \cos nt}$$

$$\frac{y}{x} = \tan 2nt \cdot \tan nt$$

$$y = x \tan 2n \tan nt, \text{ the required result.}$$

Example 2.6 : *Solve* $2\frac{dx}{dt} - x + 3y = \sin t \qquad 2\frac{dy}{dt} + 3x - y = \cos t$

Given that $x = \frac{1}{4}, \frac{dx}{dt} = \frac{1}{5}$ *when* $t = 0$

Solution :

Step 1 : Use $D = \frac{d}{dt}$, then above equations are

$$2Dx - x + 3y = \sin t$$

$$2Dy + 3x - y = \cos t$$

Step 2 : Collect the terms of x and y.

$$(2D-1)x + 3y = \sin t \quad \text{... (1)}$$

$$3x + (2D-1)y = \cos t \quad \text{... (2)}$$

Step 3 : Solving for x using cramers rule.

$$\begin{vmatrix} 2D-1 & 3 \\ 3 & 2D-1 \end{vmatrix} x = \begin{vmatrix} \sin t & 3 \\ \cos t & 2D-1 \end{vmatrix}$$

Step 4 : Simplify.

$$\left[(2D-1)^2 - 3^2\right]x = (2D-1)\sin t - 3\cos t$$

$$(2D-1+3)(2D-1-3)x = 2\cos t - \sin t - 3\cos t$$

$$(2D+2)(2D-4)x = -\sin t - \cos t$$

Step 5 : Find C.F for x.

$$(2D + 2)(2D - 4)x = 0 \quad \text{gives the roots } D = -1, 2$$

$$\therefore \quad C.F = C_1 e^{-t} + C_2 e^{2t}$$

Step 6 : Find P.I for x.

$$P.I = \frac{1}{4(D+1)(D-2)}[\sin t - \cos t]$$

$$= -\frac{1}{4}\left[\frac{1}{D^2 - D - 2}\sin t + \frac{1}{D^2 - D - 2}\cos t\right]$$

$$= -\frac{1}{4}\left[\frac{1}{-1 - D - 2}\sin t + \frac{1}{-1 - D - 2}\cos t\right]$$

$$= \frac{1}{4}\left[\frac{(D-3)}{(D+3)(D-3)}\sin t + \frac{(D-3)}{(D+3)(D-3)}\cos t\right]$$

$$= \frac{1}{4}\left[(D-3)\frac{1}{(D^2 - 9)}\sin t + (D-3)\frac{1}{(D^2 - 9)}\cos t\right]$$

$$= \frac{1}{4}\left[(D-3)\frac{1}{-1-9}\sin t + (D-3)\frac{1}{-1-9}\cos t\right]$$

$$= -\frac{1}{40}[\cos t - 3\sin t - \sin t - 3\cos t]$$

$$= \frac{1}{20}[\cos t + 2\sin t]$$

Step 7 : $\therefore$ G.S. for x is

$$x = C_1 e^{-t} + C_2 e^{2t} + \frac{1}{20}(\cos t + 2\sin t) \qquad \text{... (3)}$$

Step 8 : Use the equation where coefficient of y is simple.

i.e. $(2D - 1)x + 3y = \sin t$

$$3y = \sin t - (2D - 2)x$$

Step 9 : Substitute x, $\frac{dx}{dt}$ to find y.

$$y = \frac{1}{3}\left[\sin t + 2C_1 e^{-t} - 4C_2 e^{2t} + \frac{1}{10}\sin t + \frac{1}{5}\cos t - 2x\right]$$

Step 10 : To find C_1 and C_2 use initial conditions $x = \frac{1}{4}$, $t = 0$ in (3), to get

$$C_1 + C_2 + \frac{1}{20} = \frac{1}{4}$$

$$C_1 + C_2 = \frac{1}{5} \qquad \text{... (A)}$$

Step 11 : Differentiating (3) w.r.t 't'.

$$\frac{dx}{dt} = -C_1 e^{-t} + 2C_2 e^{2t} - \frac{1}{20}\sin t + \frac{1}{10}\cos t$$

Again, $\frac{dx}{dt} = \frac{1}{5}$, t = 0, give

$$-C_1 + 2C_2 + \frac{1}{10} = \frac{1}{5}$$

$$-C_1 + 2C_2 = \frac{1}{10} \qquad \text{... (B)}$$

Solving (A), (B) we get

$$C_2 = \frac{1}{10}, \; C_1 = \frac{1}{10}$$

Step 12 : Substituting these in (3),

$$x = \frac{1}{10}\left[e^{-t} + e^{2t} + \frac{1}{2}(\cos t + 2\sin t)\right] \qquad \text{... (4)}$$

Step 13 : Substituting for x in (1),

$$y = \frac{1}{3}\left[\sin t - (2D-1)\frac{1}{10}\left\{e^{-t} + e^{2t} + \frac{1}{2}(\cos t + 2\sin t)\right\}\right]$$

$$y = \frac{1}{10}\left[\left(e^{-t} - e^{2t}\right) + \left(4\sin t - \frac{1}{2}\cos t\right)\right] \qquad \text{... (5)}$$

Example 2.7 : *Solve*

$$\frac{dx}{dt} = x^2 + xy \qquad \text{... (1)}$$

$$\frac{dy}{dt} = y^2 + xy \qquad \text{... (2)}$$

Subject to the conditions, $x = 1$, $y = 2$ at $t = 0$

Solution : Divide (2) by (1)

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2 + xy} = \frac{y(x+y)}{x(x+y)} = \frac{y}{x}$$

$\frac{dy}{y} = \frac{dx}{x}$; its integration gives

$$\log y = \log x + \log e \Rightarrow \log\left(\frac{y}{x}\right) = \log c$$

$$\frac{y}{x} = C \quad y = Cx \qquad \text{... (3)}$$

Using conditions y = 2, x = 1 at t = 0, we have from (3), C = 2

Substituting for C in (3)

$$y = 2x \qquad \text{... (4)}$$

Putting y = 2x in (1), we have

$$\frac{dx}{dt} = x^2 + 2x = 3x^2$$

$x^{-2}dx = 3\ dt$ Integration gives

$$\frac{x^{-1}}{-1} = 3t + C_1 \quad -\frac{1}{x} = 3t + C_1$$

Since x = 1 at t = 0, we get $C_1 = -1$

$$\therefore \quad -\frac{1}{x} = 3t - 1 \quad x = \frac{1}{1-3t}$$

So that $y = 2x = \dfrac{2}{1-3t}$

Hence the solution of the given system under the given condition is

$$x = \frac{1}{1-3t}, y = \frac{2}{1-3t}$$

Example 2.8 : *In a heat exchange, the temperatures x and y of two liquids, satisfy the equations* $4\dfrac{dx}{dt} = y - x = 2\dfrac{dy}{dt}$. *Find the temperatures x and y as a function of time, given that x = 20 and y = 100 at time t = 0.*

Solution :

Step 1 : Use $D = \dfrac{d}{dt}$; then given equations are

$$4Dx + x - y = 0$$

$$x + 2Dy - y = 0$$

Step 2 : Collect the terms of x and y.

$$(4D + 1)\, x - y = 0 \qquad \text{... (1)}$$

$$x + (2D - 1)\, y = 0 \qquad \text{... (2)}$$

Step 3 : Solving for x.

$$\begin{vmatrix} 4D+1 & -1 \\ 1 & 2D-1 \end{vmatrix} x = \begin{vmatrix} 0 & -1 \\ 0 & 2D-1 \end{vmatrix}$$

Step 4 : Simplify.

$$[(2D-1)(4D+1)+1]x = 0$$

$$(8D^2-2D)x = 0$$

Step 5 : Find C.F for x.

$$8D^2-2D = 0$$

$$2D(4D-1) = 0, \quad \text{giving } D = 0, \frac{1}{4}$$

$$\therefore \qquad C.F = C_1e^{0t}+C_2e^{\frac{1}{4}t}$$

Step 6 : Here P.I for x = 0.

Step 7 :

$$\therefore \text{G.S. x is} \qquad x = C_1+C_2e^{\frac{t}{4}} \qquad \dots (3)$$

Step 8 : Use the equation where coefficient of y is simple.

$$\text{i.e.} \qquad y = (4D+1)x$$

Step 9 : Substituting for x, $\frac{dx}{dt}$ to find y.

$$y = (4D+1)\left(C_1+C_2\, e^{\frac{1}{4}t}\right)$$

$$= 4\left(\frac{1}{4}\right)C_2e^{\frac{1}{4}t}+C_1+C_2e^{\frac{1}{4}t}$$

$$y = C_1+2C_2e^{\frac{1}{4}t} \qquad \dots (4)$$

Step 10 : To find C_1 and C_2 use conditions x = 20, y = 100 at t = 0 substituting in (3) and (4), we have

$$C_1+C_2 = 20$$

$$C_1+2C_2 = 100$$

These gives $C_2 = 80$, $C_1 = -60$

Step 11 : Substituting these values in (3) and (4), we get

$$x = -60+80\,e^{\frac{1}{4}t}$$

$$y = -60+160\,e^{\frac{1}{4}t}$$

Example 2.9 : *Solve*

$$\frac{dx}{dt}+\frac{dy}{dt}+5x+7y=2$$

$$2\frac{dx}{dt}+3\frac{dy}{dt}+x+y=\sin t$$

Solution :

Step 1 : Let $D = \frac{d}{dt}$, then above equations can be written as

$$Dx+Dy+5x+7y = 2$$

$$2Dx+3Dy+x+y = \sin t$$

Step 2 : Collect the terms of x and y.

$$(D+5)\,x+(D+7)\,y = 2 \qquad \text{... (1)}$$

$$(2D+1)\,x+(3D+1)\,y = \sin t \qquad \text{... (2)}$$

Step 3 : Solving for x.

$$\begin{vmatrix} D+5 & D+7 \\ 2D+1 & 3D+1 \end{vmatrix} x = \begin{vmatrix} 2 & D+7 \\ \sin t & 3D+1 \end{vmatrix}$$

Step 4 : Simplify.

$$[(3D+1)(D+5)-(D+7)(2D+1)]x = (3D+1)(2)-(D+7)\sin t$$

$$\left(D^2+D-2\right)x = 2-\cos t-7\sin t$$

Step 5 : Find C.F for x.

$D^2+D-2=0$ giving $D = 1, -2$

$$\therefore \qquad C.F = C_1e^{t}+C_2e^{-2t}$$

Step 6 : Find P.I for x.

$$P.I = 2\frac{1}{D^2+D-2}\left(e^{0t}\right)-\frac{1}{D^2+D-2}\cos t-7\frac{1}{D^2+D-2}\sin t$$

$$= 2\frac{1}{0+0-2}(1)-\frac{1}{-1+D-2}\cos t-7\frac{1}{-1+D-2}\sin t$$

$$= -1-\frac{(D+3)}{(D^2-9)}\cos t-7\frac{(D+3)}{(D^2-9)}\sin t$$

$$= -1-(D+3)\frac{1}{-1-9}\cos t-7\frac{(D+3)}{-1-9}\sin t$$

$$= -1+\frac{1}{10}(-\sin t+3\cos t)+\frac{7}{10}(\cos t+3\sin t)$$

$$= -1 + 2\sin t + \cos t$$

Step 7 : G.S. is given by,

$$x = C_1e^t + C_2e^{-2t} + 2\sin t + \cos t - 1 \quad \text{... (3)}$$

Step 8 : Use the equation where coefficient of y is simple.

$$\text{i.e.}(D+5)x+(D+7)y = 2$$

Step 9 : Substituting for x , $\frac{dx}{dt}$ to find y.

$$y = \frac{1}{(D+7)}\left[2-(D+5)\left(C_1e^t+C_2e^{-2t}+2\sin t+\cos t-1\right)\right]$$

$$= \frac{1}{D+7}\left[2-\left\{C_1e^t-2C_2e^{-2t}+2\cos t-\sin t+5C_1e^t\right.\right.$$

$$\left.\left.+5C_2e^{-2t}+10\sin t+5\cos t-5\right\}\right]$$

$$= \frac{1}{D+7}\left[-6C_1e^t-3C_2e^{-2t}-7\cos t-9\sin t+7\right]$$

$$= -6C_1\frac{1}{D+7}e^t-3C_2\frac{1}{D+7}e^{-2t}-7\frac{(D-7)}{(D^2-49)}$$

$$\cos t-9\frac{(D-7)}{(D^2-49)}\sin t+7\frac{1}{D+7}e^{0t}$$

$$= -6C_1\frac{1}{1+7}e^t-3C_2\frac{1}{-2+7}e^{-2t}-7(D-7)\frac{1}{-1-49}\cos t$$

$$-9\frac{(D-7)}{-1-49}\sin t+7\frac{1}{0+7}(1)$$

$$= -\frac{3}{4}c_1e^t-\frac{3}{5}c_2e^{-2t}+\frac{7}{50}(-\sin t-7\cos t)+\frac{9}{50}(\cos t-7\sin t)+1$$

$$y = -\frac{3}{4}c_1e^t-\frac{3}{5}c_2e^{-2t}-\frac{7}{5}\sin t-\frac{4}{5}\cos t+1 \quad \text{... (4)}$$

Hence equations (3) and (4), give the G.S. of the system of equations.

Example 2.10 : *Solve*

$$3\frac{dx}{dt}+\frac{dy}{dt}+2x = 3\cos t$$

$$2\frac{dy}{dt}+\frac{dx}{dt}+3y = 7\cos t-4\sin t$$

given that $x = y = 0$ when $t = 0$.

Solution : Step 1 :

Use $D = \frac{d}{dt}$, the above equations can be written as

$$3Dx + Dy + 2x = 3\cos t$$

$$2Dy + Dx + 3y = 7\cos t - 4\sin t$$

Step 2 : Collect the terms of x and y

$$(3D+2)x + Dy = 3\cos t \quad \text{... (1)}$$

$$Dx + (2D+3)y = 7\cos t - 4\sin t \quad \text{... (2)}$$

Step 3 : Solving for x.

$$\begin{vmatrix} 3D+2 & D \\ D & 2D+3 \end{vmatrix} x = \begin{vmatrix} 3\cos t & D \\ 7\cos t - 4\sin t & 2D+3 \end{vmatrix}$$

Step 4 : Simplify.

$$[(2D+3)(3D+2) - D^2]x = (2D+3)(3\cos t) - D(7\cos t - 4\sin t)$$

$$[5D^2 + 13D + 6]x = \sin t + 13\cos t$$

Step 5 : Find C.F for x

$$5D^2 + 13D + 8 = 0, \quad 5D^2 + 10D + 3D + 6 = 0$$

$$5D(D+2) + 3(D+2) = 0, \quad \text{giving}$$

$$D = -2, -\frac{3}{5}$$

$$\therefore \quad \text{C.F.} = C_1 e^{-2t} + C_2 e^{-\frac{3}{5}t}$$

Step 6 : Find P.I for x.

$$\text{P.I.} = \frac{1}{5D^2 + 13D + 6}\sin t + 13\frac{1}{5D^2 + 13D + 6}\cos t$$

$$= \frac{1}{5(-1) + 13D + 6}\sin t + 13\frac{1}{5(-1) + 13D + 6}\cos t$$

$$= \frac{(13D - 1)}{(169D^2 - 1)}\sin t + 13\frac{(13D - 1)}{(169D^2 - 1)}\cos t$$

$$= (13D - 1)\frac{1}{169(-1) - 1}\sin t + 13(13D - 1)\frac{1}{169(-1) - 1}\cos t$$

$$= -\frac{1}{170}[13\cos t - \sin t - 169\sin t - 13\cos t] = \sin t$$

Step 7 :

$\therefore$ G.S. of x is given by

$$x = C_1 e^{-2t} + C_2 e^{-\frac{3}{5}t} + \sin t \qquad \text{... (3)}$$

Step 8 : Use the equation where coefficient of y is simple.

i.e. $(3D+2)x + Dy = 3\cos t$

Step 9 : Substituting for x, $\frac{dx}{dt}$ to find y.

$$y = \frac{1}{D}\left[3\cos t - (3D+2)\left(C_1 e^{-2t} + C_2 e^{-\frac{3}{5}t} + \sin t\right)\right]$$

$$= \frac{1}{D}\left[3\cos t + 6c_1 e^{-2t} - 2c_1 e^{-2t} + \frac{9}{5} c_2 e^{-\frac{3t}{5}} - 2c_2 e^{-\frac{3}{5}t} - 3\cos t - 2\sin t\right]$$

$$= 4c_1 \int e^{-2t}\, dt - \frac{1}{5} C_2 \int e^{-\frac{3}{5}t} \cdot dt - 2\int \sin t\, dt$$

$$y = -2C_1 e^{-2t} + \frac{1}{3} C_2 e^{-\frac{3}{5}t} + 2\cos t \qquad \because \frac{1}{D} = \int dx \quad \text{... (4)}$$

Step 10 : To find C_1 and C_2 using initial conditions x = 0, y = 0 at t = 0, we have from (3) and (4)

$$C_1 + C_2 = 0\,; \quad -2C_1 + \frac{1}{3} C_2 = -2$$

Giving $\qquad C_2 = -\frac{6}{7}, \quad C_1 = \frac{6}{7}$

Step 11 : Substituting these in (3) and (4), we get

$$x = +\frac{6}{7} e^{-2t} - \frac{6}{7} e^{-\frac{3t}{5}} + \sin t$$

$$y = \frac{-12}{7} e^{-2t} - \frac{2}{7} e^{-\frac{3t}{5}} + 2\cos t$$

Example 2.11 : *If acceleration components of an electron are given by* $\frac{d^2x}{dt^2} - b\frac{dy}{dt} = 0,\ \frac{d^2y}{dt^2} + b\frac{dx}{dt} - a = 0$. *Find its path if it were at rest at origin initially.*

Solution : **Step 1 :** This problem involves all derivatives $\therefore$ to reduce the powers we integrate w.r.t. t.

$$\therefore \qquad \frac{dx}{dt} - by = c_1' \qquad \frac{dy}{dt} + bx - at = c_2'$$

Step 2 : As initially the particle is at rest at origin

$\therefore$ at $t = 0 \qquad x = y = \dfrac{dy}{dt} = 0$

Substituting we get $c_1' = 0, \quad c_2' = 0$

$\therefore$ The equations reduces to

$$\frac{dx}{dt} - by = 0, \qquad \frac{dy}{dt} + bx = at$$

Step 3 :

Let $\qquad D = \dfrac{d}{dt}$

$$Dx - by = 0, \qquad bx + Dy = at$$

Step 4 : Solving for x

$$\begin{vmatrix} D & -b \\ b & D \end{vmatrix} x = \begin{vmatrix} 0 & -b \\ at & D \end{vmatrix}$$

Step 5 : Simplify

$$\left(D^2 + b^2\right) x = abt$$

Step 6 : Find C.F

$$C.F = C_1 \cos bt + C_2 \sin bt$$

Step 7 : Find P.I for x

$$\begin{aligned} P.I. &= \frac{1}{D^2 + b^2} abt \\ &= \frac{ab}{b^2} \frac{1}{\left(1 + \frac{D^2}{b^2}\right)} t \\ &= \frac{a}{b} \left\{1 - \frac{D^2}{b^2}\right\} t \\ &= \frac{a}{b} \{t - 0\} \\ &= \frac{at}{b} \end{aligned}$$

Step 8 : $\qquad x = C.F + P.I$

$$= C_1 \cos bt + C_2 \sin bt + \frac{at}{b} \qquad \text{... (1)}$$

Step 9 : Consider the equation whose coefficient in y is simple.

i.e. $$Dx - by = 0$$

$$\therefore \quad by = \frac{dx}{dt}$$

$$by = -C_1 \, b \sin bt + b\, C_2 \cos bt + \frac{a}{b}$$

$$\therefore \quad y = -C_1 \sin bt + C_2 \cos bt + \frac{a}{b^2} \qquad \text{... (2)}$$

Step 10 : Given at t = 0, x = 0, y = 0

$\therefore$ Substituting in (1) and (2)

$$C_1 + 0 + 0 = 0, \qquad 0 + C_2 + \frac{a}{b^2}$$

$$C_1 = 0 \quad \text{and} \quad C_2 = \frac{-a}{b^2}$$

Step 11 : Substituting C_1 and C_2 in we get

$$x = \frac{-a}{b^2} \sin bt + \frac{at}{b}$$

$$x = \frac{a}{b^2}(bt - \sin bt)$$

$$y = \frac{a}{b^2}(1 - \cos bt)$$

Example 2.12 : *Solve*

$$\frac{dx}{dt} + 2x - 3y = t$$

$$\frac{dy}{dt} - 3x + 2y = e^{2t}$$

Solution : Step 1 :

Use $D = \frac{d}{dt}$ hence equations becomes

$$Dx + 2x - 3y = t$$

$$Dy - 3x + 2y = e^{2t}$$

Step 2 : Collect the terms of x and y.

$$(D+2)\,x - 3y = t \qquad \text{... (1)}$$

$$(D+2)\,y - 3x = e^{2t} \qquad \text{... (2)}$$

Step 3 : Solving for x

$$\begin{vmatrix} D+2 & -3 \\ -3 & D+2 \end{vmatrix} x = \begin{vmatrix} t & -3 \\ e^{2t} & D+2 \end{vmatrix}$$

Step 4 : Simplify.

$$\left[(D+2)^2 - 3^2\right] x = (D+2)\, t + 3\, e^{2t}$$

$$= 1 + 2\, t + 3\, e^{2t}$$

$$\left(D^2 + 4\, D - 5\right) x = 1 + 2\, t + 3\, e^{2t}$$

Step 5 : Find C.F for x.

$$D^2 + 4\, D - 5 = 0 \quad \text{gives} \quad D = -5, 1 \text{ hence}$$

$$C.F = C_1\, e^{-5t} + C_2\, e^t$$

Step 6 : Find P.I for x.

$$P.I = \frac{1}{D^2 + 4\, D - 5}(1+2t) + \frac{3\, e^{2t}}{D^2 + 4\, D - 5}$$

$$= -\frac{1}{5}\left[1 - \frac{4D + D^2}{5}\right](1 + 2\, t) + \frac{3\, e^{2t}}{4 + 8 - 5}$$

$$= -\frac{1}{5}\left(1 + \frac{4D}{5}\right)(1 + 2t) + \frac{3}{7}\, e^{2t}$$

$$= -\frac{1}{5}\left(\frac{13}{5} + 2t\right) + \frac{3\, e^{2t}}{7}$$

Step 7 : Hence G.S. is given by,

$$x = C.F + P.I = C_1\, e^{-5t} + C_2\, e^t - \frac{13}{25} - \frac{2t}{5} + \frac{3\, e^{2t}}{7} \qquad \dots (6)$$

Now $$\frac{dx}{dt} = -5\, C_1\, e^{-5t} + C_2\, e^t - \frac{2}{5} + \frac{6}{7}\, e^{2t}$$

Step 8 : Use the equation where coefficient of y is simple.

i.e. $$(D+2)\, x - 3y = t$$

$$3y = (D+2)\, x - t$$

Step 9 : Putting values of x and $\frac{dx}{dt}$ to find y.

$$y = \frac{1}{3}\left[\frac{dx}{dt} + 2x - t\right]$$

$$= \frac{1}{3}\left[-5C_1 e^{-5t} + C_2 e^t - \frac{2}{5} + \frac{6}{7}e^{2t} + 2C_1 e^{-5t}\right.$$

$$\left. + 2C_2 e^t - \frac{26}{25} - \frac{4t}{5} + \frac{6e^{2t}}{7} - t\right]$$

Simplifying we get

$$y = -C_1 e^{-5t} + C_2 e^{2t} - \frac{12}{25} - \frac{3t}{5} + \frac{4e^{2t}}{7} \quad \text{... (7)}$$

Example 2.13 : *Solve the simultaneous linear differential equations with given conditions.*

$$\frac{du}{dx} + v = \sin x$$

$$\frac{dv}{dx} + u = \cos x$$

Given that when $x = 0$, $u = 1$ and $v = 0$.

Solution : Step 1 :

Use $D = \frac{d}{dx}$, then equations become

$$Du + v = \sin x \quad \text{... (1)}$$

$$Dv + u = \cos x \quad \text{... (2)}$$

Solving for u using cramers rule

$$\begin{vmatrix} D & 1 \\ 1 & D \end{vmatrix} u = \begin{vmatrix} \sin x & 1 \\ \cos x & D \end{vmatrix}$$

Simplify.

$$(D^2 - 1)x = D \sin x - \cos x$$

Find C.F for u.

$$D^2 - 1 = 0 \quad \text{gives} \quad D = \pm 1$$

$$\therefore \quad \text{C.F.} = C_1 e^x + C_2 e^{-x}$$

Find P.I for u.

Here P.I = 0

$$\therefore \quad \text{G.S. is } u = C_1 e^x + C_2 e^{-x}$$

$$\frac{du}{dx} = C_1 e^x - C_2 e^{-x}$$

Use the equation where coefficient of v is simple.

i.e. $$Du + v = \sin x$$

$$v = \sin x - C_1 e^x + C_2 e^{-x}$$

Applying initial condition at x = 0 we see that u = 1 and v = 0. hence

$$C_1 + C_2 = 1 \quad \text{and} \quad C_2 - C_1 = 0$$

Solving for C_1 and C_2 we obtain

$$C_1 = C_2 = \frac{1}{2}$$

Hence finaily $$u = \frac{1}{2}\left(e^x + e^{-x}\right)$$

i.e. $$u = \cosh x$$

$$v = \sin x - \sinh x$$

Example 2.14 : *Solve simultaneously*

$$\frac{dx}{dt} - 3x - 6y = t^2 \qquad \text{... (1)}$$

$$\frac{dy}{dt} + \frac{dx}{dt} - 3y = e^t \qquad \text{... (2)}$$

Solution : Step 1 :

Use $$D = \frac{d}{dt}$$ hence equations become

$$Dx - 3x - 6y = t^2$$

$$Dy + Dx - 3y = e^t$$

Step 2 : Collect the terms of x and y

$$(D-3)\,x - 6y = t^2 \qquad \text{... (3)}$$

$$Dx + (D-3)\,y = e^t \qquad \text{... (4)}$$

Step 3 : Solving for y.

$$\begin{vmatrix} D-3 & -6 \\ D & D-3 \end{vmatrix} y = \begin{vmatrix} D-3 & t^2 \\ D & e^t \end{vmatrix}$$

Step 4 : Simplify.

$$\left[(D-3)^2 + 6D\right] y = (D-3)\,e^t - D\,t^2$$

$$\left(D^2 + 9\right) y = e^t - 3\,e^t - 2t = -2e^t - 2t$$

Step 5 : Find C.F for y.

$$D^2+9 = 0 \quad \text{gives} \quad D = \pm 3i$$

$$\therefore \quad C.F = C_1 \cos 3x + C_2 \sin 3x$$

Step 6 : Find P.I for y.

$$P.I = \frac{1}{D^2+9} -2e^t - 2t$$

$$= -2\frac{1}{D^2+9}e^t - 2\frac{1}{D^2+9}t$$

$$= -\frac{1}{5}e^t - \frac{2t}{9}$$

Step 7 :

$\therefore$ General solution is

$$y = C_1 \cos 3t + C_2 \sin 3t - \frac{e^t}{5} - \frac{2t}{9}$$

Now as both equations involve derivative for x it is difficult to find value of x using substitution. Hence again solving for x we get,

$$\begin{vmatrix} D-3 & -6 \\ D & D-3 \end{vmatrix} x = \begin{vmatrix} t^2 & -6 \\ e^t & D-3 \end{vmatrix}$$

Step 8 : Simplify.

$$\left[(D-3)^2+6D\right]x = (D-3)t^2 + 6e^t$$

$$(D^2+9)x = 2t - 3t^2 + 6e^t$$

Step 9 : Find C.F for x.

$$D^2+9 = 0 \quad \text{gives} \quad D = \pm 3i$$

$$\therefore \quad C.F. = C_3 \cos 3t + C_4 \sin 3t$$

Step 10 : Find P.I for x.

$$P.I = \frac{1}{D^2+9}\left(2t-3t^2+6e^t\right)$$

$$= \frac{1}{D^2+9}\left(2t-3t^2\right) + \frac{1}{D^2+9}6e^t$$

$$= \frac{1}{9}\frac{1}{\left(1+\frac{D^2}{9}\right)}\left(2t-3t^2\right) + \frac{6}{10}e^t$$

$$= \frac{1}{9}\left\{1-\frac{D^2}{9}\ \ldots\right\}(2t-3t^2)+\frac{3}{5}\ e^t$$

$$= \frac{1}{9}\left\{2t-3t^2-\frac{6}{9}\right\}+\frac{3}{5}\ e^t$$

$$= -\frac{t^2}{3}+\frac{2t}{9}+\frac{2}{27}+\frac{3\,e^t}{5}$$

Step 11 :

$\therefore$ General solution is

$$x = C_3 \cos 3t + C_4 \sin 3t + \frac{3e^t}{5} - \frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27} \qquad \ldots (8)$$

Thus,
$$x = C_3 \cos 3t + C_4 \sin 3t + \frac{3\,e^t}{5} - \frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27}$$

$$y = C_1 \cos 3t + C_2 \sin 3t - \frac{e^t}{5} - \frac{2t}{9}$$

Example 2.15 : $m\,\dfrac{d^2x}{dt^2} + eH\,\dfrac{dy}{dt} = eE \;\; and \;\; m\,\dfrac{d^2y}{dt^2} = eH\,\dfrac{dx}{dt}$

Solution : **Step 1 :** Divide by m

$$\frac{d^2x}{dt^2} + \frac{eH}{m}\frac{dy}{dt} = \frac{eE}{m}, \quad \frac{d^2y}{dt^2} = \frac{eH}{m}\frac{dx}{dt}$$

Let
$$\frac{eH}{m} = b, \quad \text{and} \quad \frac{eE}{m} = a$$

$$\frac{d^2x}{dt^2} + b\,\frac{dy}{dt} = a, \quad \frac{d^2y}{dt^2} = b\,\frac{dx}{dt}$$

Step 2 : Problem involves all derivatives, hence integrating w.r.t. t.

$$\frac{dx}{dt} + by = at + C_2, \quad \frac{dy}{dt} - bx = C_1$$

Step 3 : As initially the particle is at rest at origin

$$x = y = \frac{dx}{dt} = \frac{dy}{dt} = 0$$

Substituting in above equations we get $C_1 = 0,\ C_2 = 0$.

$\therefore$ equations reduce to

$$\frac{dx}{dt} + by = at, \quad \frac{dy}{dt} - bx = 0$$

Step 4 :

$$\text{Let } \frac{d}{dt} = D, \quad Dx + by = at, \quad Dy - bx = 0$$

Step 5 : Solving for x

$$\begin{vmatrix} D & b \\ -b & D \end{vmatrix} x = \begin{vmatrix} at & b \\ 0 & D \end{vmatrix}$$

Step 6 : Simplify.

$$(D^2+b^2)\,x = D\,at - 0$$

$$(D^2+b^2)\,x = a$$

Step 7 : Find C.F for x

$$D^2+b^2 = 0 \text{ gives}$$

$$D^2 = -b^2 \qquad D = \pm\, bi$$

$$\therefore \qquad C.F = C_1 \cos bt + C_2 \sin bt$$

Step 8 : Find P.I, for x

$$P.I = \frac{1}{D^2+b^2}\, a$$

Put $D = 0$

$$\therefore \qquad P.I = \frac{a}{b^2}$$

Step 9 :

$$X = C.F + P.I$$

$$= C_1 \cos bt + C_2 \sin bt + \frac{a}{b^2} \qquad \text{... (1)}$$

Step 10 : Consider the equation where coefficient in y is simple.

i.e. $\qquad Dx + by = at$

$$\therefore \qquad y = \frac{1}{b}\left[at - \frac{dx}{dt}\right]$$

$$\therefore \qquad y = \frac{1}{b}\left[at + b\,C_1 \sin bt - b\,C_2 \cos bt + 0\right] \qquad \text{... (2)}$$

Step 11 : Given that when t = 0

$$x = y = \frac{dx}{dt} = \frac{dy}{dt} = 0$$

Substituting in (1) and (2) we get

$$C_1 + 0 + \frac{a}{b^2} = 0$$

$$C_1 = \frac{-a}{b^2} \text{ and}$$

$$\frac{1}{b}[-bC_2 + 0] = 0$$

$$C_2 = 0$$

Step 12 : Substituting C_1 and C_2 in (1) and (2)

$$x = \frac{-a}{b^2}\cos bt + \frac{a}{b^2}$$

$$= \frac{a}{b^2}(1 - \cos bt)$$

$$y = \frac{1}{b}\left[at - \frac{a}{b}\sin bt\right]$$

$$= \frac{a}{b^2}[bt - \sin bt]$$

Example 2.16 : *Solve*

$$\frac{d^2x}{dt^2} + 4x + 5y = t^2$$

$$\frac{d^2y}{dt^2} + 5x + 4y = t + 1$$

Solution : Step 1 :

Use $D = \frac{d}{dt}$; then given equations are

$$D^2x + 4x + 5y = t^2$$

$$D^2y + 5x + 4y = t + 1$$

Step 2 : Collect the terms of x and y.

$$(D^2 + 4)x + 5y = t^2 \quad \text{... (1)}$$

$$5x + (D^2 + 4)y = t + 1 \quad \text{... (2)}$$

Step 3 : Solving for x.

$$\begin{vmatrix} D^2+4 & 5 \\ 5 & D^2+4 \end{vmatrix} x = \begin{vmatrix} t^2 & 5 \\ t+1 & D^2+4 \end{vmatrix}$$

Step 4 : Simplify.

$$\left[(D^2+4)^2 - 5^2\right]x = (D^2+4)t^2 - 5(t+1)$$

$$(D^4 + 8D^2 - 9)x = 2 + 4t^2 - 5t - 5$$

$$= -3 - 5t + 4t^2$$

Step 5 : Find C.F for x.

$$D^4 + 8D^2 - 9 = 0 \qquad D^4 + 9D^2 - D^2 - 9 = 0$$

$$D^2(D^2+9) - 1(D^2+9) = 0 \qquad (D^2-1)(D^2+9) = 0$$

Giving $\qquad D = \pm 1, \pm 3i$

$$\therefore \quad C.F = C_1 e^t + C_2 e^{-t} + C_3 \cos 3t + C_4 \sin 3t$$

Step 6 : Find P.I for x.

$$\therefore \quad P.I = \frac{1}{D^4 + 8D^2 - 9}\left[-3 - 5t + 4t^2\right]$$

$$= -3\frac{1}{D^4+8D^2-9}(e^{0t}) + \frac{1}{(-9)}\left[1 - \frac{1}{9}(8D^2 + D^4)\right]^{-1}(4t^2 - 5t)$$

$$= -3\frac{1}{0+8(0)-9}(1) - \frac{1}{9}\left[1 + \frac{1}{9}(8D^2 + D^4) + \ldots\right](4t^2 - 5t)$$

$$= \frac{1}{3} - \frac{1}{9}\left[(4t^2 - 5t) + \frac{8}{9}D^2(4t^2 - 5t) + \ldots\right]$$

$$= \frac{1}{3} - \frac{1}{9}\left[4t^2 - 5t + \frac{64}{9}\right]$$

$$= -\frac{1}{9}\left[4t^2 - 5t + \frac{37}{9}\right]$$

Step 7 :

$\therefore$ G.S. for x is

$$x = C_1 e^t + C_2 e^{-t} + C_3 \cos 3t + C_4 \sin 3t - \frac{1}{9}\left[4t^2 - 5t + \frac{37}{9}\right] \qquad \ldots (3)$$

Step 8 : Use the equation where coefficient of y is simple

i.e. $\quad (D^2+4)x + 5y = t^2$

$$y = \frac{1}{5}\left[t^2 - (D^2+4)x\right]$$

Step 9 : To find y substituting for x in (1), we have

$$y = \frac{1}{5}\left\{t^2 - (D^2+4)\left[C_1 e^t + C_2 e^{-t} + C_3 \cos 3t + C_4 \sin 3t - \frac{1}{9}\left(4t^2 - 5t + \frac{37}{9}\right)\right]\right\}$$

$$= \frac{1}{5}\left[t^2 - D^2\left(C_1\ e^t + C_2\ e^{-t} + C_3\ \cos 3t + C_4\ \sin 3t - \frac{4}{9}\ t^2 + \frac{5}{9}\ t - \frac{37}{81}\right)\right.$$

$$\left. - 4\left(C_1\ e^t + C_2\ e^{-t} + C_3\ \cos 3t + C_4\ \sin 3t - \frac{4}{9}\ t^2 + \frac{5}{9}\ t - \frac{37}{81}\right)\right]$$

$$= \frac{1}{5}\left[t^2 - C_1\ e^t - C_2\ e^{-t} + 9\ C_3\ \cos 3t + 9\ C_4\ \sin 3t + \frac{8}{9} - 4\ C_1\ e^t\right.$$

$$\left. - 4\ C_2\ e^{-t} - 4\ C_3\ \cos 3t - 4\ C_4\ \sin 3t + \frac{16}{9}\ t^2 - \frac{20}{9}\ t + \frac{4 \times 37}{81}\right]$$

$$y = -C_1\ e^t - C_2\ e^{-t} + C_3\ \cos 3t + C_4\ \sin 3t + \frac{1}{9}\left(5t^2 - 4t + \frac{44}{9}\right) \quad \text{... (4)}$$

Hence equation (3) and (4) give the required solutions of system of equations.

Example 2.17 : $\frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x = e^{-t}$

$\frac{d^2 y}{dt^2} - 4\ \frac{dx}{dt} + 3y = \sin 2t$

Solution : Step 1 :

Use $D = \frac{d}{dt}$ hence system can be written as,

$$D^2x + Dy + 3x = e^{-t}$$

$$D^2y - 4\ Dx + 3y = \sin\ 2t$$

Step 2 : Collect the terms of x and y.

$$\left(D^2 + 3\right) x + Dy = e^{-t}$$

$$-4\ Dx + \left(D^2 + 3\right) y = \sin\ 2t$$

Step 3 : Solving for x by Crammer's rule.

$$\begin{vmatrix} D^2 + 3 & D \\ -4D & D^2 + 3 \end{vmatrix} x = \begin{vmatrix} e^{-t} & D \\ \sin 2t & D^2 + 3 \end{vmatrix}$$

Step 4 : Simplify.

$$\left[\left(D^2 + 3\right)^2 + 4\ D^2\right] x = 4\ e^{-t} - 2\cos 2t$$

$$\left(D^2 + 1\right)\left(D^2 + 9\right) x = 4\ e^{-t} - 2\cos 2t$$

Step 5 : Find C.F for x.

$$\left(D^2 + 1\right)\left(D^2 + 9\right) = 0 \quad \text{gives} \quad \therefore\ D = \pm\ i,\ \pm\ 3i$$

$\therefore$ C.F. is $\qquad x = C_1\ \cos t + C_2\ \sin t + C_3\ \cos 3t + C_4\ \sin 3t$

Step 6 : Find P.I for x.

$$\text{P.I} = \frac{1}{(D^2+1)(D^2+9)} 4 \cdot e^{-t} - \frac{1}{(D^2+1)(D^2+9)}(2 \cos 2t)$$

$$= \frac{1}{5} e^{-t} + \frac{2}{15} \cos 2t$$

Step 7 :

$\therefore$ General solution for x = C.F + P.I

$$\therefore \quad x = C_1 \cos t + C_2 \sin t + C_3 \cos 3t + C_4 \sin 3t + \frac{1}{5} e^{-t} + \frac{2}{15} \cos 2t$$

Step 8 : And similarly solving for y we get,

$$(D^2+1)(D^2+9)\, y = -\sin 2t - 4\, e^{-t}$$

Auxiliary equation for y is same

$$(D^2+1)(D^2+9) = 0 \quad \therefore D = \pm i, \ \pm 3i$$

Step 9 :

$\therefore$ C.F. is $\quad y = C_5 \cos t + C_6 \sin t + C_7 \cos 3t + C_8 \sin 3t$

Step 10 :

$$\text{P.I. for } y = \frac{1}{(D^2+1)(D^2+9)} (-\sin 2t - 4\, e^{-t})$$

$$= (-1) \frac{1}{(D^2+1)(D^2+9)} \sin 2t - 4 \frac{1}{(D^2+1)(D^2+9)} e^{-t}$$

$$= +\frac{1}{15} \sin 2t - \frac{1}{5} e^{-t}$$

Step 11 :

$\therefore$ General solution for y = C.F. + P.I.

$$\therefore \quad y = C_5 \cos t + C_6 \sin t + C_7 \cos 3t + C_8 \sin 3t + \frac{1}{15} \sin 2t - \frac{1}{5} e^{-t}$$

Substituting these values of x and y in any one of given equations we get,

$c_5 - 2\, c_2, \quad c_6 = -2\, c_1, \quad c_7 = -2\, c_4, \quad c_8 = 2\, c_3$

By comparing the coefficients of functions of t.

Step 12 :

$\therefore$ Required solution for the given system is,

$$x = C_1 \cos t + C_2 \sin t + C_3 \cos 3t + C_4 \sin 3t + \frac{1}{5} e^{-t} + \frac{2}{15} \cos 2t$$

and

$$y = 2C_2 \cos t - 2C_1 \sin t - 2C_4 \cos 3t + 2C_3 \sin 3t - \frac{1}{5} e^{-t} + \frac{1}{15} \sin 2t$$

Example 2.18 : *Solve the system*

$$(D+1)x + (2D+1)y = e^t \qquad \text{... (1)}$$

$$(D-1)x + (D+1)y = 1 \qquad \text{... (2)}$$

Solution : Operator $(D-1)$ on equation (1) and $(D+1)$ on equation (2) and subtract

$$\therefore \quad (D^2 - 3D - 2)\, y = -1$$

Auxiliary equation is $D^2 - 3D - 2 = 0$

$$\therefore \quad D = \frac{3 \pm \sqrt{17}}{2}$$

$$\therefore \quad \text{C.F.} = C_1 e^{\frac{1}{2}(3+\sqrt{17})} + C_2 e^{\frac{1}{2}(3-\sqrt{17})}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D - 2}(-1)\, e^{0t} \quad \text{replace } D \to 0.$$

$$\text{P.I.} = \frac{1}{2}$$

$\therefore$ General solution y = C.F. + P.I.

$$\therefore \quad y = C_1 e^{\frac{1}{2}(3+\sqrt{17})t} + C_2 e^{\frac{1}{2}(3-\sqrt{17})t} + \frac{1}{2}$$

Now from equation (2)

$$(D-1)\, x = 1 - (D+1)\, y$$

$$\therefore \quad x = \frac{1}{D-1}\left[1 - \left\{\frac{C_1}{2}\left(3+\sqrt{17}\right) + C_1\right\} e^{\frac{1}{2}(3+\sqrt{17})t}\right.$$

$$\left. - \left\{\frac{C_2}{2}\left(3+\sqrt{17}\right) + C_2\right\} e^{\frac{1}{2}(3-\sqrt{17})t} - \frac{1}{2}\right]$$

$$\therefore \quad x = \frac{1}{D-1}\frac{1}{2}e^{0t} - \left\{\frac{C_1}{2}\left(3+\sqrt{17}\right)+C_1\right\}\frac{1}{D-1}e^{\frac{1}{2}(3+17)t}$$

$$-\left\{\frac{C_2}{2}\left(3+\sqrt{17}\right)+C_2\right\}\frac{1}{D-1}e^{\left(3-\sqrt{17}\right)t}$$

$$\therefore \quad x = \frac{1}{2}\left[e^t - 1 - \frac{C_1}{2}\left(3+\sqrt{17}\right)e^{\frac{1}{2}\left(3+\sqrt{17}\right)t} + \frac{C_2}{3}\left(3-\sqrt{17}\right)e^{\frac{1}{2}\left(3-\sqrt{17}\right)t}\right]$$

$\therefore$ General solution is given by,

$$x = \frac{1}{2}\left[e^t - 1 - \frac{C_1}{2}\left(3+\sqrt{17}\right)e^{\frac{1}{2}\left(3+\sqrt{17}\right)t} + \frac{C_2}{3}\left(3-\sqrt{17}\right)e^{\frac{1}{2}\left(3-\sqrt{17}\right)t}\right]$$

$$y = C_1 e^{\frac{1}{2}\left(3+\sqrt{17}\right)t} + C_2 e^{\frac{1}{2}\left(3-\sqrt{17}\right)t} + \frac{1}{2}$$

Example 2.19 : *Solve*

$$\frac{d^2x}{dt^2} - \frac{dy}{dt} = 2x + 2t$$

$$\frac{dx}{dt} + 4\frac{dy}{dt} = 3y$$

Solution : Step 1 :

Use $D = \frac{d}{dt}$ system of equation can be written as,

$D^2x - Dy - 2x = 2t$

$Dx + 4D - 3y = 0$

Step 2 :

Collect the terms of x and y

$$\left(D^2 - 2\right)x - Dy = 2t \quad \text{... (1)}$$

$$Dx + (4D-3)\,y = 0 \quad \text{... (2)}$$

Step 3 : Solving for x using cramers rule.

$$\begin{vmatrix} D^2-2 & -D \\ D & 4D-3 \end{vmatrix} x = \begin{vmatrix} 2t & -D \\ 0 & 4D-3 \end{vmatrix}$$

Step 4 : Simplify.

$$\left[\left(D^2-2\right)(4D-3) + D^2\right]x = (4D-3)\,2t$$

Step 5 : Find C.F for x.

$$\left(2D^3 - D^2 - 4D + 3\right) x = 4 - 3t$$

$$2D^3 - D^2 - 4D + 3 = 0$$

$$\therefore \ (D-1)(D-1)\left(D+\frac{3}{2}\right) = 0 \quad \therefore \ D = 1, 1, -\frac{3}{2}$$

$$\therefore \qquad C.F = (C_1 + C_2\, t)\ e^t + C_3\, e^{-\frac{3t}{2}}$$

Step 6 : Find P.I for x.

$$P.I = \frac{1}{2\,D^3 - D^2 - 4D + 3}\,(4 - 3\,t)$$

$$\therefore \qquad P.I = 4\,\frac{1}{2\,D^3 - D^2 - 4\,D + 3}\ e^{0\,t} - 3\,\frac{1}{2\,D^3 - D^2 - 4\,D + 3}\ t$$

replace $D \to 0$

$$= \frac{4}{3} - 3\,\frac{1}{3\left[1 + \left(\dfrac{2\,D^3 - D^2 - 4D}{3}\right)\right]}\ t$$

$$= \frac{4}{3} - \left[1 + \left(\frac{2D^3 - D^2 - 4D}{3}\right)\right]^{-1} t$$

$$= \frac{4}{3} - \left[1 + \frac{4D}{3}\right] t = \frac{4}{3} - t - \frac{4}{3}$$

$$\therefore \qquad P.I = -\,t$$

Step 7 :

$\therefore$ General solution for x = C.F + P.I

$$\therefore \qquad x = (C_1 + C_2\, t)\ e^t + C_3\, e^{\frac{-3}{2}t} - t$$

$$\therefore \qquad \frac{dx}{dt} = e^t\,(C_1 + C_2\, t + C_2) - \frac{3}{2}\, C_3\, e^{-\frac{3}{2}t} - 1$$

Step 8 : Use the equation where coefficient of y is simple.

i.e. $Dx + (4D - 3)\, y = 0$

Step 9 : Thus $\quad y = \dfrac{-1}{4D-3}\,(Dx)$

$\therefore$ Substituting $\dfrac{dx}{dt}$ we get

$$y = \frac{-1}{4D-3}\left[e^t(C_1+C_2+C_2t)-\frac{3}{2}C_3e^{-3t/2}-1\right]$$

$$y = \frac{-1}{4D-3}\ e^t(C_1+C_2+C_2t)+\frac{1}{4D-3}\ \frac{3}{2}C_3e^{-3t/2}+\frac{1}{4D-3}\ 1$$

$$= -e^t\frac{1}{4(D+1)-3}(C_1+C_2+C_2t)+\frac{3}{2}C_3\frac{1}{4\left(\frac{-3}{2}\right)-3}e^{-3t/2}+\frac{1}{0-3}(1)$$

$$= -e^t\frac{1}{4D+1}(C_1+C_2+C_2t)+\frac{3}{2}C_3\frac{1}{(-9)}e^{-3t/2}-\frac{1}{3}$$

$$= -e^t\{1-4D\}(C_1+C_2+C_2t)-\frac{1}{6}C_3\ e^{-3t/2}-\frac{1}{3}$$

$$= -e^t\{C_1+C_2+C_2t-4C_2\}-\frac{1}{6}C_3\ e^{-3t/2}-\frac{1}{3}$$

$$y = -e^t(C_1-3C_2+C_2t)-\frac{1}{6}C_3e^{-3t/2}-\frac{1}{3}$$

$$\therefore \quad \text{G.S. } x = (C_1+C_2\ t)\ e^t+C_3\ e^{-\frac{3t}{2}}-t$$

$$y = -[C_1-3C_2+C_2t]\ e^t-\frac{C_3}{6}\ e^{-3t/2}-\frac{1}{3}$$

Exercise 2.1

1. *Solve* $\frac{du}{dx}+v = \sin x$

   $\frac{dv}{dx}+u = \cos x$ *given that when* $x = 0,\ u = 1,\ v = 0.$

   [Ans. : u = cosh x, v = sin x - sinh x]

2. $\frac{dx}{dt}+5x-2y = t$

   $\frac{dy}{dt}+2x+y = 0$ *given that* $x = 0,\ y = 0$ *at* $t = 0.$ **[Ans. :** $x = -\frac{1}{27}(1+6t)\ e^{-3t}+\frac{1}{27}(1+3t)$

   $y = -\frac{2}{27}(2+3t)\ e^{-3t}+\frac{2}{27}(2-3t)$**]**

3. $\frac{dx}{dt}+y = e^t$

   $-\frac{dy}{dt}+x = e^{-t}$ **[Ans. :** $x = c_1\cos t+c_2\sin t+\frac{1}{2}\left(e^t-e^{-t}\right)$

   $y = c_1\sin t-c_2\cos t+\frac{1}{2}\left(e^t-e^{-t}\right)$**]**

4. *If* $\frac{dx}{dt} - wy = a\cos pt$

$\frac{dy}{dt} + wx = a \sin pt$ *show that*

$x = A\cos wt + B\sin wt + \frac{a}{p+w}\sin pt$

$y = B\cos wt - A\sin wt - \frac{a}{p+w}\cos pt$

5. *The equations of motion of a particle are given by* $\frac{dx}{dt} + wy = 0,\ \frac{dy}{dt} - wx = 0.$

Find the path of the particle. **[Ans. : x = A cos wt + B sin wt**

y = A sin wt – B cos wt]

6. *Solve the simultaneous equations for r and* θ

$\frac{dr}{dt} - 2r - \theta = 0$

$\frac{d\theta}{dt} + r - 4\theta = 0$

Given that $\theta(0) = 0$ *and* $r'(0) = 6$ **[Ans. :** $r = 3(e^{3t} - t\,e^{3t})$

$\theta = -3t\ e^{3t}$**]**

7. *A mechanical system with two degrees 0 freedom satisfies the equations*

$2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4$ *and* $2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0$. *Obtain expressions for x and y in terms of t given that*

$x,\ y\ \frac{dx}{dt}$ *and* $\frac{dy}{dt}$ *all vanish at* $t = 0$. **[Ans. :** $x = -\frac{8}{9}\cos\frac{3}{2}t + \frac{8}{9}$

$y = \frac{4}{3}t - \frac{8}{9}\sin\frac{3t}{2}$

8. $2\frac{dx}{dt} - x + 3y = \sin t$

$2\frac{dy}{dt} + 3y - y = \cos t$

Given $x = \frac{1}{4},\ y = \frac{-1}{20}$ *at* $t = 0$. **[Ans. :** $x = \frac{1}{10}\left(e^{2t} + e^{-t}\right) + \frac{1}{20}(\cos t + 2\sin t)$

$y = \frac{-1}{10}e^{2t} + \frac{e^{-t}}{10} + \frac{2}{5}\sin t - \frac{1}{20}\cos t$**]**

9. *The small oscillations of a certain system are given by*

$D^2x + 3x - 2y = 0$

$D^2x + D^2y - 3x + 5y = 0$

Given $x = 0,\ y = 0,\ \frac{dx}{dt} = 3,\ \frac{dy}{dt} = 2$ *at* $t = 0$. *Find x and y.*

[Ans. : $x = \frac{11}{4}\sin(t) + \frac{1}{12}\sin(3t)$

$y = \frac{11}{4}\sin(t) - \frac{1}{4}\sin(3t)$**]**

Symmetrical form of simultaneous differential equations

General form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

where P, Q, R are functions of x, y, z are said to be symmetrical simultaneous differential equations. The solution of such a system of equations is given by a pair of relations $u(x, y, z) = C_1$ and $v(x, y, z) = C_2$ which are independent of each other. We can solve such a system by following methods.

a) Method of combinations or grouping :

In this method we select two groups such that the third variable gets eliminated i.e. we get the variable separable form, then integration of this gives one part of the solution.

Illustrations on Type 1

➠ **Example 2.20 :** $\frac{dx}{xy} = \frac{dy}{x^2} = \frac{dz}{xyz}$

Solution : By combinations

$$\frac{dx}{xy} = \frac{dy}{x^2}$$

$$\frac{x^2 dx}{x} = y\ dy$$

$$x\ dx = y\ dy$$

which is variable separable integrating we get

$$\frac{x^2}{2} = \frac{y^2}{2} + C$$

i.e. $$x^2 - y^2 = C_1 \quad \text{... (1)}$$

Again by combinations

$$\frac{dx}{xy} = \frac{dz}{xyz}$$

$$\frac{dx}{1} = \frac{dz}{z}$$

which is variable separable

Integrating we get

$$x = \log z + C_2$$

$$x - \log z = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.21 : $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{x^2y^2z^2}$

Solution : By combinations

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating we get

$$-\frac{1}{x} = -\frac{1}{y}+C$$

i.e. $$\frac{1}{y}-\frac{1}{x} = C_1 \qquad \text{... (1)}$$

Again by combinations

$$\frac{dx}{y^2} = \frac{dz}{x^2y^2z^2}$$

$\therefore$ $$x^2\ dx = \frac{dz}{z^2}$$

Integrating we get

$$\frac{x^3}{3} = -\frac{1}{z}+C$$

i.e. $$\frac{x^3}{3}+\frac{1}{z} = C_2 \qquad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.22 : $\frac{dx}{yz^2} = \frac{dy}{xz^2} = \frac{dz}{xy}$

Solution : By combinations

$$\frac{dx}{yz^2} = \frac{dz}{xy}$$

$\therefore$ $$x\ dx = z^2\ dz$$

Integrating we get

$$\frac{x^2}{2}-\frac{z^3}{3} = C_1 \qquad \text{... (1)}$$

Again by combinations

$$\frac{dx}{yz^2} = \frac{dy}{xz^2}$$

$$\therefore \qquad x\,dx = y\,dy$$

Integrating we get

$$\frac{x^2}{2} = \frac{y^2}{2}+C$$

i.e. $$x^2-y^2 = C_2 \qquad \dots (2)$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.23 : $\frac{dx}{y}=\frac{dy}{x}=\frac{dz}{xyz}$

Solution : By Combinations

$$\frac{dx}{y} = \frac{dy}{x}$$

$$\therefore \qquad x\,dx = y\,dy$$

Integrating we get

$$\frac{x^2}{2} = \frac{y^2}{2}+C$$

$$x^2-y^2 = C_1 \qquad \dots (1)$$

Again $$\frac{dx}{y} = \frac{dz}{xyz}$$

$$x\,dx = \frac{dz}{z}$$

Integrating

$$\frac{x^2}{2} = \log z + C_2 \qquad \dots (2)$$

Equations (1) and (2) together constitute the solution of the system.

b) Use of one solution to get the other

Example 2.24 : $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{nxy}$

Solution : By combinations

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating we get

$$\log x = \log y + \log C_1$$

$$\log\frac{x}{y} = \log C_1$$

$$\frac{x}{y} = C_1 \qquad \ldots (1)$$

Again by combinations

$$\frac{dy}{y} = \frac{dz}{nxy}$$

$$x\,dy = \frac{dz}{n}$$

Now from (1) $x = y\,C_1$

$$\therefore \qquad y\,C_1\,dy = \frac{dz}{n}$$

Integrating we get

$$\frac{C_1}{2}y^2 = \frac{z}{n}+C_2$$

Again substitute $C_1 = \dfrac{x}{y}$

$$\therefore \qquad \frac{x}{2y}\cdot y^2-\frac{z}{n} = C_2$$

$$\frac{xy}{2}-\frac{z}{n} = C_2 \qquad \ldots (2)$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.25 : $\dfrac{dx}{2x}=\dfrac{dy}{-y}=\dfrac{dz}{4xy^2-z}$

Solution : By combinations

$$\frac{dx}{2x} = \frac{dy}{-y}$$

Integrating we get

$$\frac{1}{2}\log x+\log y = \log C_1$$

$$\log y\sqrt{x} = \log C_1$$

$$y\sqrt{x} = C_1$$

i.e.

$$xy^2 = C_1 \qquad \ldots (1)$$

Again by combinations

$$\frac{dx}{2x} = \frac{dz}{4xy^2 - z}$$

From (1) $\quad xy^2 = C_1$

$$\therefore \quad \frac{dx}{2x} = \frac{dz}{4C_1 - z}$$

Integrating we get

$$\frac{1}{2}\log x = -\log(4C_1 - z) + \log C_2$$

$$\therefore \quad \log\sqrt{x}(4C_1 - z) = \log C_2$$

$$\sqrt{x}(4C_1 - z) = C_2$$

Again $\quad C_1 = xy^2$

$$\therefore \quad \sqrt{x}(4xy^2 - z) = C_2 \qquad \dots (2)$$

(1) and (2) together constitute the solution of the system.

Example 2.26 : $\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + tan(y-3x)}$

Solution : By combinations

$$\frac{dx}{1} = \frac{dy}{3}$$

$\therefore$ Integrating we get

$$x = \frac{y}{3} + C$$

$$y - 3x = C_1 \qquad \dots (1)$$

Again by combinations

$$\frac{dy}{3} = \frac{dz}{5z + \tan(y-3x)}$$

i.e.

$$\frac{dy}{3} = \frac{dz}{5z + \tan C_1}$$

Integrating we get

$$\frac{y}{3} = \frac{1}{5}\log[5z + \tan C_1] + C$$

$$5y - 3\log[5z + \tan(y-3x)] = C_2 \qquad \dots (2)$$

(1) and (2) together constitute the solution of the system.

Example 2.27 : $\dfrac{dx}{y}=\dfrac{dy}{-x}=\dfrac{dz}{\left(x^2-3y^2\right)y}$

Solution : By combinations

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$x\,dx + y\,dy = 0$$

Integrating we get

$$\frac{x^2}{2}+\frac{y^2}{2} = C$$

$$x^2+y^2 = C_1 \qquad \dots (1)$$

Again by combinations

$$\frac{dx}{y} = \frac{dz}{\left(x^2-3y^2\right)y}$$

using (1) $$y^2 = C_1-x^2$$

$$\frac{dx}{1} = \frac{dz}{\left[x^2-3\left(C_1-x^2\right)\right]}$$

$$\left(x^2-3C_1+3x^2\right)dx = dz$$

$$\left(4x^2-3\,C_1\right)dx = dz$$

Integrating we get

$$4\frac{x^3}{3}-3\,C_1\,x = z+C_2$$

$$\frac{4}{3}x^3-3\left(x^2+y^2\right)x-z = C_2 \qquad \dots (2)$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.28 : $\dfrac{dx}{x}=\dfrac{dy}{y}=\dfrac{dz}{-(x+z)}$

Solution : By combinations

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating we get

$$\log x = \log y+\log C_1$$

$$\therefore \quad x = C_1 y \quad \text{... (1)}$$

Again by combinations

$$\frac{dx}{x} = \frac{dz}{-(x+z)}$$

$$\therefore \quad -(x+z)\,dx = xdz$$

$$-xdx = xdz + zdx$$

$$-xdx = d(xz)$$

$$d(xz) = -x\,dx$$

Integrating we get

$$xz = -\frac{x^2}{2}+C$$

$$xz+\frac{x^2}{2} = C_2 \quad \text{... (2)}$$

Another method

$$\frac{dy}{y} = \frac{dz}{-(x+z)}$$

Using (1) $\quad x = C_1 y$

$$\frac{dy}{y} = \frac{dz}{-(C_1 y+z)}$$

$$(C_1 y+z)\,dy + y\,dz = 0$$

$$C_1 ydy + zdy + ydz = 0$$

$$C_1 y\,dy + d(yz) = 0$$

Integrating we get

$$\frac{C_1 y^2}{2}+yz = C_2$$

Put $C_1 = \frac{x}{y}$

$$\frac{xy}{2}+yz = C_2 \quad \text{... (3)}$$

Any two relations from (1), (2), (3) constitute the solution of the system.

Example 2.29 : $\dfrac{dx}{xy} = \dfrac{dy}{y^2} = \dfrac{dz}{xyz - 2x^2}$

Solution : By combinations

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating we get

$$\log x - \log y = \log C_1$$

$$\frac{x}{y} = C_1 \qquad \dots (1)$$

Again by combinations

$$\frac{dx}{xy} = \frac{dz}{xyz - 2x^2}$$

From (1) $\qquad y = \dfrac{x}{C_1}$

$$\therefore \qquad \frac{dx}{x \cdot \frac{x}{C_1}} = \frac{dz}{x \cdot \frac{x}{C_1} \cdot z - 2x^2}$$

$$C_1 \, dx = \frac{dz}{\left(\frac{z}{C_1} - 2\right)}$$

$$C_1 \, dx = \frac{C_1 \, dz}{z - 2\,C_1}$$

$$\frac{dx}{1} = \frac{dz}{z - 2\,C_1}$$

Integrating we get

$$x = \log(z - 2\,C_1) + C_2$$

$$x - \log\left(z - 2\frac{x}{y}\right) = C_2 \qquad \dots (2)$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.30 : $\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{x e^{x^2+y^2}}$

Solution : By combinations

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$x\,dx + y\,dy = 0$$

Integrating

$$\frac{x^2}{2} + \frac{y^2}{2} = C$$

$$x^2 + y^2 = C_1 \quad \text{... (1)}$$

Again by combination

$$\frac{dy}{-x} = \frac{dz}{x\,e^{x^2+y^2}}$$

$$\frac{dy}{-1} = \frac{dz}{e^{C_1}}$$

$$e^{C_1} \cdot dy = -\,dz$$

Integrating we get

$$y\,e^{C_1} = -z + C_2$$

$$y\,e^{x^2+y^2} + z = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.31 : $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2+y^2+z^2}}$

Solution : By combinations

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating we get

$$\log x = \log y + \log C_1$$

$$x = y\,C_1 \quad \text{... (1)}$$

Again by combinations

$$\frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2+y^2+z^2}}$$

Using (1) $x = y C_1$

$$\frac{dy}{y} = \frac{dz}{z-a\sqrt{y^2C_1^2+y^2+z^2}}$$

$$\frac{dz}{dy} = \frac{z-a\sqrt{(1+C_1^2)\,y^2+z^2}}{y}$$

which is homogeneous $\therefore$ put $z = uy$

$$\therefore \quad \frac{dz}{dy} = u+y\frac{du}{dy}$$

$\therefore$ The equation becomes

$$u+y\frac{du}{dy} = \frac{uy-a\sqrt{(1+C_1^2)\,y^2+u^2y^2}}{y}$$

$$= u-a\sqrt{(1+C_1^2)+u^2}$$

$$\frac{du}{\sqrt{(1+C_1^2)+u^2}} = \frac{a\,dy}{-y}$$

which is variable separable

$\therefore$ Integrating we get

$$\log\left[u+\sqrt{(1+C_1^2)+u^2}\right] = -a\log y+\log C_2$$

$$u+\sqrt{(1+C_1^2)+u^2} = \frac{C_2}{y^a}$$

Put $u = \frac{z}{y}$ and $C_1 = \frac{x}{y}$

$$\frac{z}{y}+\sqrt{1+\left(\frac{x}{y}\right)^2+\left(\frac{z}{y}\right)^2} = \frac{C_2}{y^a}$$

$$\frac{z+\sqrt{x^2+y^2+z^2}}{y} = \frac{C_2}{y^a}$$

$$z+\sqrt{x^2+y^2+z^2} = C_2\,y^{1-a}$$

c) Use of Multipliers

We know the property of ratio and proportion

$$\text{if } \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l\,dx+m\,dy+n\,dz}{lP+mQ+nR}$$

$$=\frac{l_1dx+m_1dy+n_1dz}{l_1P+m_1Q+n_1R}$$

where (l, m, n) and (l_1, m_1, n_1) are the multipliers constants or variables.

We choose l, m, n and l_1, m_1, n_1 in such a way that

$lP + mQ + nR = 0$ and $l_1P + m_1Q + n_1R = 0$ then we have

$l dx + m dy + n dz = 0$ and $l_1 dx + m_1 dy + n_1 dz = 0$

integrating we get

$$u(x, y, z) = C_1 \qquad \text{... (1)}$$

and
$$v(x, y, z) = C_2 \qquad \text{... (2)}$$

(1) and (2) together constitute the solution of the system.

Note : Standard sets of multipliers (l, m, n), (lx, my, nz) (x, y, z) $\left(\frac{l}{x}, \frac{m}{y}, \frac{n}{z}\right)$, (lx^2, my^2, nz^2), $\left(\frac{l}{x^2}, \frac{m}{y^2}, \frac{n}{z^2}\right)$ (lx^3, my^3, nz^3) and so on where l, m, n are any constants.

Example 2.32 : $\dfrac{dx}{mz-ny} = \dfrac{dy}{nx-lz} = \dfrac{dz}{ly-mx}$

Solution : Use multipliers (l, m, n)

$$\therefore \quad \text{each ratio} = \frac{l dx + m dy + n dz}{lmz - lny + mnx - lmz + lny - mnx}$$

$$= \frac{l dx + m dy + n dz}{0}$$

$$\Rightarrow \quad l dx + m dy + n dz = 0$$

Integrating we get

$$lx + my + nz = C_1 \qquad \text{... (1)}$$

Again use multipliers (x, y, z)

$$\therefore \quad \text{each ratio} = \frac{x dx + y dy + z dz}{mzx - nxy + nxy + lzy - lyz - mxz}$$

$$= \frac{x dx + y dy + z dz}{0}$$

$$\Rightarrow \quad x dx + y dy + z dz = 0$$

Integrating we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C$$

$$x^2 + y^2 + z^2 = C_2 \qquad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.33 : $\dfrac{dx}{x(2y^4-z^4)}=\dfrac{dy}{y(z^4-zx^4)}=\dfrac{dz}{z(x^4-y^4)}$

Solution :

Use multipliers $\left(\frac{1}{x},\frac{1}{y},\frac{2}{z}\right)$

$$\therefore \quad \text{each ratio} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{2}{z}dz}{2y^4-z^4+z^4-2x^4+2x^4-2y^4}$$

$$= \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{2dz}{z}}{0}$$

$$\Rightarrow \quad \frac{dx}{x}+\frac{dy}{y}+\frac{2}{z}dz = 0$$

Integrating we get

$$\log x+\log y+2\log z = \log C_1$$

$$\log xyz^2 = \log C_1$$

$$xyz^2 = C_1 \qquad \text{... (1)}$$

Again use multipliers x^3, y^3, z^3

$$\therefore \quad \text{each ratio} = \frac{x^3dx+y^3dy+z^3dz}{x^4(2y^4-z^4)+y^4(z^4-2x^4)+z^4(x^4-y^4)}$$

$$= \frac{x^3dx+y^3dy+z^3dz}{0}$$

$$\Rightarrow \quad x^3dx+y^3dy+z^3dz = 0$$

Integrating we get

$$\frac{x^4}{4}+\frac{y^4}{4}+\frac{z^4}{4} = C$$

$$x^4+y^4+z^4 = C_2 \qquad \text{... (2)}$$

(1) and (2) constitute the solution of the system.

Example 2.34 : $\dfrac{dx}{x(y^2-z^2)}=\dfrac{dy}{-y(z^2+x^2)}=\dfrac{dz}{z(x^2+y^2)}$

Solution :

Use multipliers $\left(\frac{-1}{x},\frac{1}{y},\frac{1}{z}\right)$

$$\therefore \qquad \text{each ratio} = \frac{-\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{-y^2+z^2-z^2-x^2+x^2+y^2}$$

$$= \frac{-\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\Rightarrow \quad \frac{-dx}{x}+\frac{dy}{y}+\frac{dz}{z} = 0$$

Integrating we get

$$-\log x+\log y+\log z = \log C_1$$

$$\frac{yz}{x} = C_1 \qquad \dots (1)$$

Again use multipliers (x, y, z)

$$\therefore \qquad \text{each ratio} = \frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow xdx + ydy + zdz = 0$$

Integrating we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = C$$

$$x^2+y^2+z^2 = C_2 \qquad \dots (2)$$

(1) and (2) together constitute the solution of the system.

Example 2.35 : $\dfrac{x\,dx}{z^2-2yz-y^2} = \dfrac{dy}{y+z} = \dfrac{dz}{y-z}$

Solution : By combinations

$$\frac{dy}{y+z} = \frac{dz}{y-z}$$

$$ydy - zdy = ydz + zdz$$

$$ydz + z\,dy = ydy - zdz$$

$$d\,(yz) = ydy - zdz$$

Integrating

$$yz = \frac{y^2}{2} - \frac{z^2}{2} + C$$

$$2yz - y^2 + z^2 = C_1 \qquad \dots (1)$$

Use multipliers (1, y, z)

$$\text{each ratio} = \frac{xdx + ydy + zdz}{0}$$

$$\Rightarrow xdx + ydy + zdz = 0$$

Integrating we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C$$

$$x^2 + y^2 + z^2 = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Use of property of ratio and proportion :

Example 2.36 : $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{(1 + 2xy + 3x^2y^2)(x+y)z}$

Solution : By combination

$$\frac{dx}{1} = \frac{dy}{1}$$

Integrating we get

$$x - y = C_1 \quad \text{... (1)}$$

Also from first two groups

$$\text{each ratio} = \frac{ydx + xdy}{y + x}$$

Equating with the third group

$$\frac{ydx + xdy}{x + y} = \frac{dz}{(1 + 2xy + 3x^2y^2)(x+y)\,z}$$

$$(1 + 2xy + 3x^2y^2)\, d\,(xy) = \frac{dz}{z}$$

Put xy = u

$$(1 + 2u + 3u^2)\, du = \frac{dz}{z}$$

Integrating we get

$$u + u^2 + u^3 = \log z + C_2$$

$$xy + x^2y^2 + x^3y^3 - \log z = C_2 \quad \text{... (2)}$$

(1) and (2) together constitute the solution of the system.

Example 2.37 : $\dfrac{dx}{y+z}=\dfrac{dy}{x+z}=\dfrac{dz}{x+y}$

Solution :

$$\text{each ratio} = \frac{dx+dy+dz}{2(x+y+z)} \quad \text{... (A)}$$

Also $$\text{each ratio} = \frac{dx-dy}{y-x} \text{ i.e. } \frac{dx-dy}{-(x-y)} \quad \text{... (B)}$$

Again $$\text{each ratio} = \frac{dy-dz}{-(y-z)} \quad \text{... (C)}$$

$$\text{each ratio} = \frac{dx-dz}{-(x-z)} \quad \text{... (D)}$$

(A) = (B) gives

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{-(x-y)}$$

Integrating we get

$$\frac{1}{2}\log(x+y+z)+\log(x-y) = \log C_1$$

$$(x-y)\sqrt{x+y+z} = C_1 \quad \text{... (1)}$$

(C) = (D) gives

$$\frac{dy-dz}{y-z} = \frac{dx-dz}{x-z}$$

Integrating we get

$$\log(y-z) = \log(x-z)+\log C_2$$

$$\frac{y-z}{x-z} = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.38 : $\dfrac{dx}{x^2-yz}=\dfrac{dy}{y^2-zx}=\dfrac{dz}{z^2-xy}$

Solution :

$$\text{each ratio} = \frac{dx-dy}{(x+y+z)(x-y)} \quad \text{... (A)}$$

$$= \frac{dy - dz}{(x+y+z)(y-z)} \quad \text{... (B)}$$

$$= \frac{dx - dz}{(x+y+z)(x-z)} \quad \text{... (C)}$$

$$= \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx} \quad \text{... (D)}$$

$$= \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz} \quad \text{... (E)}$$

(A) = (B) gives

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$$

Integrating we get

$$\log(x - y) - \log(y - z) = \log C_1$$

$$\frac{x - y}{y - z} = C_1 \quad \text{... (1)}$$

(D) = (E) gives

$$\frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx} = \frac{xdx + ydy + zdz}{(x^2 + y^2 + z^2 - xy - yz - zx)(x + y + z)}$$

As $\quad x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)$

Thus $\quad (x+y+z)\cdot[dx + dy + dz] = xdx + ydy + zdz$

Integrating we get

$$\frac{(x+y+z)^2}{2} - \frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = C$$

i.e. $\quad (x+y+z)^2 - x^2 - y^2 - z^2 = C_2$

i.e. $\quad x^2 + y^2 + z^2 + 2xy + 2yz + 2zx - x^2 - y^2 - z^2 = C_2$

$$2(xy + yz + zx) = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.39 : $\dfrac{dx}{x^2 + y^2 - yz} = \dfrac{dy}{xz - x^2 - y^2} = \dfrac{dz}{xz - yz}$

Solution : Using multipliers (1, 1, – 1)

$$\text{each ratio} = \frac{dx+dy-dz}{0}$$

$$\Rightarrow \quad dx + dy - dz = 0$$

Integrating we get

$$x + y - z = C_1 \qquad \dots (1)$$

$$\text{Again each ratio} = \frac{xdx+ydy}{x^3+xy^2-x^2y-y^3}$$

$$= \frac{xdx+ydy}{x(x^2+y^2)-y(x^2+y^2)}$$

$$= \frac{xdx+ydy}{(x-y)(x^2+y^2)}$$

Equating with the third

$$\frac{dz}{z(x-y)} = \frac{xdx+ydy}{(x-y)(x^2+y^2)}$$

Integrating we get

$$\log z = \frac{1}{2}\log(x^2+y^2)+\log C_2$$

$$\log\frac{z}{\sqrt{x^2+y^2}} = \log C_2$$

$$\frac{z}{\sqrt{x^2+y^2}} = C_2 \qquad \dots (2)$$

Equation (1) and (2) together constitute the solution of the system.

Example 2.40 : $\dfrac{dx}{y^3x-2x^4} = \dfrac{dy}{2y^4-x^3y} = \dfrac{dz}{9z(x^3-y^3)}$

Solution :

$$\frac{dx}{x(y^3-2x^3)} = \frac{dy}{y(2y^3-x^3)} = \frac{dz}{9z(x^3-y^3)}$$

Use multipliers $\left(\frac{1}{x},\frac{1}{y},\frac{1}{3z}\right)$

$$\text{each ratio} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{3z}}{0}$$

$$\Rightarrow \quad \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{3z} = 0$$

Integrating we get

$$\log x+\log y+\frac{1}{3}\log z = \log C$$

$$\log x\cdot y\cdot z^{1/3} = \log C$$

$$x\,y\,z^{1/3} = C$$

$$x^3y^3z = C_1 \quad \text{... (1)}$$

$$\text{each ratio} = \frac{x^2dx+y^2dy}{-2x^6+2y^6}$$

equate with the third group

$$\frac{dz}{9z(x^3-y^3)} = \frac{x^2dx+y^2dy}{-2(x^3-y^3)(x^3+y^3)}$$

$$\frac{-2}{3}\frac{dz}{z} = \frac{3x^2dx+3y^2dy}{x^3+y^3}$$

Integrating we get

$$-\frac{2}{3}\log z = \log(x^3+y^3)+\log C$$

$$\log(x^3+y^3)+\frac{2}{3}\log z = -\log C$$

$$3\log(x^3+y^3)+2\log z = \log C_2$$

$$\log(x^3+y^3)^3\cdot z^2 = \log C_2$$

$$(x^3+y^3)^3\cdot z^2 = C_2 \quad \text{... (2)}$$

(1) and (2) together constitute the solution of the system.

Example 2.41 : $\frac{dx}{x^2+y^2}=\frac{dy}{2xy}=\frac{dz}{z(x+y)^3}$

Solution :

$$\text{each ratio} = \frac{dx+dy}{x^2+y^2+2xy}$$

$$= \frac{dx+dy}{(x+y)^2} \quad \text{... (A)}$$

Also each ratio $= \dfrac{dx-dy}{(x-y)^2}$ ... (B)

From (A) and (B)

$$\frac{dx+dy}{(x+y)^2} = \frac{dx-dy}{(x-y)^2}$$

Let x + y = u, x − y = v

dx + dy = du, dx − dy = dv

$$\therefore \quad \frac{du}{u^2} = \frac{dv}{v^2}$$

Integrating

$$\frac{-1}{u} = \frac{-1}{v}+C$$

i.e. $$\frac{-1}{x+y}+\frac{1}{x-y} = C_1$$

Also $$\frac{dz}{z(x+y)^3} = \frac{dx+dy}{(x+y)^2}$$

i.e. $$\frac{dz}{z\,u^3} = \frac{du}{u^2}$$

$$\frac{dz}{z} = u\,du$$

Integrating

$$\log z = \frac{u^2}{2}+C_2$$

$$\log z-\frac{(x+y)^2}{2} = C_2 \quad \text{... (2)}$$

(1) and (2) together constitute the solution of the system.

Exercise 2.2

Solve the following

1. $\dfrac{dx}{yz^2} = \dfrac{dy}{xz^2} = \dfrac{dz}{xy^2}$ **[Ans. : $x^2 - y^2 = C_1$, $y^3 - z^3 = C_2$]**

2. $\dfrac{dx}{y^2z} = \dfrac{dy}{zx^2} = \dfrac{dz}{xy^2}$ **[Ans. : $x^3 - y^3 = C_1$, $x^2 - z^2 = C_2$]**

3. $\frac{dx}{xz} = \frac{dy}{yz} = \frac{2dz}{(x+y)^2}$ **[Ans. : $y = C_1x, (x+y)^2 - 2z^2 = C_2$]**

4. $\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{2x-3y}$ **[Ans. : $x^2 + y^2 = C_1, 3x + 2y + z = C_2$]**

5. $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$ **[Ans. : $x^2 - y^2 - 2xy = C_1, x^2 - y^2 - z^2 = C_2$]**

6. $\frac{x^2dx}{y^3} = \frac{y^2dy}{x^3} = \frac{dz}{z}$ **[Ans. : $x^6 - y^6 = C_1, x^3 + y^3 = C_2z^3$]**

7. $\frac{adx}{(b-c)yz} = \frac{bdy}{(c-a)xz} = \frac{cdz}{(a-b)xy}$ **[Ans. : $ax^2 + by^2 + cz^2 = C_1, a^2x^2 + b^2y^2 + c^2z^2 = C_2$]**

8. $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$ **[Ans. : $y = C_1z, x^2 + y^2 + z^2 = y\,C_2$]**

9. $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{(x+y)\left[e^{xy} + \sin xy + x^2y^2\right]}$ **[Ans. : $x - y = C_1, 3e^{xy} - 3\cos xy + (xy)^3 - 3z = C_2$]**

10. $\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$ **[Ans. : $x + y + z = C_1, x^2 + y^2 + z^2 = C_2$]**

11. $\frac{dx}{3z-4y} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x}$ **[Ans. : $2x + 3y + 4z = C_1, x^2 + y^2 + z^2 = C_2$]**

12. $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$ **[Ans. : $x + y + z = C_1, xyz = C_2$]**

13. $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{(x^2+y^2)\sqrt{1-x^2y^2}}$ **[Ans. : $x^2 - y^2 = C_1, z^2 + \sin^{-1}xy + xy\sqrt{1-x^2y^2} = C_2$]**

14. $\frac{dx}{x^3+3xy^2} = \frac{dy}{y^3+3x^2y} = \frac{dz}{2(x^2+y^2)z}$ **[Ans. : $z^2 = C_1y, z = C_2(x^2 - y^2)$]**

15. $\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{z^2(x+y)^2}$ **[Ans. : $\frac{1}{x+y} - \frac{1}{x-y} = C_1, x + y + \frac{1}{z} = C_2$]**

16. $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{(x+y)(z^2+1)}$ **[Ans. : $x^2 - y^2 = C_1, 2(x+y) = \log(z^2+1) + C_2$]**

17. $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$ **[Ans. : $y^2 - 2xy - x^2 = C_1, x^2 - y^2 - z^2 = C_2$]**

18. $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2-y^2)}$ **[Ans. : $x^2 - y^2 = C_1, (x^2 - y^2)x^2 + \frac{2}{z} = C_2$]**

University Questions

Dec. - 98

1. *Solve the simultaneous linear differential equation*

$$\frac{du}{dx} + v = \sin x, \quad \frac{dv}{dx} + u = \cos x \quad \text{provided at } x = 0,\ u = 1,\ v = 0.$$

[6 Marks]

May - 99

1. *The acceleration component of a particle moving in a plane are given by :*

$$\frac{d^2 x}{dt^2} = b\frac{dy}{dt}$$

$$\frac{d^2 y}{dt^2} = a - b\frac{dx}{dt}$$

where a and b are constant. If the particle is initially at rest at the origin then show that the path of the particle is the cycloid. **[6 Marks]**

Dec. - 99

1. *Solve :* $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} + \dfrac{dz}{2xz}$ **[4 Marks]**

2. *Solve :*

$$L\frac{dx}{dt} + Rx + R(x - y) = E$$

$$L\frac{dy}{dt} + Ry - R(x - y) = 0$$

Given that $x = y = 0$ *at* $t = 0$. **[7 Marks]**

May - 2000

1. *Solve the simultaneous equations :*

$$\frac{d^2x}{dt^2} = b\frac{dy}{dt} \ :$$

$$\frac{d^2 y}{dt^2} = a - b\frac{dx}{dt},$$

where, $x = 0,\ y = 0,\ \dfrac{dx}{dt} = 0,\ \dfrac{dy}{dt} = 0$ *at* $t = 0$. **[6 Marks]**

Dec. - 2000

1. *Solve the simultaneous equations :*

$$\left.\begin{aligned} &\frac{dx}{dt} + \frac{dy}{dt} + 5x + 7y = 2 \\ &2\frac{dx}{dt} + 3\frac{dy}{dt} + x + y = \sin t \end{aligned}\right\}$$

under the conditions $x = 0,\ y = 0$ *at* $t = 0$. **[8 Marks]**

2. *Solve*

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{(1 + 2xy + 3x^2y^2)(x + y)z}$$

[2 Marks]

May - 2001

1. *A mechanical system with two degrees of freedom satisfies the equations :*

$$2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4\ ;$$

$$2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0$$

Obtain the expressions for x, y in terms of 't'. Given x, y, $\frac{dx}{dt}, \frac{dy}{dt}$ *are all zero at* $t = 0$. **[8 Marks]**

Dec. - 2001

1. *The small oscillations of a certain system with two degrees of freedom are given by two simultaneous equations*

$$D^2x + 3x - 2y = 0$$

$$D^2x + D^2y - 3x + 5y = 0$$

If $x = y = 0$ *and* $\frac{dx}{dt} = 3,\ \frac{dy}{dt} = 2,$ *when* $t = 0$, *find x and y when* $t = \frac{\pi}{2}$. **[7 Marks]**

2. *Solve :* $xy''' + 3y'' = e^x$

$$xy''' + 3y'' = e^x$$ **[5 Marks]**

May - 2002

1. *Solve the following simultaneous linear differential equations :*

$$\frac{dx}{dt} - 2x - y = 0, \quad \frac{dy}{dt} + x - 4y = 0$$

where $y(0) = 0,\ x'(0) = 6.$ **[6 Marks]**

Dec. - 2002

1. *Solve the following simultaneous linear differential equation :*

$$(D+2)x + (D+1)y = t$$

$$5x + (D+3)y = t^2$$ **[5 Marks]**

2. *Solve the following symmetrical simultaneous equation :*

$$\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z(x^3 - y^3)}$$ **[5 Marks]**

May - 2003

1. *Solve :*

$$4\frac{dx}{dt} = y - x = 2\frac{dy}{dt}.$$

Given that :

$x = 20,\ y = 100$ *at* $t = 0.$ **[5 Marks]**

2. *Solve :*

i) $$\frac{dx}{x^2 + y^2} = \frac{dy}{2xy} = \frac{dz}{z(x+y)^3}$$

ii) $$\frac{dx}{mz - ny} = \frac{dy}{nx - mz} = \frac{dz}{ny - nx}$$ **[8 Marks]**

Dec. - 2003

1. *Solve* **[5 Marks]**

   i) $\frac{dx}{dt} + 2x - 3y = t$

   ii) $\frac{dy}{dt} - 3x + 2y = e^{2t}$

2. *Solve* **[8 Marks]**

   i) $\frac{dx}{3z - 4y} = \frac{dy}{4x - 2z} = \frac{dz}{2y - 3x}$

   ii) $\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$

May - 2004

1. *Solve* **[5 Marks]**

   $\frac{dx}{dt} + y = \sin t$

   $\frac{dy}{dt} + x = \cos t$

   $x = 2,\ y = 0$ *at* $t = 0$.

2. *Solve* **[5 Marks]**

   $\frac{dx}{x\left(y^2 - z^2\right)} = \frac{dy}{y\left(z^2 - x^2\right)} = \frac{dz}{z\left(x^2 - y^2\right)}$

Dec. - 2004

1. *Solve (any one) :* **[4 Marks]**

   i) $\frac{dx}{xz} = \frac{dy}{yz} = \frac{2dz}{(x + y)^2}$

   ii) $\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{4xy^2 - 2z}$

2. *The small oscillations of a certain system with two degrees of freedom are given by* **[6 Marks]**

   $D^2x + 3x - 2y = 0$

   $D^2x + D^2y - 3x + 5y = 0$

   If $x = y = 0$ *and* $Dx = 3,\ Dy = 2$ *when* $t = 0$, *find* x *and* y *when* $t = \frac{1}{2}$.

May - 2005

1. *Solve the simultaneous equations :* **[4 Marks]**

   $\frac{dx}{dt} - 2x - y = 0$

   $\frac{dy}{dt} + x - 4y = 0$

2. *Solve :* **[4 Marks]**

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

Dec. - 2005

1. *Solve the system of equations :*

$$L\frac{dx}{dt} + Rx + R(x - y) = E$$

$$L\frac{dy}{dt} + Ry - R(x - y) = 0$$

[5 Marks]

2. *Solve the equation :*

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

[5 Marks]

May - 2006

1. *Solve :*

$$\frac{dx}{dt} - wy = a\cos pt,$$

$$\frac{dy}{dt} + wx = a\sin pt,\ w \neq p$$

[5 Marks]

2. *Solve :*

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y}$$

[5 Marks]

3 Applications of Differential Equations

3.1 Introduction

The linear differential equation plays an important role for different theories of electrical and mechanical systems simply by renaming the variables. This mechanical-electrical analogy has an important practical applications. As the Electrical circuits are easy to assemble, less expensive and more accuracy in measure, complicated mechanical systems can be studied approximately by constructing equivalent Electrical models.

3.2 Vibration of Spring (or an Elastic String)

Type 1 Free Oscillations

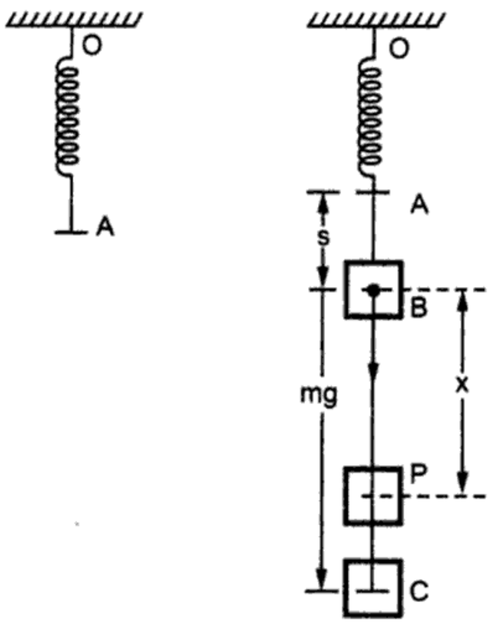


Fig. 3.1

Step 1 : Consider a spring OA suspended vertically form a fixed support at 'O'. Let a body of mass 'm' be attached to the lower end at A of the spring. The mass of the body being so large as compared to mass of the spring, that the later may be neglected.

Step 2 : If the body comes to rest at the point B then point B is called the position of **static equilibrium** and the distance **AB = s (static extension).**

$\therefore$ By Hook's law, we have

mg = ks (k is the spring constant)

Step 3 : Let mass 'm' be pulled down to a point 'C' such that BC = l and released slowly. The mass m will start moving up and down with the point B as the center of motion.

Let P be the position of body at any time 't' such that BP = x.

Take the point B as origin.

The forces acting on the body are

i) It's weight 'mg' acting vertically downward.

ii) the spring force 'k (s + x) (where k is the spring constant)

acting vertically upward to restore the position of the body to the point B.

Step 4 : Hence the equation of motion is

$$m\frac{d^2x}{dt^2} = mg - k(s+x)$$

or

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad (\because mg = ks) \quad \dots (1)$$

Step 5 : Let $\frac{k}{m} = \omega^2$, then equation (1) can be written as

$$\frac{d^2x}{dt^2} = -\omega^2 x \quad \dots (2)$$

Step 6 : The equation (2) represents SHM with period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$= 2\pi\sqrt{\frac{s}{g}} \qquad \left(\because k = \frac{mg}{s}\right)$$

$$= 2\pi\sqrt{\frac{\text{Static extension}}{\text{Acceleration due to Gravity}}}$$

Remark :

The model of vibrating mass spring considered in this section is not very realistic, because the amplitude of vibration does not decrease as one would expect. In actual practice the amplitude of vibration decreases due to frictional forces and air resistance and finally system comes to rest.

A more realistic model could be where **damping force** is considered. This damping force depends upon many variable factors, Hence an exact law to determine this force is not easy.

However, it has been established experimentally, that for small speeds this force is proportional to instantaneous speed of the mass moving with the spring.

Speed of the mass moving with the spring.

Thus magnitude of damping force can be taken as "$\lambda\frac{dx}{dt}$", where 'λ' is damping constant.

3.3 Type 2 Damped Oscillations

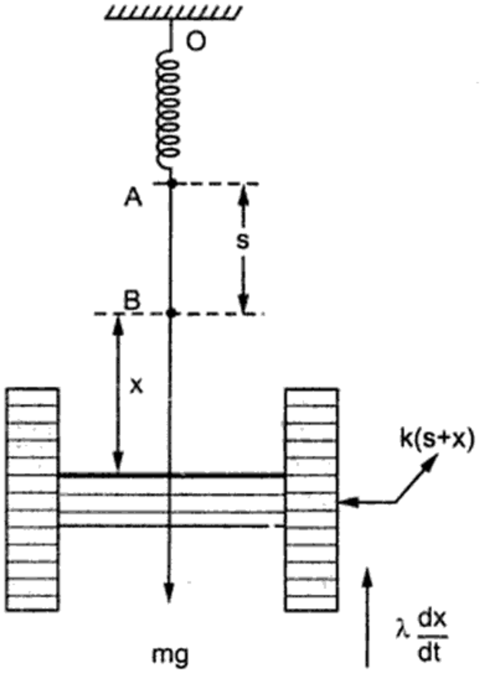


Fig. 3.2

Step 1 : Consider the vibrations of mass, as in the previous case, along with a damping force, which is proportional to instantaneous velocity of the mass. The forces acting on mass are (Refer Fig. 3.2)

i) It's weight 'mg' acting vertically downward.

ii) The spring force k (s + x), acting vertically upward.

iii) The damping force '$\lambda \frac{dx}{dt}$' opposing the motion.

Step 2 : Hence the equation of motion is

$$m\frac{d^2x}{dt^2} = mg - k(s+x) - \lambda\frac{dx}{dt}$$

or $$\frac{d^2x}{dt^2} = -\frac{k}{m}x - \frac{\lambda}{m}\frac{dx}{dt} \qquad (\because mg = ks)$$

or $$\frac{d^2x}{dt^2} + \frac{\lambda}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Step 3 : Let $\frac{\lambda}{m} = 2p$ and $\frac{k}{m} = \omega^2$

$\therefore$ The above equation can be written as

$$\frac{d^2x}{dt^2} + 2p\frac{dx}{dt} + \omega^2 x = 0$$

Step 4 : which is Linear differential equation with constant coefficient.

i.e. $$\left(D^2 + 2pD + \omega^2\right)x = 0 \qquad \ldots (1)$$

A.E. is $$D^2 + 2pD + \omega^2 = 0$$

$$\Rightarrow \qquad D = -p \pm \sqrt{p^2 - \omega^2}$$

Step 5 : The solution of (1) depends upon the relative values of p and ω.

$\therefore$ Consider the following cases.

Case (I) when $p > \omega$

The roots are real and distinct.

Let the roots be α_1 and α_2

$\therefore$ The solution is

$$x = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} \quad \text{... (2)}$$

To determine the constants C_1 and C_2, let the mass be pulled down to the position C, below the static equilibrium position, and then released.

Let BC = l, so that initially,

when $t = 0$, $x = l$

and $\frac{dx}{dt} = v = 0$

From (2), we get

$$C_1 + C_2 = l \quad \text{... (3)}$$

Differentiating (ii) w.r.t 't'.

$$\frac{dx}{dt} = C_1 \alpha_1 e^{\alpha_1 t} + C_2 \alpha_2 e^{\alpha_2 t}$$

Now $t = 0$, $\frac{dx}{dt} = 0$

$$\therefore \quad 0 = C_1 \alpha_1 + C_2 \alpha_2 \quad \text{... (4)}$$

From (3) and (4)

$$C_1 = -\frac{l\alpha_2}{\alpha_1 - \alpha_2}, \; C_2 = \frac{l\alpha_1}{\alpha_1 - \alpha_2}$$

$$\therefore \quad x = \frac{l}{\alpha_1 - \alpha_2}\left(\alpha_1 e^{\alpha_2 t} - \alpha_2 e^{\alpha_1 t}\right) \quad \text{... (5)}$$

The equation (5) shows that x is always positive and hence the motion is **non-oscillatory.**

Further as $t \to \infty$, x decreases and $\to 0$.

The restoring force in this case is too large.

The non-oscillatory motion in this case is called **over damped motion.**

Case (II) when $p = \omega$

The roots of auxilliary equation are real and equal, each being equal to – P.

$\therefore$ The solution is

$$x = (C_1 + C_2 t) e^{-pt} \quad \text{... (6)}$$

Taking the same conditions as in Case I, we get

$$\boxed{C_1 = l} \qquad \boxed{C_2 = pl}$$

From (6) $$x = l(1+pt)\, e^{-pt} \quad \text{... (7)}$$

The equation (7) shows that x is always positive and the motion is **non-oscillatory.**

Further as $t \to \infty$, x decreases to 0 the motion of mass m in this case is called **Critically Damped** or **Dead Beat Motion.**

Case (III) when $p < \omega$

The roots of Auxilliary equation are complex i.e. $D = -P \pm i\alpha$ where $\alpha = \sqrt{\omega^2 - p^2}$

Hence the solution is

$$x = e^{-pt}(C_1 \cos\alpha t + C_2 \sin\alpha t) \quad \text{... (8)}$$

with the same conditions in previous case, we get

$$C_1 = l, \quad C_2 = \frac{pl}{\alpha} \qquad \text{From (8)}$$

$$x = l\, e^{-pt}\left(\cos\alpha t + \frac{p}{\alpha}\sin\alpha t\right)$$

Let $1 = r\cos\theta$, $\frac{p}{\alpha} = r\sin\theta$

$$\therefore \quad x = l\, e^{-pt}[r\cos\theta\cos\alpha t + r\sin\theta\sin\alpha t]$$

$$\therefore \quad x = l\, e^{-pt}\, r\cos[at - \theta]$$

where $$r = \sqrt{1+\frac{p^2}{\alpha^2}}$$

$$\theta = \tan^{-1}\left(\frac{p}{\alpha}\right)$$

or $$x = l\sqrt{1+\frac{p^2}{\alpha^2}}\; e^{-pt}\cos\left[\alpha t - \tan^{-1}\left(\frac{p}{\alpha}\right)\right] \quad \text{... (9)}$$

From (9), it is observed that motion is oscillatory due to the presence of the factor $\cos\left[\alpha t - \tan^{-1}\left(\frac{p}{\alpha}\right)\right]$

The amplitude of motion is $l\sqrt{1+\frac{p^2}{\alpha^2}} \cdot e^{-pt}$

which decreases to zero as $t \to \infty$ and system comes to rest.

The time period 'T' of two motion is

$$T = \frac{2\pi}{\alpha}$$

Now $\alpha = \sqrt{\omega^2 - p^2}$

The time T_1 of force motion is $\frac{2\pi}{\omega}$ Hence $T > T_1$

Thus a mass moving under the effect of damping force increases the time period of oscillations, thus reducing the speed of motion. The motion of this kind is called **oscillatory motion with damping.**

The results of the above three types of motion can be summarized by the following graph :

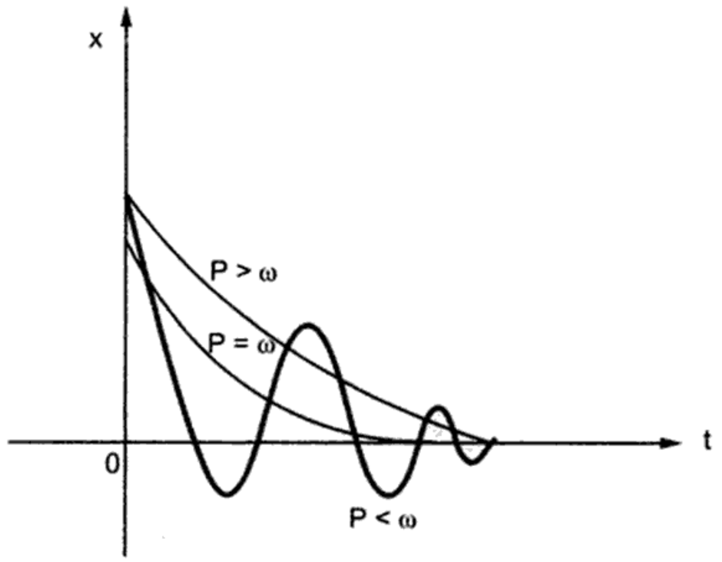


Fig. 3.3

3.4 Type 3 Forced vibrations (free from damping)

The motion of a Mass-spring system under the action of an external periodic force is called **forced vibration.**

Step 1 : Let us consider the vibrating system as in Fig. 3.3 with the modification that the support itself vibrates under the action of an external periodic force **"m F cos nt".** The forces acting on the mass are

i) It's weight 'mg' acting vertically downward.

ii) the spring force "k (s + x)", acting vertically upwards.

iii) the external force **"mF cos nt"** acting vertically downward.

Step 2 : The equation of motion is

$$m\frac{d^2x}{dt^2} = mg - k(s+x) + m\,F\cos nt$$

or $$\frac{d^2x}{dt^2} = -\frac{k}{m}x + F\cos nt \qquad (\because mg = ks)$$

or $$\frac{d^2x}{dt^2} + \omega^2 x = F\cos nt \qquad \text{... (1)}$$

(where $\frac{k}{m} = \omega^2$)

Step 3 : The equation (1) is linear differential equation with constant coefficients A.E. is $D^2 + \omega^2 = 0 \Rightarrow D = \pm i\omega$

$\therefore$ The complementary function is

$$x = C_1 \cos\omega t + C_2 \sin\omega t$$

or $$x = r\cos(\omega t - \phi)$$

where $C_1 = r\cos\phi$, $C_2 = r\sin\phi$

Step 4 : $$\text{P.I.} = \frac{1}{D^2+\omega^2} F\cos nt$$

which depends upon the relative values of ω and n as under :

Case (I) when $\omega \neq n$

$$\text{P.I.} = F\frac{1}{\omega^2 - n^2}\cos nt$$

$\therefore$ The general solution is

$$x = \text{C.F.} + \text{P.I.}$$

$$= r\cos(\omega t - \phi) + \frac{F}{\omega^2 - n^2}\cos nt \qquad \text{... (2)}$$

The equation (2) shows that the motion of mass consists of two oscillatory motions, one due to free oscillations and other due to forced vibration. The periods of free and forced oscillations are $\frac{2\pi}{\omega}$ and $\frac{2\pi}{n}$ respectively.

Whereas the respective amplitudes are r and $\frac{F}{\omega^2 - n^2}$.

It is also observed that if frequency free vibration is very high. (i.e. ω is too large) the amplitude $\frac{F}{\omega^2 - n^2}$, of the forced vibration will be too small.

Case (II) when $\omega = n$, then

$$\text{P.I.} = F\frac{1}{D^2+\omega^2}\cos\omega t \qquad (\because \omega = n)$$

$$= F\cdot t\frac{1}{2D}\cos\omega t \qquad \textbf{(Case of failure)}$$

$$= \frac{F\,t}{2\omega}\sin nt \qquad \left(\because \frac{1}{D}\cos\omega t = \int \cos\omega t\,dt\right)$$

Hence general solution of the equation (1) is

$$x = C_1\cos\omega t + C_2\sin\omega t + \frac{F\,t}{2\omega}\sin\omega t$$

$$= C_1\cos\omega t + \left(C_2 + \frac{F\,t}{2\omega}\right)\sin\omega t \qquad \dots (3)$$

Let $C_1 = r\cos\phi,\ C_2 + \frac{F\,t}{2\omega} = r\sin\phi$

where $$r = \sqrt{C_1^2 + \left(C_2 + \frac{F\,t}{2\omega}\right)^2}$$

and $$\phi = \tan^{-1}\left(\frac{C_2 + \frac{Ft}{2\omega}}{C_1}\right)$$

$\therefore$ The equation (3) may be written as

$$x = r\cos(\omega t - \phi) \qquad \dots (4)$$

The equation (4) gives oscillatory motion in this case, with amplitude

$$r = \sqrt{C_1^2 + \left(C_2 + \frac{Ft}{2\omega}\right)^2}$$

and time period $T = \frac{2\pi}{\omega}$

The amplitude contains t, the time and it is obvious that amplitude increases with time. Hence after a long time the amplitude becomes abnormally large and may case of rupture of system. This phenomenon in which applied frequency due to external periodic force becomes equal to the frequency of free oscillations is known as **Resonance.**

Note : When designing a mechanical system, the phenomenon of resonance is avoided, so that the system does not fall apart.

3.5 Type 4 Forced vibrations with damping

Step 1 : Consider the vibration of the mass as in the previous case, along with a damping force, which is proportional to the instantaneous velocity of the mass. The forces acting on the mass are

i) It's weight 'mg' acting downward.

ii) the spring force 'k (s + x)' acting vertically upward.

iii) the external force 'm F cos nt' acting vertically downward.

iv) the damping force '$\lambda \frac{dx}{dt}$' opposing the motion.

Step 2 : The equation of motion is

$$m\frac{d^2x}{dt^2} = mg - k(s+x) + m F \cos nt - \lambda \frac{dx}{dt}$$

or $$\frac{d^2x}{dt^2} + \frac{\lambda}{m}\frac{dx}{dt} + \frac{k}{m}x = F \cos nt \qquad (\because mg = ks) \quad \dots (1)$$

Step 3 : Taking $\frac{\lambda}{m} = 2p$ and $\frac{k}{m} = \omega^2$ the equation (1) reduces to

$$\frac{d^2x}{dt^2} + 2p\frac{dx}{dt} + \omega^2 x = F \cos nt \qquad \dots (2)$$

The equation (2) is linear differential equation with constant coefficient.

Step 4 : Hence A.E. is $D^2 + 2pD + \omega^2 = 0$

or $$D = \frac{-2p \pm \sqrt{4p^2 - 4\omega^2}}{2}$$

$$= -p \pm \sqrt{p^2 - \omega^2}$$

$$= -p \pm \alpha \qquad (\text{where } \alpha = \sqrt{p^2 - \omega^2})$$

$\therefore$ C.F. is $$x = e^{-pt}\left(C_1 e^{\alpha t} + C_2 e^{-\alpha t}\right)$$

Step 5 : Now, $$\text{P.I.} = \frac{1}{D^2 + 2pD + \omega^2} F \cos nt$$

$$= F \cdot \frac{1}{-n^2 + 2pD + \omega^2} \cos nt$$

$$\text{P.I.} = F \cdot \frac{1}{(\omega^2 - n^2) + 2pD} \cos nt$$

$$= F \frac{(\omega^2 - n^2) - 2pD}{(\omega^2 - n^2)^2 - 4p^2D^2} \cos nt$$

$$= F \cdot \frac{(\omega^2 - n^2)\cos nt + 2pn \sin nt}{(\omega^2 - n^2)^2 + 4p^2n^2}$$

Let $\omega^2 - n^2 = r \cos \phi$, and $2pn = r \sin \phi$

So that $$r = \sqrt{(\omega^2 - n^2)^2 + 4p^2n^2}$$

and $$\phi = \tan^{-1}\left(\frac{2pn}{\omega^2 - n^2}\right)$$

Thus $$\text{P.I.} = \frac{F\,r\cos(nt-\phi)}{r^2}$$

$$= F\cdot\frac{\cos\left\{nt - \tan^{-1}\left(\frac{2pn}{\omega^2 - n^2}\right)\right\}}{\sqrt{\left(\omega^2 - n^2\right)^2 + 4p^2n^2}}$$

Step 6 : Thus the complete solution is

$$x = e^{-pt}\left\{C_1 e^{\alpha t} + C_2 e^{-\alpha t}\right\} + \frac{F\cdot\cos\left\{nt - \tan^{-1}\left(\frac{2pn}{\omega^2 - n^2}\right)\right\}}{\sqrt{\left(\omega^2 - n^2\right)^2 + 4p^2n^2}}$$

Thus the motion consists of two oscillations, first term represents free oscillations, which due away after a long time as $t \to \infty$.

The second term gives forced oscillations of amplitude.

$$\frac{F}{\sqrt{\left(\omega^2 - n^2\right)^2 + 4p^2n^2}}$$

and time period $\frac{2\pi}{n}$, which is same as that of the applied force and we get a steady state motion.

3.6 Illustrated Examples

Example 3.1 : *A body weighing 4.9 kg, is hung from a spring. A pull of 10 kg will stretch the spring to 5 cm. The body is pulled down 6 cm below the static equilibrium position and then released. Find the displacement of the body from it's equilibrium position at time 't' seconds, the maximum velocity and the period of oscillation.*

Solution : Step 1 : Let 'O' be the fixed end and A the free end of the spring. A force of 10 kg stretches the spring by 0.05 meters and if 'k' is the spring constant then

$$10 = k \times 0.05$$

$$\Rightarrow \quad k = 200 \text{ kg/metre}$$

Step 2 : If B denotes the equilibrium position of the body, when it is hung from the end A, then

$$4.9 = k\cdot AB = 200\ AB$$

$$\therefore \quad AB = \frac{4.9}{200} = 0.0245 \text{ meters}$$

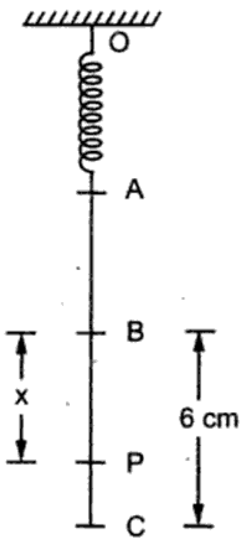


Fig. 3.4

Step 3 : The body is pulled down to the position C, where BC = 0.06 meters. If 't' seconds after release from 'C', the body be at the position 'P', where BP = x, then the forces acting on the body are

i) it's weight 4.9 kg, vertically downward

ii) the spring force k (AB + x) vertically upward.

Step 4 : The equation of motion is

$$\frac{4.9}{g}\frac{d^2x}{dt^2} = 4.9 - k(AB + x)$$

or $$\frac{4.9}{g}\frac{d^2x}{dt^2} = 4.9 - 200\,(0.0245 + x)$$

$$= -200\,x$$

Step 5 : Taking $g = 9.8 \text{ m/sec}^2$

The above equation can be written as

$$\frac{4.9}{9.8}\frac{d^2x}{dt^2} = -200\,x$$

or $$\frac{d^2x}{dt^2} = -400\,x$$

or $$\frac{d^2x}{dt^2} + 400\,x = 0 \qquad \text{... (1)}$$

Step 6 : The equation (1) is LDE with constant coefficient.

i.e. $$\left(D^2 + 400\right)x = 0$$

A.E. is $$D^2 + 400 = 0$$

$$\Rightarrow \quad D^2 = -400 \Rightarrow D = \pm 20\,i$$

Step 7 : Thus the solution of (1) is

$$x = C_1 \cos 20\,t + C_2 \sin 20\,t \qquad \text{... (2)}$$

Step 8 : Differentiating (2) w.r.t 't', we get

$$\frac{dx}{dt} = -20\,C_1 \sin 20\,t + 20\,C_2 \cos 20\,t \qquad \text{... (3)}$$

Step 9 : Initially, when $t = 0$, $x = 0.06$

From (2), we get $C_1 = 0.06$

Also initially, when $t = 0$, $\frac{dx}{dt} = 0$

$\therefore$ From (3) we get

$$C_2 = 0$$

Step 10 : Substituting the values of C_1 and C_2 in (2), we get

$$x = 0.06 \cos 20\, t$$

which gives the displacement of the body from the position of static equilibrium at any time 't'.

Step 11 : Time period $= \frac{2\pi}{20} = \frac{\pi}{10}$ seconds

Step 12 : Maximum velocity $= \omega \times$ (amplitude)

$$= 20 \times 0.06$$

$$= 1.2 \text{ meters/sec}$$

Example 3.2 : *A mass of 200 gm is tied at the end of a spring which extends to 4 cm under a force of 196000 dynes. This spring is pulled by 5 cm and released. Find the displacement, t seconds after release; if there be a damping force of 2000 dynes per cm per second. What would be the damping force for the dead beat motion ?*

Solution : Step 1 : The force 196000 dynes $= \frac{196,000}{980} = 200$ gm wt.

A force 200 gm wt stretches the spring by 4 cm and if 'k' is the spring constant the

$$196,000 = k \cdot 4$$

or

$$k = 49,000 \text{ gm/cm}$$

Step 2 : If B denotes the equilibrium position of the mass after mass of 200 gm is hung from the free end A of the spring OA (O is fixed end), then

$$mg = k \cdot AB$$

$$200 \times 980 = k \cdot AB$$

$$\therefore \quad AB = 4 \text{ cm} \qquad (\because k = 49000 \text{ gm/sec})$$

Step 3 : The mass is pulled down to position C, where BC = 5 cm. If 't' seconds after the release from the position C the mass be at the position P, where BP = x.

Then the forces acting on the mass are :

i) it's weight 200 gms acting vertically downward.

ii) the spring force k (AB + x) acting vertically upwards.

iii) the damping force 2000 $\frac{dx}{dt}$ gm opposing the motion.

Step 4 : The equation of motion is

$$m\frac{d^2x}{dt^2} = mg - k(AB+x) - \lambda\frac{dx}{dt}$$

Substituting m = 200, k = 49000, AB = 4, λ = 2000 we get

$$200\frac{d^2x}{dt^2} = 200\times 980 - 49000\,(4+x) - 2000\,\frac{dx}{dt} \qquad (\because AB = 4 \text{ cm})$$

Dividing by 200

$$\therefore \quad \frac{d^2x}{dt^2} = -245x - 10\,\frac{dx}{dt}$$

$$\text{or} \quad \frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 245x = 0$$

$$\text{or} \quad \left(D^2 + 10D + 245\right)x = 0 \qquad \left(\frac{d}{dt} = D\right) \quad \text{... (1)}$$

Step 5 : which is a differential equation with constant coefficients.

A.E. is $D^2 + 10\,D + 245 = 0$

$$\therefore \quad D = \frac{-10 \pm \sqrt{100-980}}{2}$$

$$D = -5 \pm i\sqrt{220}$$

Step 6 : The solution of (1) is

$$x = e^{-5t}\left(C_1 \cos\sqrt{220}\,t + C_2 \sin\sqrt{220}\,t\right) \quad \text{... (2)}$$

Step 7 : Differentiating equation (2) w.r.t

$$\frac{dx}{dt} = e^{-5t}\left\{-\sqrt{220}\,C_1 \sin\sqrt{220}\,t + \sqrt{220}\,C_2 \cos\sqrt{220}\,t\right\}$$

$$-5\,e^{-5t}\left\{\left(C_1 \cos\sqrt{220}\,t + C_2 \sin\sqrt{220}\,t\right)\right\} \quad \text{... (3)}$$

Step 8 : Initially t = 0, x = 5

From (2), we get

$$\boxed{C_1 = 5}$$

Also, at $t = 0$, $\frac{dx}{dt} = 0$ Substituting in (3)

$$0 = \sqrt{220}\, C_2 - 5\, C_1$$

or

$$\boxed{C_2 = \frac{5\, C_1}{\sqrt{220}} = \frac{5 \times 5}{\sqrt{220}} = \frac{25}{\sqrt{220}}}$$

Step 9 : Substituting the values of C_1 and C_2 in equation (2), we get

$$x = e^{-5t}\left\{5 \cos\sqrt{220}\, t + \frac{25}{\sqrt{220}} \sin\sqrt{220}\, t\right\}$$

Step 10 : Let the damping force for the dead beat motion be λ dynes per second. Replacing $2000\frac{dx}{dt}$ by $\lambda\frac{dx}{dt}$ in the equation of motion, we get

$$200\frac{d^2x}{dt^2} = -49{,}000\, x - \lambda\frac{dx}{dt}$$

or $$200\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + 49{,}000\, x = 0$$

or $$\left(200\, D^2 + \lambda D + 49{,}000\right) x = 0$$

A.E. is $$200\, D^2 + \lambda D + 49{,}000 = 0$$

For dead beat motion the roots of A.E. must be equal which gives $\left(b^2 = 4ac\right)$

i.e. $$\lambda^2 = 4 \times 200 \times 49{,}000$$

$\therefore$ $$\lambda = 2800\sqrt{5} \text{ dynes}$$

Examples 3.3 : *A spring which stretches by an amount e under a force mk^2e is suspended from a support p and has a mass m at it's lower end. At time t = 0, the mass is at rest in it's equilibrium position at a point A below P. A vertical oscillation is now given to the support P such that at any time (t > 0) it's displacement below it's initial position is **'a sin nt'**. Show that the displacement c of the mass below A is given by*

$$\frac{d^2x}{dt^2} + k^2x = k^2a \sin nt$$

Hence show that if $n \neq k$, the displacement is given by

$$x = \frac{ka}{k^2 - n^2}(k \sin nt - n \sin kt)$$

what happens if $n = k$?

Solution : Step 1 : If 'λ' is spring constant, then

$$mk^2e = \lambda e$$

or $$\lambda = mk^2$$

Step 2 : If the mass 'm' produces a stretch 'l' in the given spring them

$$mg = \lambda l$$

Step 3 : Let the mass be at the position B at any time ($t > 0$) such that $AB = x$, This includes the displacement **'a sin nt'** given to the support

Thus at point B, the stretch in the spring is

$(l + x - a \sin nt)$

The forces acting on the mass are

i) it's weight 'mg' acting vertically downward.

ii) the spring force $\lambda(l + x - a \sin nt)$, acting vertically upwards

Step 4 : The equation of motion is

$$m\frac{d^2x}{dt^2} = mg - \lambda(l + x - a \sin nt)$$

$$= -\lambda x + \lambda a \sin nt$$

Step 5 : Substituting the value of 'λ' and simplifying, we get

$$\frac{d^2x}{dt^2} + k^2x = k^2 a \sin nt \qquad \ldots (1)$$

which is the required equation of the motion.

Step 6 : The equation (1) is linear differential equation with constant co-efficient.

Hence A.E. is

$$D^2 + k^2 = 0 \quad \text{or} \quad D = \pm ki$$

Step 7 : Complementary function of (1) is

$$x = C_1 \cos kt + C_2 \sin kt$$

Step 8 : and $$\text{P.I.} = \frac{1}{D^2 + k^2} k^2 a \sin nt$$

Case (I) when $n \neq k$

$$\text{P.I.} = k^2 a \frac{1}{k^2 - n^2} \sin nt$$

Hence the complete solution is

$$x = \text{C.F.} + \text{P.I.}$$

$$x = C_1 \cos kt + C_2 \sin kt + \frac{k^2 a}{k^2 - n^2} \sin nt \qquad \text{... (2)}$$

Initially when $t = 0$, $x = 0$

$$\therefore \quad \boxed{C_1 = 0}$$

Differentiating (2) w.r.t. 't', we get

$$\frac{dx}{dt} = -k C_1 \sin kt + k C_2 \cos kt + \frac{k^2 an}{k^2 - n^2} \cos nt$$

Initially, when $t = 0$, $v = 0$

$$0 = k C_2 + \frac{k^2 an}{k^2 - n^2}$$

or

$$\boxed{C_2 = -\frac{k\,an}{k^2 - n^2}}$$

Substituting the values of C_1 and C_2 in equation (2), we get

$$x = \frac{k^2 a}{k^2 - n^2} \sin nt - \frac{k\,an}{k^2 - n^2} \sin kt$$

or

$$x = \frac{k\,a}{k^2 - n^2} (k \sin nt - n \sin kt)$$

Case (II) : when $n = k$

$$\text{P.I.} = \frac{1}{D^2 + k^2} k^2 a \sin nt$$

$$= n^2 a \frac{1}{D^2 + n^2} \sin nt \qquad (\because n = k)$$

$$= n^2 at \frac{1}{2D} \sin nt$$

$$\text{P.I.} = -\frac{n^2 at}{2n} \cos nt$$

$$= -\frac{n\,at}{2} \cos nt$$

Hence, the solution of equation (1) is

$$x = C_1 \cos nt + C_2 \sin nt - \frac{n\,at}{2} \cos nt \qquad \text{... (3)}$$

Initially, when $t = 0$, $x = 0$

$$\therefore \quad \boxed{C_1 = 0}$$

Differentiating (3) w.r.t. 't'

$$\frac{dx}{dt} = v = -n C_1 \sin nt + n C_2 \cos nt - \frac{na}{2}(\cos nt - nt \sin nt)$$

Initially, when t = 0, v = 0

$$\therefore \quad \boxed{C_2 = \frac{a}{2}}$$

From (3)

$$x = \frac{a}{2}\sin nt - \frac{n\,at}{2}\cos nt$$

$$= \frac{a}{2}(\sin nt - nt \cos nt)$$

Let $1 = r\cos\phi$, $nt = r\sin\phi$ and $\phi = \tan^{-1}(nt)$

$$\therefore \quad x = \frac{ar}{2}\sin(nt-\phi) \qquad \text{where } r = \sqrt{1+n^2t^2}$$

Example 3.4 : *A spring for which the spring constant k = 700 Nm^{-1} hangs in a vertical position with it's upper end fixed to a support. A mass of 28 kg is attached to the lower end and the system brought to rest. Find the position of the mass at time 't' of a force 70sin2t N is applied to the support.*

Solution : Step 1 : Since k is given, the equation of motion can be written as

$$28\frac{d^2x}{dt^2} = -700\,x + 70\sin 2t \qquad \text{... (1)}$$

where 'x' is the distance of the mass at time 't', from the static equilibrium position.

Step 2 : The equation (1) reduces to

$$\frac{d^2x}{dt^2} + 25\,x = \frac{5}{2}\sin 2t$$

or $$(D^2+25)\,x = \frac{5}{2}\sin 2t$$

which is LDE with constant coefficient

Step 3 :

A.E is $$D^2 + 25 = 0$$

$$\Rightarrow \quad D = \pm 5i$$

C.F. is $$\text{C.F.} = C_1\cos 5t + C_2 \sin 5t$$

Step 4 : Consider

$$\text{P.I.} = \frac{1}{D^2+25}\,\frac{5}{2}\sin 2t$$

$$= \frac{5}{2}\,\frac{1}{D^2+25}\sin 2t$$

$$= \frac{5}{2}\,\frac{1}{-4+25}\sin 2t$$

$$= \frac{5}{42}\sin 2t$$

Step 5 : Hence the solution of (1) is

$$x = \text{C.F.} + \text{P.I.}$$

$$= C_1\cos 5t + C_2\sin 5t + \frac{5}{42}\sin 2t \quad \text{... (2)}$$

Initially, $t = 0$, $x = 0$

$$\therefore \quad \boxed{C_1 = 0}$$

Step 6 : Differentiating (2), we get

$$\frac{dx}{dt} = -5C_1\sin 5t + 5C_2\cos 5t + \frac{5}{21}\cos 2t$$

Initially, when $t = 0$, $v = 0$, which gives

$$0 = 5C_2 + \frac{5}{21}$$

$$\Rightarrow \quad \boxed{C_2 = -\frac{1}{21}}$$

Step 7 : $\therefore$ $x = \frac{1}{42}(5\sin 2t - 2\sin 5t)$ meters

which is displacement of mass at any time 't'.

Example 3.5 : *The differential equation* $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + n^2x = 0$; $(k < n)$ *represents the damped harmonic oscillations of a particle. Solve this equation and show that the ratio of the amplitude of any oscillation to that of the preceding one is constant. i.e. it's amplitude form G.P.*

Solution : Step 1 : The given DE is

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + n^2x = 0$$

$$\text{or}\left(D^2 + 2kD + n^2\right)x = 0 \quad \text{... (1)}$$

is LDE with constant co-efficients

Step 2 : Hence A.E. is

$$D^2 + 2kD + n^2 = 0$$

or
$$D = \frac{-2k \pm \sqrt{4k^2 - 4n^2}}{2}$$

$$= -k \pm i\omega \qquad \text{where } \omega = \sqrt{n^2 - k^2}$$

Step 3 : $\therefore$ The solution of (1) is

$$x = e^{-kt}(C_1 \cos\omega t + C_2 \sin\omega t) \qquad \dots (2)$$

Let $C_1 = r\cos\phi$, $C_2 = r\sin\phi$

Step 4 : Equation (2) can be written as

$$x = r\, e^{-kt}\cos(\omega t - \phi)$$

$$= \sqrt{C_1^2 + C_2^2}\; e^{-kt}\cos\left(\omega t - \tan^{-1}\left(\frac{C_2}{C_1}\right)\right)$$

Step 5 : which shows the motion is oscillatory with time period $T = \frac{2\pi}{\omega}$

Step 6 : The amplitude a of the motion is given by

$$a = \sqrt{C_1^2 + C_2^2}\; e^{-kt}$$

Step 7 : The particle will be at the extreme positions in the same direction at the times.

Thus, $\frac{2\pi}{\omega}, \frac{4\pi}{\omega}, \frac{6\pi}{\omega} \dots$

Let the amplitudes corresponding to these time be $a_1, a_2, a_3 \dots$

when $t = \frac{2\pi}{\omega}$, $a_1 = \sqrt{C_1^2 + C_2^2}\; e^{-\frac{2\pi k}{\omega}}$

when $t = \frac{4\pi}{\omega}$, $a_2 = \sqrt{C_1^2 + C_2^2}\; e^{-\frac{4\pi k}{\omega}}$

and at $t = \frac{6\pi}{\omega}$, $a_3 = \sqrt{C_1^2 + C_2^2}\; e^{-\frac{6\pi k}{\omega}}$

and so on .....

Step 8 : The ratio of the successive amplitudes $a_1, a_2, a_3 \dots$ is constant and equal to $e^{-\frac{2\pi k}{\omega}}$. Hence the successive amplitudes form G.P.

Example 3.6 : *A spring stretches 1 cm under the tension of 2 kgs and has a negligible weight. It is fixed at one end is attached to a weight W kgs at the other. It is found that resonance occures when an axial periodic force '2 cos 2t' kgs acts on the weight show that when the free vibrations have died out, the forced vibrations are given $x = ct \sin 2t$ and find the values of W and C.*

Solution : A weight of 2 kgs stretches the spring by 1/100 meters.

$$\therefore \qquad 2 = T = k\,\frac{1}{100}$$

$$\Rightarrow \qquad k = 200 \text{ kg/met}$$

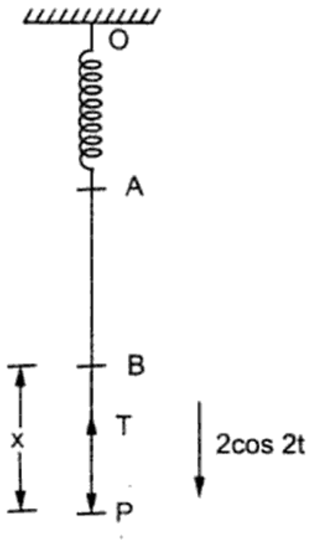


Fig. 3.5

Let B be the equilibrium position of the weight W attached to A, then

$$W = T_B = k \cdot AB$$

$$= 200 \cdot AB$$

$$\Rightarrow \qquad AB = \frac{W}{200} \text{ meters.}$$

At any time t, let the weight be at P where BP = x,

$$\text{Tension T at P} = k \cdot AP$$

$$= 200\left(\frac{W}{200} + x\right)$$

$$T = W + 200\,x$$

$\therefore$ The equation of motion is

$$\frac{W}{g}\,\frac{d^2x}{dt^2} = -T + W + 2\cos 2t$$

Substituting T = W + 200 x

or
$$\frac{W}{g}\,\frac{d^2x}{dt^2} = -W - 200\,x + W + 2\cos 2t$$

Multiply both sides by g.

$$\Rightarrow \qquad W\frac{d^2x}{dt^2} + 200\,gx = 2g\cos 2t \qquad \text{... (1)}$$

The phenomenon of resonance will occur when the frequency of free oscillations is equal to the frequency of forced oscillations. If we write equation (1) as

$$\frac{d^2x}{dt^2} + \omega^2 x = \frac{2g}{W}\cos 2t$$

where $\omega^2 = \dfrac{200\,g}{W}$, the period of free oscillations is found to $\dfrac{2\pi}{\omega}$ and the period of the forced oscillations is π .

Hence $\frac{2\pi}{\omega} = \pi$

or $\frac{200\text{ g}}{W} = \omega^2 = 4$

$\therefore$ $W = 50\text{ g}$

$\therefore$ $\frac{g}{W} = \frac{1}{50}$

Taking this value of W in equation (1), we get

$$\frac{d^2x}{dt^2} + 4x = \frac{1}{25}\cos 2t \quad \text{... (2)}$$

Now, free oscillations are given by C.F. and forced oscillations by the P.I. Hence when the free oscillations have died out, the forced oscillations are give by the P.I. of (2).

Now, $\text{P.I.} = \frac{1}{25}\frac{1}{D^2+4}\cos 2t$

$= \frac{1}{100} t \sin 2t$

Hence $C = \frac{1}{100}$

Exercise 3.1

1. *A spring is such that it would be stretched 72 mm by a mass of 15 kg. A mass of 30 kg is attached and brought to rest. The resistance of medium is numerically equal to $20\frac{dx}{dt}$ Newtons. Find the equation of the motion of the weight if it is pulled down 140 mm and given an upward velocity 3 m/sec.* **[Ans. : $x = e^{-\frac{1}{2}t}(0.14\cos 8t - 0.366\sin 8t)$]**

2. *A spring is stretched 77 mm by a mass of 1.5 kg. This mass is attached to the free end of the spring and brought to rest. The mass is then pulled down 4 cm and released. Determine the displacement of the mass below the equilibrium position at any time 't', if a periodic for '7 sin 6t' acts on the spring.* **[Ans. : $0.04\cos 8t - \frac{1}{8}\sin 8t + \frac{1}{6}\sin 6t$]**

3. *Solve the differential equation $\frac{d^2x}{dt^2} + 2h\frac{dx}{dt} + \left(h^2+p^2\right)x = ke^{-ht}\cos pt$ completely. Show that the particular integral represents an oscillation of variable amplitude which maximum when ht = 1.*

   [Ans. : $x = A\,e^{-ht}\cos\left(pt+B\right) + \frac{k}{2p} + e^{-ht}\sin pt$]

4. *A mass M suspended from the end of a hellical spring is subjected to a periodic force f = F cos wt in the direction of it's length. The force f is measured positive vertically downwards and at zero time M is at rest. If the spring stiffness is S prove that the displacement of M at time t from the*

commencement of motion is given by $x = \dfrac{F}{M(p^2 - w^2)}(\cos wt - \cos pt)$ *where* $p^2 = \dfrac{S}{M}$ *and damping effects are neglected.*

5. *A spring is loaded with 9.8 kg weight, it's point of support has a motion given by sin wt and the resistance to the motion of the load is ten times the instantaneous speed. If the tension of the spring be 50 times the stretch and it's mass be neglected. Show that the motion is given by* $\dfrac{d^2x}{dt^2} + 10\dfrac{dx}{dt} + 50x = 50 \sin wt$. *Find the amplitude of steady motion when* $w^2 = 50$.

6. *A body executes damped forced vibrations given by the equation* $\dfrac{d^2x}{dt^{-2}} + 2k\dfrac{dx}{dt} + b^2x = e^{-kt}\sin wt$ *solve the differential equation for both cases when* $w^2 \neq b^2 - k^2$ *and when* $w^2 = b^2 - k^2$.

[Ans. : If $w^2 \neq b^2 - k^2$, then

i) $x = A\,e^{-kt}\cos(pt + B) + \dfrac{e^{-kt}}{p^2 - w^2}\sin wt$ where $p = \sqrt{b^2 - k^2}$

ii) If $w^2 = b^2 - k^2$ then $x = A\,e^{-kt}\cos(wt + b) - \dfrac{e^{-kt} + \cos wt}{2w}$]

7. *A 3 kgs weight stretches a certain spring 15 cm. The weight is pulled 10 cms below the equilibrium position and released. A damping force in pounds equal to 5.75 v, where v being instantaneous velocity in m/sec is working on the spring. Find x as function of time 't'.*

[Ans. : $x = 0.148\,e^{-(4.615)t} - (0.048)\,e^{-(14.65)t}$]

8. *A spring is fixed at upper end supports a weight of 980 gms. at it's lower end. The spring stretched* $\frac{1}{2}$ *cm under a load of 10 gm and resistance (in gm wt) to the motion of the weight in numerically equal to* $\frac{1}{10}$ *of the speed of the weight in cm/sec. The weight is pulled down* $\frac{1}{4}$ *cm. below it's equilibrium position and then released. Find the expression for the distance of weight from it's equilibrium position at time 't' during it's first upward motion. Also find the time it takes the damping factors to drop* $\frac{1}{10}$ *of it's initial value.*

[Ans. : $x = e^{-(0.05)t}\{0.25\cos(4.5)t + 0.003\sin(4.5)t\}$ $t = 46$ sec

9. *A body weighing 20 kg is hung from a spring. A pull of 40 kg wt will stretch the spring to 10 cm. The body is pulled down to 20 cm. below the static equilibrium position and then released. Find the displacement of the body from it's equilibrium position at time 't' seconds the maximum velocity and the period of oscillation.* **[Ans. : Max velocity = 2.8 m/sec**

period of oscillation = $\dfrac{2\pi}{w} = 0.45$ sec]

10. *The differential equation for damped vibrating system under the action of t periodic force is* $\dfrac{d^2x}{dt^2} + 2r\dfrac{dx}{dt} + n^2x = a\cos pt$. *Show that in* $n > r > 0$; *the complementary function represents vibration which are soon damped out. Prove that the particular integral is of the form* $b\cos(pt - \alpha)$, *where* $b^2 = \dfrac{a^2}{(n^2 - p^2)^2 + 4r^2p^2}$.

3.7 Coupled Masses

In coupled masses, we get a good application of simultaneous equations, an example will explain better.

Example 3.7 : *Two particles, each of mass m gram are suspended from two springs of same stiffness k as in the following Fig. 3.6. After the system comes to rest, the lower mass is pulled 1 cm, downward and released. Discuss their motion.*

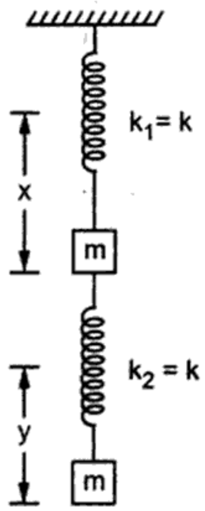


Fig. 3.6

Solution : Step 1 : Let x and y be the displacements of the upper and lower masses, at time t from their respective positions of equilibrium.

Then the stretch of the upper spring is x and that of the lower spring is y – x.

$\therefore$ Restoring force acting on the upper mass

$$= -kx + k(y - x) = k(y - 2x)$$

and that on the lower mass = – k (y – x)

Step 2 : Hence their equations of motion are :

$$m\frac{d^2x}{dt^2} = k(y - 2x) \text{ and}$$

$$m\frac{d^2y}{dt^2} = -k(y - x)$$

or $(mD^2 + 2k)x - ky = 0$ ... (1)

and $\quad (mD^2+k)\,y-kx = 0 \quad$ ... (2)

which are simultaneous differential equations.

Step 3 : Using Cramer's rule

i.e. solving for x and we get

$$\begin{vmatrix} mD^2+2k & -k \\ -k & mD^2+2k \end{vmatrix} x = \begin{vmatrix} 0 & -k \\ 0 & mD^2+2k \end{vmatrix}$$

$$\left[(mD^2+k)(mD^2+2k)-k^2\right] x = 0$$

or $$\left(D^4+3\lambda D^2+\lambda^2\right) x = 0, \quad \text{where } \lambda=\frac{k}{m}$$

Step 4 : Its auxiliary equation is

$$D^4+3\lambda D^2+\lambda^2 = 0$$

which gives $$D^2 = \frac{-3\lambda\pm\sqrt{9\lambda^2-4\lambda^2}}{2} = -2.62\lambda \text{ or } -0.38\lambda$$

$$= -a^2, -b^2 \quad \text{(say)}$$

So that $$D = \pm\, ia, \pm\, ib$$

Hence $$x = C_1 \cos at + C_2 \sin at + C_3 \cos bt + C_4 \sin bt \qquad \text{... (3)}$$

Step 5 : Also from equation (1)

$$y = \left(\frac{D^2}{\lambda}+2\right)x$$

$$y = \left(2-\frac{a^2}{\lambda}\right)(C_1\cos at + C_2 \sin at) + \left(2-\frac{b^2}{\lambda}\right)(C_3 \cos bt + C_4 \sin bt) \qquad \text{... (4)}$$

Step 6 : But initially when t = 0, x = y = l, $\frac{dx}{dt} = 0 = \frac{dy}{dt}$

Hence from (3), $C_1 + C_3 = l$ and $C_2 a + C_4 b = 0$

and from (4) $$l = \left(2-\frac{a^2}{\lambda}\right)C_1 + \left(2-\frac{b^2}{\lambda}\right)C_3$$

$$0 = \left(2-\frac{a^2}{\lambda}\right)\alpha\, C_2 + \left(2-\frac{b^2}{\lambda}\right)\beta\, C_4$$

when $$C_1 = \frac{l(\lambda-b^2)}{a^2-b^2}, \quad C_3 = \frac{l(\lambda-a^2)}{b^2-a^2}, \quad C_2 = C_4 = 0$$

Substituting these values of the constants in (3) and (4), we get x and y which show that the motion of the string is a combination of two simple harmonic motions of periods $\frac{2\pi}{a}$ and $\frac{2\pi}{b}$.

3.8 Electro-Mechanical Analogy

With the illustrations of the examples cited earlier, it is learnt that how the differential equations play a dominant role in differential theories of electrical and mechanical system. While making an electrical equivalent of mechanical system, the following correspondence between the electrical and mechanical quantities should be taken into account while formulating mathematical model :

Mechanical system	Electrical circuit (series connection)
Mass (M)	Inductance (C)
Damping constant	Resistance (R)
Spring stiffness (k)	Reciprocal of capacitance $\left(\frac{1}{C}\right)$
Driving force ϕ cos nt	Electromotive force (EMF) E
Displacement x (t)	Current i (t) or charge 'q'

3.9 L-C-R Circuit

Consider the electrical circuit given in Fig. 3.7

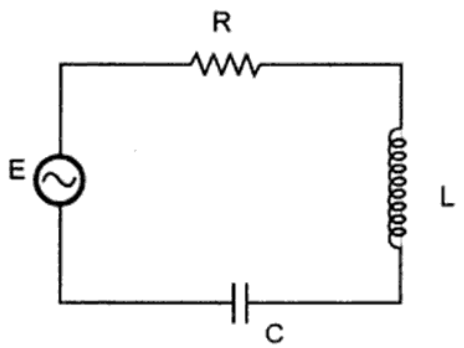


Fig. 3.7

Let $\quad I$ = Instantaneous current

Q = Instantaneous charge

L = Inductance

C = Capacitor of capacity

R = Resistance

$\therefore$ Voltage drop across R = RI

Voltage drop across L = $L\dfrac{dI}{dt}$

Voltage drop across C = $\dfrac{Q}{C}$

We know that $I = \dfrac{dQ}{dt}$ $\therefore$ $Q = \int I\,dt$

By Kirchoffs law we get

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int I\,dt = E(t)$$

i.e.
$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t)$$

which is a Linear differential equation in Q and C.

3.10 Illustrations

Example 3.8 : *In an L-C-R circuit, the charge q on the condenser is given by* $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = E \sin \omega t$. *The circuit is tuned to resonate so that* $\omega^2 = \dfrac{1}{LC}$. *If initially the current and charge be zero, show that for small values of* $\dfrac{R}{L}$ *the current in the circuit at time t is given by* $\dfrac{Et}{2L} \sin \omega t$.

(May-2001)

Solution : Step 1 : The given differential equation is $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = E \sin \omega t$

i.e.
$$L D^2 q + R D q + \frac{q}{C} = E \sin \omega t, \quad \text{where } D = \frac{d}{dt}$$

i.e.
$$\left(D^2 + \frac{R}{L} D + \frac{1}{LC}\right) q = \frac{E}{L} \sin \omega t$$

i.e.
$$\left(D^2 + \omega^2\right) q = \frac{E}{L} \sin \omega t \quad \text{... (1)}$$

Step 2 : Note we can neglect $\left(\dfrac{RD}{L}\right)$ as $\dfrac{R}{L}$ is small and $\because$ $\dfrac{1}{LC} = \omega^2$

Step 3 : To find C.F.

A.E. is $D^2 + \omega^2 = 0$ i.e. $D^2 = -\omega^2$ $D = \pm \omega i$

$\therefore$
$$\text{C.F.} = C_1 \cos \omega t + C_2 \sin \omega t$$

Step 3 : To find P.I.

$$\text{P.I.} = \frac{E}{L} \cdot \frac{1}{(D^2 + \omega^2)} \sin \omega t$$

$$= \frac{-Et}{2L\omega} \cos \omega t$$

Step 4 : $\therefore$ complete solution is

$$q = C_1 \cos \omega t + C_2 \sin \omega t - \frac{E}{2L\omega} t \cos \omega t \quad \dots (2)$$

Step 5 :

$$\therefore i = \frac{dq}{dt} = -C_1 \omega \sin \omega t + C_2 \omega \cos \omega t - \frac{E}{2L\omega}(-t\omega \sin \omega t + \cos \omega t) \quad \dots (3)$$

Step 6 : Now initially, at t = 0, q = 0 and i = 0.

$\therefore$ from (2) and (3) we get at t = 0

$$0 = C_1 + 0 + 0 \quad \therefore C_1 = 0 \text{ and}$$

$$0 = 0 + C_2\omega - \frac{E}{2L\omega}(0 + 1) \quad \therefore \quad C_2\omega = \frac{E}{2L\omega}$$

$\therefore$ Substituting in (3) we get

$$i = 0 + \frac{E}{2L\omega} \cos \omega t - \frac{E}{2L\omega}(-t\omega \sin \omega t + \cos \omega t)$$

$$= \left(\frac{E}{2L\omega} - \frac{E}{2L\omega}\right) \cos \omega t + \frac{Et}{2L} \sin \omega t = \frac{Et}{2L} \sin \omega t$$

Example 3.9 : *An uncharged condenser of capacity C is charged by applying an e.m.f. of value* $E \sin \frac{t}{\sqrt{LC}}$ *through the leads of inductance L and negligible resistance. The charge Q on the plate of the condenser satisfies the differential equation* $\frac{d^2Q}{dt^2} + \frac{Q}{LC} = \frac{E}{L} \sin \frac{t}{\sqrt{LC}}$.

Prove that the charge at any time t is given by $Q = \frac{EC}{2}\left[\sin \frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}} \cos \frac{t}{\sqrt{LC}}\right]$

Solution : Step 1 : Let $p^2 = \frac{1}{LC}$ $\therefore$ The given differential equation becomes

$$\frac{d^2Q}{dt^2} + p^2Q = \frac{E}{L} \sin pt$$

i.e.

$$(D^2 + p^2)\, Q = \frac{E}{L} \sin pt \quad \dots (1)$$

Step 2 : To find C.F. A.E. is $D^2 + p^2 = 0$ i.e. $D^2 = -p^2$ $\therefore D = \pm p$

$$\therefore \quad \text{C.F.} = C_1 \cos pt + C_2 \sin pt$$

Step 3 : To find P.I.

$$\text{P.I.} = \frac{E}{L} \frac{1}{(D^2 + p^2)} \sin pt = \frac{E}{L} \cdot \frac{(-t)^1}{(2p)^1 \, 1\,!} \sin\left(pt + 1\frac{\pi}{2}\right)$$

$$= -\frac{Et}{2\,pL} \cos pt$$

Step 4 : The complete solution is

$$\therefore \quad Q = C_1 \cos pt + C_2 \sin pt - \frac{Et}{2\,pL} \cos pt \qquad \text{... (2)}$$

Step 5 : at $t = 0$, $Q = 0$ and $i = 0$

$$\therefore \text{from (2)} \quad 0 = C_1 + 0 + 0 \quad \therefore C_1 = 0$$

$$\therefore \quad Q = C_2 \sin pt - \frac{Et}{2pL} \cos pt \qquad \text{... (3)}$$

$$\therefore \quad i = \frac{dQ}{dt} = C_2\, p \cos pt - \frac{E}{2\,pL} (-\,tp \sin pt + \cos pt)$$

Step 6 : Now, at $t = 0$, $i = 0$

$$\therefore \quad 0 = C_2 p\,(1) - \frac{E}{2pL}(0 + 1)$$

$$\therefore \quad C_2 p = \frac{E}{2pL} \quad \therefore C_2 = \frac{E}{2p^2 L}$$

Step 7 : Substituting in Q we get

$$Q = \frac{E}{2p^2 L} \sin pt - \frac{Et}{2pL} \cos pt$$

$$= \frac{E}{2\frac{1}{LC} L} \sin pt - \frac{E}{2\frac{1}{\sqrt{LC}} L} t \cos pt \qquad \text{...} \because p = \frac{1}{\sqrt{LC}}$$

$$= \frac{EC}{2} \sin pt - \frac{E}{2}\sqrt{\frac{C}{L}}\, t \cos pt = \frac{EC}{2}\left(\sin pt - \frac{t}{\sqrt{LC}} \cos pt\right)$$

$$= \frac{EC}{2}\left(\sin \frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}} \cos \frac{t}{\sqrt{LC}}\right) \qquad \text{...} \because p = \frac{1}{\sqrt{LC}}$$

Example 3.10 : *The charge Q of a condenser of capacity C, discharged in a circuit of resistance R and self inductance L, satisfies* $L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{Q}{C} = 0$. *Solve the equation given that* $Q = Q_0$ *and* $\dfrac{dQ}{dt} = 0$ *when* $t = 0$ *and that* $CR^2 < 4L$ **(Dec.-2001)**

Solution : Step 1 : We have the differential equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

i.e. $$\left(LD^2 + RD + \frac{1}{C}\right) Q = 0 \quad \text{where } D \equiv \frac{d}{dt}$$

Step 2 : To find C.F.

A. E. is $$LD^2 + RD + \frac{1}{C} = 0$$

$$\therefore \quad D = \frac{-R \pm \sqrt{R^2 - \dfrac{4L}{C}}}{2L} = \frac{-R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{4L}{4L^2C}}$$

$$= \frac{-R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{-R}{2L} \pm \sqrt{-\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

Now, $\because \; CR^2 < 4L \; \therefore \dfrac{R^2}{4L} < \dfrac{1}{C} \quad \therefore \dfrac{R^2}{4L^2} < \dfrac{1}{LC}$ so that $\dfrac{1}{LC} - \dfrac{R^2}{4L^2} > 0$

Now, let $\dfrac{1}{LC} - \dfrac{R^2}{4L^2} = \omega^2$ and $\therefore \quad \omega^2 > 0$

$$\therefore \quad D = \frac{-R}{2L} \pm \sqrt{-\omega^2} = \frac{-R}{2L} \pm i\omega$$

$$\therefore \quad \text{C.F.} = e^{\frac{-Rt}{2L}} (C_1 \cos\omega t + C_2 \sin\omega t), \text{ PI} = 0 \text{ as RHS is zero.}$$

Step 3 : Complete solution is

$$Q = e^{\frac{-Rt}{2L}} (C_1 \cos\omega t + C_2 \sin\omega t) \qquad \text{... (1)}$$

Step 4 :

$$\therefore i = \frac{dQ}{dt} = e^{\frac{-Rt}{2L}} (-C_1\omega \sin\omega t + C_2\omega \cos\omega t) - \frac{R}{2L} e^{\frac{-Rt}{2L}} (C_1\cos\omega t + C_2 \sin\omega t) \quad \text{... (2)}$$

Step 5 : Now, at $t = 0$, $Q = Q_0$ and $\dfrac{dQ}{dt} = i = 0$ (given)

$\therefore$ Using (1) and (2) we get

$$Q_0 = (1)(C_1 + 0) \quad \therefore C_1 = Q_0$$

and
$$0 = (1)(0 + C_2\omega) - \frac{R}{2L}(1)(C_1 + 0)$$

$$\therefore \quad C_2\omega = \frac{R}{2L} C_1 = \frac{R}{2L} Q_0 \qquad \ldots \because C_1 = Q_0$$

$$\therefore \quad C_2 = \frac{RQ_0}{2\,L\omega}$$

Step 6 : $\therefore$ Substituting for C_1 and C_2 in (1) we get

$$Q = e^{\frac{-Rt}{2L}}\left(Q_0 \cos\omega t + \frac{R\,Q_0}{2\,L\omega}\sin\omega t\right)$$

$$= Q_0 e^{\frac{-Rt}{2L}}\left(\cos\omega t + \frac{R}{2\,L\omega}\sin\omega t\right) \qquad \text{where } \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Exercise 3.2

1. *An electric circuit consists of an inductance L condenser of capacity C and e.m.f $E_0 \cos\omega t$ so that the charge Q satisfies the differential equation $\frac{d^2Q}{dt^2} + \frac{Q}{CL} = \frac{E_0}{L}\cos\omega t$. If $\omega = \frac{1}{\sqrt{LC}}$ and initially at $t = 0$, $Q = Q_0$ and $i = i_0$ find the charge and current at any time t.*

   [Ans. : $Q = Q_0 \cos\omega t + \frac{i_0}{\omega}\sin\omega t + \frac{E_0}{2L\omega}\, t \sin\omega t$]

2. *An inductor of 0.5 H is connected in series with a resistance of 6 Ω and a capacitor of 0.02 farad and a voltage generator having alternating voltage 24 sin 10t, $t > 0$. Find the charge and current at time t if the charge is zero when the switch is closed at $t = 0$.*

   [Ans. : $Q = \frac{e^{-6t}}{10}(4\cos 8t + 3\sin 8t) - \frac{2}{5}\cos 10t$]

3. *For an electric circuit with circuit constants R-L-C, the charge Q on the plate of the condenser is given by :*

   $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E\sin\omega t$ *and current by* $I = \frac{dQ}{dt}$. *The circuit is tuned to resonate so that* $\omega^2 = \frac{1}{LC}$. *If* $R^2 < \frac{4L}{C}$ *and* $Q = I = 0$ *at* $t = 0$ *show that*

   $Q = \frac{E}{R\omega}\left[-\cos\omega t + e^{\frac{-Rt}{2L}}\left(\cos pt + \frac{R}{2Lp}\sin pt\right)\right]$ *where* $p^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$

   [Dec.-2004]

4. *The voltage V and current i at a distance x from the transmission end satisfy the equations* $\frac{dV}{dx} = -Ri$ *and* $\frac{di}{dx} = -GV$ *where R and G are constants. If* $V = V_0$ *at transmission end* $(x = 0)$ *and* $V = 0$ *at the receiving end* $(x = l)$ *show that* $V = V_0 \frac{\sinh n(l - x)}{\sinh nl}$

   and $i = V_0\sqrt{\frac{G}{R}}\frac{\cosh n(l - x)}{\sinh nl}$ *where* $n^2 = RG$

   [Dec. 2000]

   Hint : $\frac{d^2V}{dx^2} = -R\frac{di}{dx} = -R(-GV) = RGV = n^2 V$

5. *An emf E sin pt is applied at t = 0 to a circuit containing a condenser C and an inductance L in series. The current I satisfies the equation :*

$$L\frac{dI}{dt} + \frac{1}{C}\int I dt = E \sin pt \text{ where } I = -\frac{dQ}{dt} \text{ and } p = \frac{1}{\sqrt{LC}}$$

Find the current at any time t if initially both current I and charge Q are zero.

[May-2001, Dec.- 2001, Dec.-2004] [Ans. : $I = \frac{Et}{2L} \sin pt$]

6. *A circuit consists of inductance L and condenser of capacity C in series. An e.m.f. E sin nt is applied to it at time t = 0, the initial current and charge on the condenser being zero. Find the current flowing in the circuit at any time t for (i) $\omega \neq n$ (ii) $\omega = n$ where $\omega^2 = \frac{1}{LC}$*

[Dec.- 2002, Dec.-2005]

[Ans. : (i) $\frac{nE}{L(\omega^2 - n^2)}(\cos nt - \cos \omega t)$ (ii) $\frac{Et}{2L} \sin nt$]

7. *A circuit consists of an inductance of 0.05 henry, a resistance of 20 Ω and a condenser of capacity 100 microfarad and an e.m.f. of E = 100 V. Find i and q given the initial conditions q = 0, i = 0 when t = 0.* **[Dec.- 2003]**

[Ans. : $q = \frac{-e^{-200t}}{100}\left(\cos 400t + \frac{1}{2}\sin 400t\right) + \frac{1}{100}$, $i = 5e^{-400t} \sin 400t$]

University Questions

Dec. - 98

1. *Two particles, each of mass m gram are suspended from two springs of same stiffness k as shown in Fig. 3.8. After the system comes to rest, the lower mass is pulled l cm downward and released. Discuss their motion.* **[8 Marks]**

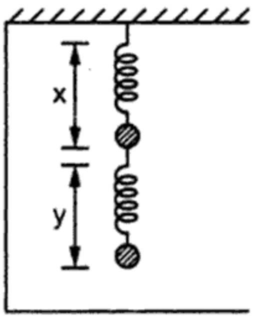


Fig. 3.8

May - 99

1. *A spring stretches 1 cm under the tension of 2 kg. and has a negligible weight. It is fixed at one end and is attached to a weight W kg at the other. It is found that resonance occurs when an axial periodic force 2 cos 2t kg acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by x = ct sin 2t and find the values of W and C.* **[8 Marks]**

Dec. - 99

1. *A spring stretches 1 cm under the tension of 2 kg and has a negligible weight. It is fixed at one end and is attached to a weight w kg at the other. It is found that resonance occurs when an axial periodic force 2 cos 2t kg acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by* $x = Ct \sin 2t$. *Find w and c.* **[7 Marks]**

May - 2000

1. *A spring streches 1 inch under the tension of 2 kg and has negligible weight. It is fixed at one end and is attached to a weight W kg at the other. It is found that resonance occurs when an axial periodic force 2 cos 2t kg acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by* $x = ct \sin 2t$ *and find values of W and C.* **[8 Marks]**

Dec. - 2000

1. *A mass m suspended from the end of helical spring is subjected to a periodic force* $f = F \sin \omega t$ *in the direction of its length. The force f is measured positive vertically downward and at zero time m is at rest. If the spring stiffness is s, show that the displacement of m at time 't' from the commencement of motion is given by*

$$x = \frac{F}{m\left(p^2 - \omega^2\right)}\left\{\sin \omega t - \frac{\omega}{p} \sin pt\right\} \text{ where } p^2 = \frac{s}{m}.$$

[7 Marks]

May - 2001

1. *A spring under the action of force "mk^2e" is suspended from a fixed point P having mass 'm' attached to its lower end, would stretch by amount 'e'. Initially, the mass is at rest at a point A below P in its equilibrium position. A vertical oscillation is given to the support P. Such that at any time 't' $(t > 0)$ its displacement below initial position is "a sin at". Show that the displacement 'x' of mass 'm' below 'A' is given by*

$$x = \frac{ka}{k^2 - n^2}(k \sin nt - a \sin kt) \text{ if } (n \neq k).$$

[8 Marks]

Dec. - 2001

1. *Find the motion of the mass-spring system with mass 0.125 kg damping zero, spring constant 1.125 kg / sec^2, and driving force cos t − 4 sin t Nt, assuming zero initial displacement and velocity. For what frequency of driving force would you get resonance ?* **[8 Marks]**

May - 2002

1. *Two particles, each of mass m gram suspended from two springs of same stiffness K as in the following Fig. 3.9. After the system comes to rest, the lower mass is pulled 1 cm downward and released. Discuss their motion.* **[8 Marks]**

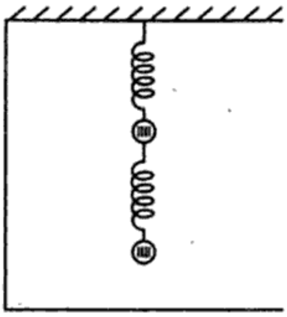


Fig. 3.9

Dec. - 2002

1. *A spring stretches 1 inch under the tension of 2 lbs and has a negligible weight. It is fixed at one end and is attached to a weight W lbs at the other. It is found that resonance occurs when an axial periodic force (2 cos t) lbs acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by* $x = Ct\sin 2t$ *and find the values of W and C.* **[8 Marks]**

May - 2003

1. *A spring which stretches by an amount e under a force* $mk^2 e$ *is suspended from a support P and has a mass m at its lower end. At time t = 0, the mass is at rest in its equilibrium position at a point A below P. A vertical oscillation is now given to the support P such that at any time its displacement below its initial position is 'a sin nt'. Show that the displacement x of the mass below A is :*

$$\frac{d^2x}{dt^2} + k^2x = k^2 a\sin nt.$$

Hence show that if $n \neq k$, *the displacement is given by*

$$x = \frac{ka}{k^2 - n^2}(k\sin nt - a\sin kt).$$

[8 Marks]

Dec. - 2003

1. *It is found experimentally that a 3 kg weight stretches a spring 15 cm. If the weight is pulled 10 cm below the equilibrium position and released :* **[8 Marks]**

   a) Find the differential equation and associated condition describing the motion.

   b) Find the position of the weight as a function of time.

   c) Find amplitude, period and frequency of the motion.

   d) Determine the position, velocity and acceleration of the weight $\frac{1}{2}$ *sec. after it has been released.*

May - 2004

1. *A body weighing 4.9 kg is hung from a spring. A pull of 10 kg will stretch the spring to 5 cm. The body is pulled down 6 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position in time t seconds, the maximum velocity and period oscillation.* **[8 Marks]**

Dec. - 2004

1. *A spring stretches 1 cm under the tension of 2 kgs and has a negligible weight. It is fixed at one end and is attached to a weight W kgs at the other. It is found that resonance occurs when an axial periodic force 2 cos 2t kgs acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by x = ct sin 2t and find the value of W and C.* **[8 Marks]**

May - 2005

1. *System of differential equations of an undamped mechanical system is given by :*

   $\ddot{x}_1 = -x_1 - 3(x_1 - x_2)$

   $3\ddot{x}_2 = 3(x_1 - x_2) - 3x_2$

   Assuming that there is no friction, find the natural frequencies of the system and corresponding normal modes of vibration using matrix method. **[8 Marks]**

2. *A spring under the action of force 'mk²e' is suspended from a fixed point P having mass 'm' attached to its lower end, would stretch by amount 'e'. Initially the mass is at rest at a point A below P in its equilibrium position. A vertical oscillation is given to the support P. Such that at any time 't' (t > 0) its displacement below initial position is a sin nt'. Show that the displacement 'x' of mass 'm' below A is given by :*

   $x = \frac{ka}{k^2 - n^2}(k \sin nt - a \sin kt)$ *if* $n \neq k$. **[8 Marks]**

Dec. - 2005

1. *Weight of 1 kg stretches a spring 5 cm. A weight of 3 kg is attached to the spring and weight is pulled 10 cm below the equilibrium position and released. Determine the position, velocity and acceleration of the weight* $\frac{1}{2}$ *sec. after it has been released.* **[8 Marks]**

May - 2006

1. *A mass of 160 gm is attached at one end of an elastic spring of which other end is fixed. A damping force acting on mass is 64 times the velocity at that instant. If mass of 170 gm produces elongation 2.45 cm in the spring, find the equation of motion of the spring.*

   If x = 0, and v = 30 m/s at t = 0,

   determine :

   i) Periodic time of oscillation

   ii) Maximum amplitude. **[5 Marks]**

4 Applications of Partial Differential Equations

4.1 Introduction

In fluid mechanics we come across the differential equation involving more independent variables, such a differential equation is known as partial differential equation. The partial differential equation involving boundary condition is known as a boundary value problem. Most of the partial differential equations are solved by the method of separation of variables, and we can use Fourier series for finding arbitrary functions.

4.2 Fourier Series

Definition : If f(x) is a periodic function with period 2L, defined in the interval $0 \leq x \leq 2L$ and satisfies the Dirichlet's conditions then f(x) can be represented by a series.

$$f(x) = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{L} x + b_1 \sin\frac{\pi x}{L}\right)$$

$$+ \left(a_2 \cos\frac{2\pi x}{L} + b_2 \sin\frac{2\pi x}{L}\right) + \ldots\ldots$$

i.e.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

This representation of f(x) is called Fourier series and a_o, a_n, b_n are called the Fourier coefficients.

Note 1 : $\sin^{-1}x$ cannot have a Fourier series expansion in any interval, since it is not a single valued function.

Note 2 : tan x cannot have a Fourier series expansion in $(0, 2\pi)$, since it becomes infinite at $x = \pi/2$.

Dirichlet's conditions :

i) f(x) and its integrals are finite and single valued.

ii) f(x) has at most finite number of finite discontinuities.

iii) f(x) has at most finite number of maxima and minima.

<table>
<tr><th>Name of the series</th><th>Interval</th><th>Fourier constants</th><th>Expression for the Fourier series</th></tr>
<tr><td>Fourier series</td><td>$0 \le x \le 2L$</td><td>$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$
$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$
$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$</td><td>$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$</td></tr>
<tr><td>Fourier series</td><td>$-L \le x \le L$</td><td>$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$
$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$
$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$</td><td>$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$</td></tr>
<tr><td>Fourier series</td><td>$-L \le x \le L$ f(x) is an even function</td><td rowspan="2">$a_0 = \frac{2}{L}\int_0^{L} f(x)\,dx$
$a_n = \frac{2}{L}\int_0^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$
$b_n = 0$</td><td rowspan="2">$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right)\right)$</td></tr>
<tr><td>Half range cosine series</td><td>$0 \le x \le L$</td></tr>
<tr><td>Fourier series</td><td>$-L \le x \le L$ f(x) is an odd function</td><td rowspan="2">$a_0 = 0$
$a_n = 0$
$b_n = \frac{2}{L}\int_0^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$</td><td rowspan="2">$f(x) = \sum_{n=1}^{\infty}\left(b_n\sin\left(\frac{n\pi x}{L}\right)\right)$</td></tr>
<tr><td>Half range sine series</td><td>$0 \le x \le L$</td></tr>
</table>

Note : If n is any positive integer then,

1) $\cos n\pi = (-1)^n$, $\sin n\pi = 0$

2) $\cos(2n \pm 1)\pi = -1$, $\sin(2n \pm 1)\pi = 0$

3) $\cos 2n\pi = 1$, $\sin 2n\pi = 0$

4) $(-1)^n = (-1)^{-n}$ $\therefore \cos(n+1)\pi = \cos(n-1)\pi$

5) $\cos\frac{n\pi}{2}$, $\sin\frac{n\pi}{2}$ depends on n.

4.3 Formation of One Dimensional Heat Flow Equation

We know that

1. Heat flows from higher temperature to lower temperature.
2. The rate of flow of heat through an area is proportional to the area into the temperature gradient $\left(\frac{\partial u}{\partial t}\right)$, where u (x, t) is temperature distribution normal to the area. Constant of proportionality is called the thermal conductivity (k) of the material.
3. The amount of heat required to change the temperature through the given range is proportional to the mass of the given the body and the change of temperature. The constant of proportionality is termed as specific heat (S).

   Consider a homogeneous bar of uniform cross-section, sides are insulated. It is assumed that the loss of heat from the sides by conduction or radiation is negligible. One end of the bar treated as the origin and the direction of heat flow as positive X-axis. Let ρ be the density (gm/cm^3) , 'S' the specific heat (cal/cm-deg.sec). The temperature at any point of the bar depend on the distance x of the point from one end and time 't' and is denoted by u (x, t).

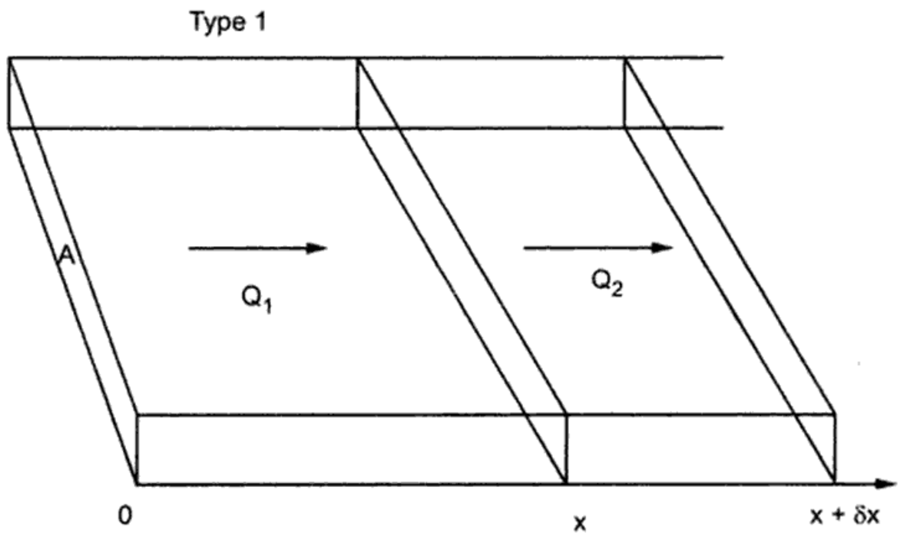


Fig. 4.1

Step 1 : As the quantity of heat crossing any section of the bar is proportional to the area and the temperature gradient normal to the area, the quantity 'Q_1' flowing into the section at a distance x is,

$$Q_1 = -kA\left(\frac{\partial u}{\partial t}\right)_x$$

Step 2 : The quantity 'Q_2' flowing out of the section at a distance $x+\delta x$ is,

$$Q_2 = -kA\left(\frac{\partial u}{\partial t}\right)_{x+\delta x}$$

Step 3 : $\therefore$ Quantity of heat retained by the slab with thickness δx is,

$$Q_1 - Q_2 = kA\left[\left(\frac{\partial u}{\partial t}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial t}\right)_x\right] \quad \text{... (1)}$$

Step 4 : The amount of heat required to change the temperature through the given range is proportional to the mass of the given body and the change of temperature. The constant of proportionality is termed as specific heat (S).

Therefore the rate of increase of heat in the slab

$$= S\rho\, A\,\delta\, x \frac{\partial u}{\partial t} \quad \text{... (2)}$$

$\therefore$ From equation (1) and (2)

$$S\rho\, A\,\delta\, x \frac{\partial u}{\partial t} = kA\left[\left(\frac{\partial u}{\partial t}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial t}\right)_x\right]$$

$$S\rho \frac{\partial u}{\partial t} = k\left[\frac{\left(\frac{\partial u}{\partial t}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial t}\right)_x}{\delta x}\right]$$

Step 5 : Taking limit as $\delta x \to 0$

$$S\rho \frac{\partial u}{\partial t} = k \lim_{\delta x \to 0}\left[\frac{\left(\frac{\partial u}{\partial t}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial t}\right)_x}{\delta x}\right]$$

By the definition of derivative by first principal

$$\frac{\partial u}{\partial t} = \frac{k}{S\rho}\frac{\partial^2 u}{\partial x^2}$$

Step 6 : For $\frac{k}{S\rho} = c^2$, it reduces to $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ and is called one dimensional heat flow equation. The constant $c^2 = \frac{k}{S\rho}$ is known as diffusivity of the material of the bar.

4.4 Type 1 : Solution of Heat Equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Consider
$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{... (1)}$$

Step 1 : Let u = X T where $X = f(x)$ and $T = \phi(t)$ be the solution of the given equation.

Step 2 : Differentiating partially we get

$$\frac{\partial u}{\partial t} = XT' \text{ and } \frac{\partial^2 u}{\partial x^2} = X''T$$

Step 3 : Substituting in given equation we have

$$XT = a^2 X''T$$

$$\frac{X''}{X} = \frac{T}{a^2 T} = \text{constant}$$

As x and t are independent variables L.H.S. = R.H.S. is possible only if they are equal to a constant.

Step 4 : Case 1 :

If constant = m^2 then

$$\frac{T}{a^2 T} = m^2 \Rightarrow T - a^2 m^2 T = 0 \text{ gives}$$

$$T = c_1 e^{a^2 m^2 t}$$

as t increases, T will also increase and $u = XT \to \infty$ u becomes infinite as t increases, hence constant m^2 is not suitable.

Step 5 : Case 2 :

If constant = 0 then T = 0 ⇒ constant i.e. T is independent of 't' which is absurd hence constant cannot be zero.

Step 6 : Case 3 :

If constant = $-m^2$ then $T = c_1 e^{-a^2 m^2 t}$ and here as t increases T remains finite. Hence constant is to be chosen negative in case of conduction of heat through a rod.

Now in this case $\frac{X''}{X} = -m^2$ = constant then

$$X'' + m^2 X = 0$$

$$\therefore \quad X = (c_2 \cos mx + c_3 \sin mx)$$

Thus the solution u = X T becomes

$$\therefore \quad X = (c_2 \cos mx + c_3 \sin mx)\, c_1 e^{-a^2 m^2 t}$$

Rearranging the constants we get

$$u = (c_4 \cos mx + c_5 \sin mx)\, e^{-c^2 m^2 t} \quad \dots (2)$$

Thus

$$u = (c_4 \cos mx + c_5 \sin mx)\, e^{-c^2 m^2 t}$$

is the suitable solution of equation (1).

4.5 Illustrations

Example 4.1 : *Solve* $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ *if* ... (1)

i) u is finite for all t *ii)* $u(o,t) = 0, \ \forall \ t$

iii) $u(L,t) = 0, \ \forall \ t$ *iv)* $u(x,0) = u_0$ *for* $0 \le x \le L$ *where l is being the length of the bar.*

Solution : Step 1 : The solution of given equation is

$$u = (c_4 \cos mx + c_5 \sin mx) \ e^{-c^2 m^2 t} \quad \text{... (2)}$$

Condition (i) is automatically satisfied.

Step 2 : The condition $u(0, t) = 0$ means at $x = 0$, $u = 0$.

Substituting we get

$$0 = (c_4 + 0) \, e^{-c^2 m^2 t}$$

which is possible only if $c_4 = 0$

Substituting in (2) we get

$$u = c_5 \sin mx \, e^{-c^2 m^2 t} \quad \text{... (3)}$$

Step 3 : The condition $u(L, t) = 0$ means at $x = L$, $u = 0$.

Substituting we get $0 = \sin mL \ e^{-c^2 m^2 t}$

As $e^{-c^2 m^2 t} \ne 0$, $c_5 \ne 0$ otherwise u becomes zero, which gives absurd result.

$\therefore \quad \sin mL = 0 \Rightarrow mL = n\pi \quad$ for $n = 1, 2, 3 ...$

Substituting $\quad m = \frac{n\pi}{L}$ in (3) we get

$$u = c_5 \sin \frac{n\pi x}{L} \, e^{\frac{-c^2 n^2 \pi^2 t}{L^2}} \quad \text{for } n = 1, 2, 3, ...$$

Step 4 : Taking $n = 1, 2, 3$ .... and varying the constant c_5 we can take the general solution as

$$u = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{L} \, e^{-\frac{c^2 n^2 \pi^2 t}{L^2}} \quad \text{... (4)}$$

Step 5 : Finally at $t = 0$, $u(x, 0) = u_0$ in $0 \le x \le L$

Substituting the value of $u(x, 0)$ in equation (4) we get,

$$u_0 = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{L}$$

This represents Fourier half range sine series for $u(x, 0)$ in $0 < x < L$

Step 6 : The formula for b_n is

$$b_n = \frac{2}{L}\int_0^L u(x,0)\sin\frac{n\pi x}{L}\,dx$$

Substituting the value of $u(x,0) = u_0$

$$b_n = \frac{2}{L}\int_0^L u_0 \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left\{u_0\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right)\right\}_0^L$$

$$= \frac{2}{L}\left[u_0\left(-\frac{L}{n\pi}\cos n\pi + \frac{L}{n\pi}\right)\right]$$

$$= \frac{2u_0}{n\pi}[1-\cos n\pi] = \frac{2u_0}{n\pi}\left[1-(-1)^n\right]$$

Step 7 : Substituting in (4) we get

$$u = \sum_{n=0}^{\infty}\frac{2u_0}{n\pi}\left[1-(-1)^n\right]\sin\frac{n\pi x}{L}\, e^{-\frac{c^2 n^2 \pi^2 t}{L^2}}$$

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} u_0 \frac{\left[1-(-1)^n\right]}{n}\sin\frac{n\pi x}{L}\, e^{\frac{-c^2 n^2\pi^2 t}{L^2}}$$

Example 4.2 : *Solve the equation* $\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$ *where* $u(x,t)$ *satisfies the following conditions :*

i) $u(0,t) = 0$ *ii)* $u(L,t) = 0$ *for all t*

iii) $u(x,0) = x \quad 0 \le x \le L/2$

$\quad = L-x \quad L/2 \le x \le L$

iv) $u(x,\infty)$ *is finite.*

Solution : We have to solve

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} \quad \text{... (1)}$$

Subject to the conditions

i) $u(0,t) = 0$, ii) $u(l,t) = 0$ and iii) $u(x,\infty)$ is finite

iv) $u(x,0) = \begin{cases} x, & 0\le x\le L/2 \\ L-x, & L/2\le x\le L\end{cases}$

As the first three conditions of this problem are same as that of the first problem the solution of this problem will be same till step 4 i.e.

$$u = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-\frac{a^2 n^2 \pi^2 t}{L^2}} \quad \text{... (4)}$$

Step 5 : The last condition at t = 0

$$u(x, o) = \begin{cases} x, & 0 \le x \le L/2 \\ L-x, & L/2 \le x \le L \end{cases}$$

Substituting t = 0 in equation (4) we get,

$$u(x, 0) = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ in } 0 < x < L$$

This represents Fourier half range sine series for $u(x, 0)$ in $(0, L)$

Step 6 : The formula for b_n is

$$b_n = \frac{2}{L} \int_0^L u(x, 0) \sin \frac{n\pi x}{L} dx$$

$$= \frac{2}{L} \int_0^{L/2} x \sin \frac{n\pi x}{L} dx + \frac{2}{L} \int_{L/2}^{L} (L-x) \sin \frac{n\pi x}{L} dx$$

$$= \frac{2}{L^2} \left\{ x \left(-\frac{\cos \frac{n\pi x}{L}}{\frac{n\pi}{L}} \right) - (1) \left(-\frac{\sin \frac{n\pi x}{L}}{\frac{n^2 \pi^2}{L^2}} \right) \right\}_0^{L/2}$$

$$+ \left\{ (L-x) \left(-\frac{\cos \frac{n\pi x}{L}}{\frac{n\pi}{L}} \right) - (-1) \left(\frac{\sin \frac{n\pi x}{L}}{\frac{n^2 \pi^2}{L^2}} \right) \right\}_{L/2}^{L}$$

$$= \frac{2}{L} \left[\frac{-L^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{L^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$$= \frac{4L}{n^2 \pi^2} \sin \frac{n\pi}{2}$$

Step 7 : Substituting in (4) we get

$$u(x, t) = \frac{4L}{\pi^2} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{L} e^{-\frac{a^2 n^2 \pi^2 t}{L^2}}$$

Example 4.3 : *Solve* $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ *if*

i) $u(x,t)$ *is bounded* *ii)* $u(0,t) = 0$

iii) $u(L,t) = 0$ *iv)* $u(x,0) = \dfrac{u_0 x}{L}$, $0 \le x \le L$

Solution : We have to solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{... (1)}$$

Subject to the conditions

i) $u(x, t)$ is bounded ii) $u(0, t) = 0$

iii) $u(L, t) = 0$ iv) $u(x, 0) = \dfrac{u_0 x}{L}$, $0 \le x \le L$

As the first three conditions of this problem are same as that of the first problem the solution of this problem will be same till step 4 i.e.

$$u = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L}\, e^{-\frac{kn^2\pi^2 t}{L^2}} \quad \text{... (4)}$$

Step 5 : The last condition at $t = 0$

$$u(x,0) = \frac{u_0 x}{L},\ 0 \le x \le L$$

Substituting $t = 0$ in equation (4) we get

$$u(x,0) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L} \ \text{ in } 0 < x < L$$

This represents Fourier half range sine series for $u(x, 0)$ in $(0, L)$

Step 6 : The formula for b_n is

$$b_n = \frac{2}{L}\int_0^L u(x,0)\sin\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{2}{L}\int_0^L \frac{u_0 x}{L}\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2u_0}{L}\left\{x\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (1)\left(-\frac{\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right\}_0^L$$

$$= \frac{2u_0}{L}\left[\frac{-L}{n\pi}\cos n\pi + \frac{L^2}{n^2\pi^2}\sin n\pi\right]$$

$$= -\frac{2u_0}{n\pi}\cos n\pi = -\frac{2u_0(-1)^n}{n\pi}$$

Step 7 : Substituting in (4) we get

$$u(x,t) = -\frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n \sin\frac{n\pi x}{L}\, e^{-\frac{kn^2\pi^2 t}{L^2}}$$

Example 4.4 : *The equation for the condition of heat along a bar of length L is $\frac{\partial\theta}{\partial t} = k\frac{\partial^2\theta}{\partial x^2}$ neglecting radiation. Find an expression for θ if the ends of the bar are maintained at zero temperature and if initially the temperature is T at the centre of the bar and falls uniformly to zero at its ends.*

Solution :

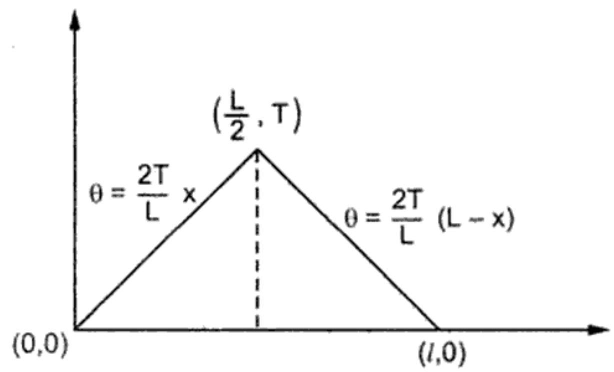


Fig. 4.2

We have to solve

$$\frac{\partial\theta}{\partial t} = k\frac{\partial^2\theta}{\partial x^2} \qquad \text{... (1)}$$

Subject to the conditions i) $\theta(0,t)=0$ ii) $\theta(L,t)=0$ and iii) At t = 0

$$\theta = \begin{cases}\frac{2T}{L}x, & 0\le x\le \frac{L}{2}\\ \frac{2T}{L}(L-x), & \frac{1}{2}\le x\le L\end{cases}$$

As the first three conditions of this problem are same as that of the first problem the solution of this problem will be same till step 4 i.e.

$$\theta = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L}\, e^{-\frac{kn^2\pi^2 t}{L^2}} \qquad \text{... (4)}$$

Step 5 : The last condition at t = 0, gives

$$\theta(x,0) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{in} \quad 0<x<L$$

This represents Fourier half range sine series for $\theta(x,0)$ in $(0,L)$

Step 6 : The formula for b_n is

$$b_n = \frac{2}{L}\int_0^L \theta(x,0)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\int_0^L \frac{2T}{L}\,x\sin\frac{n\pi x}{L}\,dx+\frac{2}{L}\int_0^L \frac{2t}{L}(L-x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{4T}{L^2}\left\{x\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right)-(1)\left(-\frac{\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right\}_0^{L/2}$$

$$+\left\{(L-x)\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right)-(-1)\left(\frac{\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right\}_{L/2}^{L}$$

$$= \frac{4T}{L^2}\left[\frac{-L^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{L}{2n\pi}\cos\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{8\,T}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Step 7 : Substituting in (4) we get

$$\theta(x,t) = \frac{8T}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\,e^{\frac{kn^2\pi^2 t}{L^2}}$$

Example 4.5 : *A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is*

$$u(x,0) = x \qquad 0\le x\le 50$$
$$= 100-x \qquad 50\le x\le 100$$

Find the temperature $u(x,t)$ at any time.

Solution : We have to solve

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{subject to conditions} \qquad \dots (1)$$

i) $u(0, t) = 0$ ii) $u(100, t) = 0$ and

iii) $u(x, 0) = \begin{cases} x, & 0 \le x \le 50 \\ 100 - x, & 50 \le x \le 100 \end{cases}$

After using first two conditions the solution of given equation is

$$u = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{100} e^{\frac{-c^2 n^2 \pi^2 t}{(100)^2}}, \quad n = 1, 2, 3 \ldots \qquad \ldots (2)$$

Using condition (iii) we have,

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{100}$$

This represents Fourier half range sine series for $u(x, 0)$ in (0, 100)

$$b_n = \frac{2}{100} \int_0^{100} u(x, 0) \sin \frac{n\pi x}{100} dx$$

$$= \frac{1}{50} \int_0^{50} x \sin \frac{n\pi x}{100} dx + \int_{50}^{100} (100 - x) \sin \frac{n\pi x}{100} dx$$

$$\frac{1}{50} \left\{ x \left(-\frac{\cos \frac{n\pi x}{100}}{\frac{n\pi}{100}} \right) - (1) \left(-\frac{\sin \frac{n\pi x}{100}}{\frac{n^2 \pi^2}{100^2}} \right) \right\}_0^{50}$$

$$+ \left\{ (100 - x) \left(-\frac{\cos \frac{n\pi x}{100}}{\frac{n\pi}{100}} \right) - (-1) \left(\frac{\sin \frac{n\pi x}{100}}{\frac{n^2 \pi^2}{100^2}} \right) \right\}_{50}^{100}$$

$$= \frac{400}{n^2 \pi^2} \sin \frac{n\pi}{2}$$

Substituting in (2) we get

$$u(x, t) = \frac{400}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{100} e^{-\frac{c^2 n^2 \pi^2 t}{100^2}}$$

Replacing n by 2n + 1 we get

$$= \frac{400}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \frac{(2n+1)\pi x}{2} e^{\frac{-c^2 (2n+1)^2 \pi^2 t}{100^2}}$$

Example 4.6 : *Solve* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

if i) u is finite $\forall t$ *ii)* $u(o,t) = 0$

iii) $u(\pi,t) = 0$ *iv)* $u(x,o) = \pi x - x^2, \; 0 \le x \le \pi$ *when* $t = 0$.

Solution : Step 1 : The solution of given equation is

$$u = (c_1 \cos mx + c_2 \sin mx) c_3 e^{-m^2 t} \quad \text{... (1)}$$

$$u = (c_4 \cos mx + c_5 \sin mx) e^{-m^2 t} \quad \text{... (2)}$$

Step 2 : The condition $u(0, t) = 0$ gives $c_4 = 0$

Substituting in (2) we get

$$u = c_5 \sin mx\, e^{-km^2 t} \quad \text{... (3)}$$

Step 3 : Condition $u(\pi, t) = 0 \Rightarrow 0 = \sin m\pi\, e^{-m^2 t}$

$e^{-m^2 t} \neq 0$, $c_5 \neq 0$ otherwise u becomes zero, which gives absurd result.

$\therefore \sin m\pi = 0 \Rightarrow m\pi = n\pi$. Hence $m = n$, for $n = 1, 2, 3, \ldots$

Substituting m =n in (2) we get

$$u = c_5 \sin nx e^{-n^2 t}, \quad \text{for } n = 1, 2, 3 \ldots.$$

Step 4 : Taking n = 1, 2, 3 ... and varying the constant c_5 we can take the general solution as

$$u = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-\frac{kn^2 \pi^2 t}{L^2}} \quad \text{... (4)}$$

Step 5 : Finally the last condition at t = 0,

$$u(x, 0) = \pi x - x^2 \text{ in } 0 \le x \le \pi$$

gives

$$u(x, 0) = \sum_{n=0}^{\infty} b_n \sin nx; \quad n = 1, 2, 3 \ldots$$

This represents Fourier half range sine series for

$$u(x, 0) = \pi x - x^2 \text{ in } (0, \pi)$$

Step 6 : The formula for b_n is

$$b_n = \frac{2}{L} \int_0^L u(x, 0) \sin \frac{n\pi x}{L} dx$$

Substituting L = π, we get

$$b_n = \frac{2}{\pi}\int_0^{\pi} u(x,0)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\,dx$$

$$\frac{2}{\pi}\left\{(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) - 2\left(-\frac{\cos nx}{n^3}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^3}(\cos n\pi - 1)\right]$$

$$= \frac{4}{n^3\pi}\left[1-(-1)^n\right]$$

Step 7 : Substituting in (4) we get

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^3}\sin nx\, e^{-n^2 t}$$

Replacing n by 2n + 1 we get

$$u(x,t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3}\sin(2n+1)x\, e^{-(2n+1)^2 t}$$

Example 4.7 : *Solve* $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ *for the conduction of heat along a rod without radiation subject to the following conditions :*

i) u is not infinite as $t \to \infty$

ii) $\frac{\partial u}{\partial x} = 0$ *for x = 0, x = L (i.e. no heat flows through the ends i.e. ends are insulated) and*

iii) $u = Lx - x^2$ *for t = 0 between x = 0, x = L.*

Solution : Step 1 : The solution of given equation is

$$u = (c_4\cos mx + c_5\sin mx)\,e^{-km^2 t} \quad \text{... (1)}$$

Step 2 : Condition (ii) is $\frac{\partial u}{\partial x} = 0$ for x = 0, x = L.

To apply condition (ii) differentiate u partially w.r.t. x

i.e. $$\frac{\partial u}{\partial x} = (-mc_4\sin mx + mc_5\cos mx)\,e^{-km^2 t} \quad \text{... (2)}$$

Now $$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0 \quad \text{gives}$$

$$0 = m\,c_5\,e^{-km^2 t} \Rightarrow c_5 = 0$$

Step 3 : Substituting $c_5 = 0$ in (2) we get

$$\frac{\partial u}{\partial x} = -m\, c_4 \sin mx\, e^{-km^2 t} \quad \text{... (3)}$$

Also condition $\left(\frac{\partial u}{\partial x}\right)_{x=L} = 0$ gives

$$0 = -mc_4 \sin mL e^{-km^2 t}$$

This requires $\sin mL = 0 \Rightarrow mL = n\pi, \quad n = 1, 2, 3 \ldots$

$$\therefore \quad m = \frac{n\pi}{L}$$

Step 4 : Substituting $c_5 = 0$ and $m = \frac{n\pi}{L}$ in (1) we get

$$u = c_4 \cos \frac{n\pi x}{L} e^{-\frac{kn^2\pi^2 t}{L^2}}, \quad n = 1, 2, 3 \ldots$$

Step 5 : Taking $n = 1, 2, 3$ ... and varying the constant c_4 we can take the general solution is

$$u = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L} e^{-\frac{k^2 n^2 \pi^2 t}{L^2}} \quad \text{... (4)}$$

i.e.

$$u = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} e^{-\frac{k^2 n^2 \pi^2 t}{L^2}} \quad \text{... (5)}$$

Step 6 : Finally from condition

$$u = Lx - x^2 \quad \text{for } t = 0 \text{ in } 0 < x < L \text{ we have from (4)}$$

$$Lx - x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad \text{for } 0 < x < L$$

This represents Fourier cosine series for $u(x, 0)$ in $(0, L)$

Step 7 : The formula for a_0 and a_n is given by

$$a_0 = \frac{1}{L}\int_0^L u(x)\,dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L u(x) \cos \frac{n\pi x}{L}\,dx$$

Now

$$a_0 = \frac{1}{L}\int_0^L u(x,0)\,dx = \frac{1}{L}\int_0^L (Lx - x^2)\,dx$$

$$= \frac{1}{L}\left[L\frac{x^2}{2} - \frac{x^3}{3}\right]_0^L = \frac{1}{L}\left[\frac{L^3}{2} - \frac{L^3}{3}\right] = \frac{L^2}{6}$$

And
$$a_n = \frac{2}{L}\int_0^L u(x,0)\cos\frac{n\pi x}{L}\,dx$$
$$= \frac{2}{L}\int_0^L \left(Lx - x^2\right)\cos\frac{n\pi x}{L}\,dx$$
$$\frac{2}{L}\left\{\left(Lx-x^2\right)\left(\frac{\sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right)-(L-2x)\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)-2\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n^3\pi^3}{L^3}}\right)\right\}_0^L$$
$$= \frac{2}{L}\left[\frac{L^2}{n^2\pi^2}\{-L\cos n\pi - L\}\right]$$
$$= -\frac{2L^2}{n^2\pi^2}\left[1+(-1)^n\right]$$

Step 7 : Substituting in (5) we get
$$u = \frac{L^2}{6} - \frac{2L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[1+(-1)^n\right]}{n^2}\cos\frac{n\pi x}{L}\,e^{-\frac{kn^2\pi^2 t}{L^2}}$$

Replacing n by 2n we have
$$u = \frac{L^2}{6} - \frac{2L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{L}\,e^{\frac{4n^2\pi^2 t}{L^2}}$$

Example 4.8 : *A bar with insulated sides is initially at temperature 0°C throughout. The end $x = 0$ is kept at 0°C for all time and the heat is suddenly applied so that $\frac{\partial u}{\partial x} = 10$ at $x = L$ for all time. Find the temperature function $u(x,t)$.*

Solution : We have to solve the P.D.E.
$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} \qquad \text{... (1)}$$

Step 1 : Let u = x T where $X = f(x)$ and $T = \phi(t)$ be the solution of the given equation.

Step 2 : Differentiating partially we get
$$\frac{\partial u}{\partial t} = XT' \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T$$

Step 3 : Substituting in given equation we have
$$XT = a^2 X'' T$$

$$\frac{X''}{X} = \frac{T}{a^2 T} = \text{Constant}$$

As x and t are independent variables L.H.S = R.H.S. is possible only if they are equal to a constant.

Step 5 : If constant $= -m^2$ then $T = c_1 e^{-a^2 m^2 t}$ and here as t increases T remains finite. Hence constant is to be chosen negative in case of conduction of heat through a rod.

Now in this case $\frac{X''}{X} = -m^2$ = constant then

$$X'' + m^2 X = 0$$

$$\therefore \quad X = (c_2 \cos mx + c_3 \sin mx)$$

Thus the solution u = x T becomes

$$\therefore \quad X = (c_2 \cos mx + c_3 \sin mx) c_1 e^{-a^2 m^2 t}$$

Rearranging the constants we get

$$u = (c_4 \cos mx + c_5 \sin mx) e^{-c^2 m^2 t} \quad \text{... (2)}$$

Thus $\quad u = (c_4 \cos mx + c_5 \sin mx) e^{-c^2 m^2 t}$

is the suitable solution of equation (1)

Step 6 : Also if $\frac{X''}{X} = \frac{T}{a^2 T} = 0$

Then the solution is

$$u(x, t) = c_6 + c_7 x \quad \text{... (3)}$$

Step 7 : Condition are

i) $u(x, 0) = 0$

ii) $u(0, t) = 0$

iii) $\left(\frac{\partial u}{\partial x}\right)_{x = L} = 10$, for all t

Since the above condition of the problem are such that any one of the above solutions (i.e. (2) and (3)) does not satisfy them. We use the combination of the solution to satisfy the given condition i.e. sum of (2) and (3).

$$\therefore \quad u(x, t) = c_6 + c_7 x + (c_1 \cos mx + c_2 \sin mx) e^{-m^2 a^2 t} \quad \text{... (4)}$$

Step 8 : Now (ii) $\Rightarrow c_6 = 0, \ c_1 = 0$

Substituting in (4) we get

$$u(x,t) = c_7\, x + c_2 \sin mx \cdot e^{-m a^2 t} \quad \dots (5)$$

Step 9 : Now to apply (iii) differentiate (5) w.r.t. x.

$$\frac{\partial u}{\partial x} = c_7 + mc_2 \cos mx\, e^{-m^2 a^2 t}$$

(iii) $\Rightarrow \ 10 = c_7 + mc_2 \cos mL\, e^{-m^2 a^2 t}$

$\Rightarrow \quad c_7 = 10$ and $\cos mL = 0$

i.e. $\quad mL = \dfrac{(2n+1)\pi}{2}$

Substituting in (5) we get

$$u(x,t) = 10\,x + c_2 \sin \frac{(2n+1)\pi x}{2\,L} \cdot e^{\frac{-a^2 (2n+1)^2 \pi^2 t}{4L^2}}$$

Step 10 : Taking n = 1, 2, 3, 4 ... and varying c_2 as b_1, b_2, b_3 ..... the general solution is given by

$$u(x,t) = 10\,x + \sum_{n=0}^{\infty} b_{2n+1} \cdot \sin \frac{(2n+1)\pi x}{2L}\, e^{\frac{-a^2 (2n+1)^2 \pi^2 t}{4L^2}} \quad \dots (6)$$

Step 11 : Now (i) $\Rightarrow t = 0, \ u = 0$

Substituting in (6) we get

$$-10\,x = \sum_{n=0}^{\infty} b_{2n+1} \sin \frac{(2n+1)\pi x}{2L}$$

which is fourier sine series. Thus

$$b_{2n+1} = \frac{2}{L}\int_0^L (-10\,x) \sin \frac{(2n+1)\pi x}{2L} \cdot dx$$

$$= -\frac{20}{L}\left\{(x)\left(-\frac{2L}{(2n+1)\pi} \cos \frac{(2n+1)\pi x}{2L}\right) + \frac{4L^2}{(2n+1)^2 \pi^2} \sin \frac{(2n+1)\pi x}{2L}\right\}_0^L$$

$$b_{2n+1} = -\frac{80\,L}{(2n+1)^2\,\pi^2} \sin \frac{(2n+1)\pi}{2}$$

Step 12 : Substituting b_{2n+1} in (6) we get the complete solution

$$u(x, t) = 10x - \frac{80\,L}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \sin \frac{(2n+1)\pi}{2}$$

$$. \sin \frac{(2n+1)\pi x}{2L}\; e^{\frac{-a^2 (2n+1)^2 \pi^2 t}{4L^2}}$$

Example 4.9 : *An insulated rod of length 'L' has its ends A and B maintained at 0°C and 100°C respectively. Until steady state condition prevail. If B is suddenly reduced to 0°C and maintained at 0°C,*

a) Find the temperature at a distance x from A at time t.

b) Also find the temperature if the change consist of raising the temperature of A to 20°C and reducing that of B to 80°C.

Solution : a) **Step 1 :** The temperature function $u(x, t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{... (1)}$$

Step 2 : Before the temperature at B change, i.e. when t = 0 heat flow was independent of time (steady state condition)

$\therefore$ as temperature function is independent of time, equation reduces to $\frac{\partial^2 u}{\partial x^2} = 0$

$\therefore$ its solution is, $u = ax + b$ ... (2)

where a and b are arbitrary constant.

Step 3 : As u = 0 for x = 0 and u = 100 for x = L

Substituting in equation (2) we get

$b = 0$ and $a = \frac{100}{L}$.

$\therefore$ The initial condition is expressed by

$$u(x, 0) = \frac{100}{L}\, x \quad \text{... (3)}$$

Step 4 : Also the boundary conditions are $U(0, t) = 0 = u(L, t)$ for all t.

We can take the general solution of (1) as

$$u = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{L}\; e^{-\frac{c^2 n^2 \pi^2 t}{L^2}} \quad \text{... (4)}$$

Step 5 : Finally at t = 0, $u(x,0)=\frac{100\,x}{L}$ in $0 \le x \le L$

Substituting the value of $u(x,0)$ in equation (4) we get,

$$\frac{100\,x}{L} = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L}$$

This represents Fourier half range sine series for $u(x,0)$ in $0<x<L$.

Step 6 : The formula for b_n is

$$b_n = \frac{2}{L}\int_0^L u(x,0)\sin\frac{n\pi x}{L}\,dx$$

Substituting the value of $u(x,0)=\frac{100\,x}{L}$

$$b_n = \frac{2}{L}\int_0^L \frac{100\,x}{L}\sin\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{200}{L^2}\int_0^L x\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{200}{L^2}\left\{x\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right)-(1)\left(-\frac{\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right\}_0^L$$

$$= \frac{200}{L^2}\left[\frac{-L}{n\pi}\cos n\pi+\frac{L^2}{n^2\pi^2}\sin n\pi\right]$$

$$= -\frac{200}{n\pi}\cos n\pi=\frac{200(-1)^{n+1}}{n\pi}$$

Step 7 : Substituting in (4) we get

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1}\sin\frac{n\pi x}{L}\,e^{-\frac{c^2n^2\pi^2t}{L^2}}$$

b) Step 1 : Here initial conditions are same :

i.e. $u(x,0) = \frac{100}{L}x$

Step 2 : The boundary conditions are,

$$u(0,t) = 20 \text{ for all t.}$$

$$u(L,t) = 80 \text{ for all t.}$$

Here the boundary values are non-zero, we write

$$u(x,t) = u_s(x) + u_t(x,t) \quad \text{... (5)}$$

where $u_s(x)$ is solution of equation (1) involving x alone and which satisfy boundary condition. $u_s(x)$ is a steady state solution and $u_t(x,t)$ may be regarded as a transient part of solution which decreases as time increases.

Step 3 : As $u_s(0) = 20$ and $u_s(L) = 80$.

From equation (2) we have b = 20 and aL + b = 80

$$\therefore \quad a = \frac{60}{L}$$

$$\therefore \quad u_s(x) = \frac{60}{L}x + 20$$

Step 4 : for x = 0, equation (5) gives

$$u_t(0,t) = u(0,t) - u_s(0) = 20 - 20 = 0 \quad \text{... (6)}$$

Step 5 : for x = L equation (5) gives

$$u_t(L,t) = u(L,t) - u_s(L) = 80 - 80 = 0 \quad \text{... (7)}$$

Step 6 : Also at t = 0 equation (5) gives

$$u_t(x,0) = u(x,0) - u_s(x) = \frac{100}{L}x - \left(\frac{60}{L}x + 20\right)$$

$$\therefore \quad u_t(x,0) = \frac{40}{L}x - 20 \quad \text{... (8)}$$

Step 7 : Therefore equation (6) and (7) gives boundary conditions and equation (8) gives initial condition relative to transient solution. As $u_t(0,t) = u_t(L,t) = 0$. We can take the general solution of $u_t(x,t)$ as

$$u_t(x,t) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-\frac{c^2 n^2 \pi^2 t}{L^2}} \quad \text{... (9)}$$

Step 8 : Substituting t = 0 in (9) we get

$$u_t(x,0) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Which is Fourier sine series

where $$b_n = \frac{2}{L}\int_0^L u_t(x,0)\sin\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{2}{L}\int_0^L \left(\frac{40\,x}{L}-20\right)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left\{\left(\frac{40x}{L}-20\right)\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right)-\left(\frac{40}{L}\right)\left(-\frac{\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right\}_0^L$$

$$= \frac{-40}{n\pi}\,[1+\cos n\pi]$$

$$= \begin{cases} 0 & \text{if n is odd} \\ \dfrac{-80}{n\pi} & \text{if n is even} \end{cases}$$

Therefore equation (9) becomes

$$u(x,t) = \frac{-80}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{L}\,e^{-\frac{c^2 n^2 \pi^2 t}{L^2}}$$

for n as an even number.

$\therefore$ the required solution is $u(x,t) = u_s(x) + u_t(x,t)$

$$u(x,t) = \frac{60}{L}\,x + 20 + \frac{-80}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{L}\,e^{-\frac{c^2 n^2 \pi^2 t}{L^2}}$$

Exercise 4.1

Solve the one-dimensional heat flow equation $\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$ *for function* $u(x,t)$, *subject to the conditions.*

a) $u(x,\infty)$ *is finite (b)* $u(0,t) = 0$, *(c)* $u(L,t) = 0$, *for all t. The above three conditions are common for the following problems only the fourth condition will change.*

i) $(d)\,u(x,0) = x$ *for* $0<x<L$

[Ans. : $u(x,t) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,e^{\frac{n^2 a^2 \pi^2}{L^2}}\sin\frac{n\pi x}{L}$**]**

ii) $(d)\,u(x,0) = Lx - x^2$ *for* $0<x<L$

[Ans. : $u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{4L^2}{n^2\pi^2}\left[1-(-1)^n\right]\sin\frac{n\pi x}{L}\,e^{-\frac{kn^2\pi^2 t}{L^2}}$**]**

iii) $(d)\,u(x,0) = L$ *for* $0<x<L$

[Ans. : $u(x,t) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}\,e^{-\frac{kn^2\pi^2 t}{L^2}}$**]**

iv) $(d)\,u(x,0) = \begin{cases} x & \text{for } 0<x<L/2 \\ L-x & \text{for } L/2<x<L \end{cases}$

[Ans. : $u(x,t) = \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\,e^{-\frac{a^2 n^2\pi^2 t}{L^2}}$**]**

3) *The equation for the conduction of heat along a bar of length L is* $\dfrac{\partial\theta}{\partial t} = k\dfrac{\partial^2\theta}{\partial t^2}$ *, neglecting radiation. Find an expression for* θ *if the ends of the bar are maintained at zero temperature and if initially the temperature is T at the centre of the bar and falls uniformly to zero at its ends.*

$$\left[\text{Ans.: } \theta(x,t) = \frac{8T}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\, e^{-\frac{kn^2\pi^2 t}{L^2}}\right]$$

4) *Solve the one-dimensional heat flow equation* $\dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2}$ *for function* $u(x,t)$, *subject to the conditions (a)* $u(0,t) = 0$ *and (b)* $\dfrac{\partial}{\partial x}u(L,t) = 0$, *for all t (c)* $u(x,0) = x$ *d)* $u(x,\infty)$ *is finite.*

$$\left[\text{Ans.: } u(x,t) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n+1)\pi x}{L}\cdot e^{-\frac{(2n-1)^2 a^2\pi^2 t}{L^2}}\right]$$

5) *Solve the one-dimensional heat flow equation* $\dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2}$ *for function* $u(x,t)$, *subject to the conditions (a)* $\dfrac{\partial}{\partial x}u(0,t) = 0$ *and (b)* $\dfrac{\partial}{\partial x}(L,t) = 0$, *for all t (c)* $u(x,t) = x^2, 0<x<L$ *is (d)* $u(x,\infty)$ *finite.*

$$\left[\text{Ans.: } u(x,t) = \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{L}\, e^{-\frac{n^2 a^2\pi^2 t}{L^2}}\right]$$

6) *Solve* $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ *for the condition of heat along a rid without radiation, subject to the following condition :*

1) *u is not infinite as* $t \to \infty$

2) $\dfrac{\partial u}{\partial x} = 0$ *for* $x = 0$, $x = L$ *(i.e. ends are insulated i.e. no heat flows through the ends) and*

3) $u = Lx - x^2$ *for* $t = 0$ *between* $x = 0$, $x = L$.

$$\left[\text{Ans.: } u(x,t) = \frac{L^2}{6} - \frac{L^2}{\pi}\sum_{1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{L}\, e^{-\frac{4k\pi^2 t}{L^2}}\right]$$

7) *The temperatures at the ends* $x = 0$ *and* $x = 50$ *cm in length of a rod are held at* $0°c$ *and* $50°c$ *respectively until steady condition prevail. The two ends of the rod are suddenly insulated. Find the temperature distribution of the rod assuming that the surface of the rod is impervious to heat.*

$$\left[\text{Ans.: } u(x,t) = 25 - \frac{200}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{50}\, e^{\frac{-(2n-1)^2 a^2\pi^2 t}{2500}}\right]$$

8) *A rod of length L is insulated along its length so that no heat is transformed from its sides, the uniform temperature of the rod is* $50°C$ *. Suddenly the end* $x = 0$ *is cooled to* $0°c$ *and the end* $x = L$ *heated to* $100°c$ *and these are maintained afterwards. Find the subsequent temperature distribution of the rod.*

$$\left[\text{Ans.: } u(x,t) = \frac{100x}{L} + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{L}\, e^{-\frac{4n^2 a^2\pi^2 t}{L^2}}\right]$$

9) *A rod of length L has its ends A and B maintained at* $20°c$ *and* $40°c$ *respectively until steady state condition prevail. The temperature at A is suddenly raised to* $50°c$ *while that at B is lowered to* $10°c$ *and maintained thereafter. Find the subsequent temperature distribution of the rod.*

$$\left[\text{Ans.: } u(x,t) = 50 - \frac{40}{L}x - \frac{60}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{L}\, e^{-\frac{4n^2 a^2\pi^2 t}{L^2}}\right]$$

10) *A rod of length L has its ends A and B kept at 0°c and 75°c, until steady state condition prevail. If the temperature of A is suddenly raised to 75°c and that of B to 175°c and maintained thereafter, find the subsequent temperature distribution of the rod.*

$$\left[\text{Ans.: } u(x,t) = 75 + \frac{100x}{L} - \frac{300}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin\frac{(2n-1)\pi x}{L}\, e^{\frac{-(2n-1)^2 a^2 \pi^2 t}{L^2}}\right]$$

11) *A rod of length L has one end kept at 0°c and other end B at 100°c until steady state condition prevail. The temperature of A is suddenly raised to 50°c while the end B is insulated. These condition are maintained thereafter, find the subsequent temperature distribution of the rod.*

$$\left[\text{Ans.: } u(x,t) = 50 + \sum_{n=1}^{\infty}\left[\frac{(-1)^{n-1}\,800}{(2n-1)^2\pi^2} - \frac{200}{(2n-1)\pi}\right] \sin\frac{(2n-1)\pi x}{2L}\, e^{\frac{-(2n-1)^2 a^2 \pi^2 t}{L^2}}\right]$$

12) *A uniform rod of length L whose surface is thermally insulated, is initially at temperature θ_0. At time $t = 0$, one end is suddenly cooled to temperature 0°c and subsequently maintained at this temperature and at the same time, the other end is thermally insulated. Find the temperature at end $x = L$ at any time t.*

13) *The ends A and B of a insulated rod of length l have their temperature at 20°c and 80°c respectively until steady state condition prevail. The temperature at these ends are changed suddenly to 40°c and 60°c respectively. Find the temperature distribution of the rod at time t.*

$$\left[\text{Ans.: } u(x,t) = \frac{20x}{L} + 40 - \frac{40}{\pi}\sum_{1}^{\infty} \frac{1}{n} \sin\frac{2n\pi x}{L}\, e^{-\frac{4c^2 n^2 \pi^2 t}{L^2}}\right]$$

14) *Find the solution of the equation* $\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$ *such that*

1) $v \neq \infty$ *when* $t \to \infty$

2) $\frac{\partial v}{\partial x} = 0$ *when* $x = 0, \ \forall\, t$

3) $v = 0$ *when* $x = L, \ \forall\, t$

4) $v = v_0$ *when* $t = 0$ *for all values of x between 0 and L.*

$$\left[\text{Ans.: } V(x,t) = \frac{4v_0}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos\frac{(2n+1)\pi x}{2L} \cdot e^{\frac{-k(2n+1)^2 \pi^2 t}{4L^2}}\right]$$

15) *A rod of length L has its ends A and B maintained at 20°c and 40°c respectively until steady state condition prevail. The temperature at A is suddenly raised to 50°c while that at B is lowered to 10°c and maintained thereafter. Find the subsequent temperature distribution of the rod.*

$$\left[\text{Ans.: } u(x,t) = 50 - \frac{40x}{L} - \frac{60}{\pi}\sum_{1}^{\infty} \frac{1}{n} \sin\frac{2n\pi x}{L}\, e^{-\frac{4c^2 n^2 \pi^2 t}{L^2}}\right]$$

Type 2 : Wave Equations

4.6 Formation of Wave Equation

Step 1 : Consider a uniform elastic string of length 'L' stretched tightly between two point o and A with 'O' as the origin and horizontal and vertical lines through 'O' as co-ordinate axes, we shall discuss the displacement y as a function of distance x and time 't' under the following assumptions.

Step 2 : i) The motion is entirely in XY-plane the equilibrium position of the string is OA and each particle of the string moves perpendicular to OA.

ii) The string is perfectly flexible and does not have any resistance to bending.

iii) As the tension in the string is large so that the forces due to weight of the string can be neglected.

iv) The displacement y and slope $\frac{dy}{dx}$, are small, so that their higher powers can be neglected.

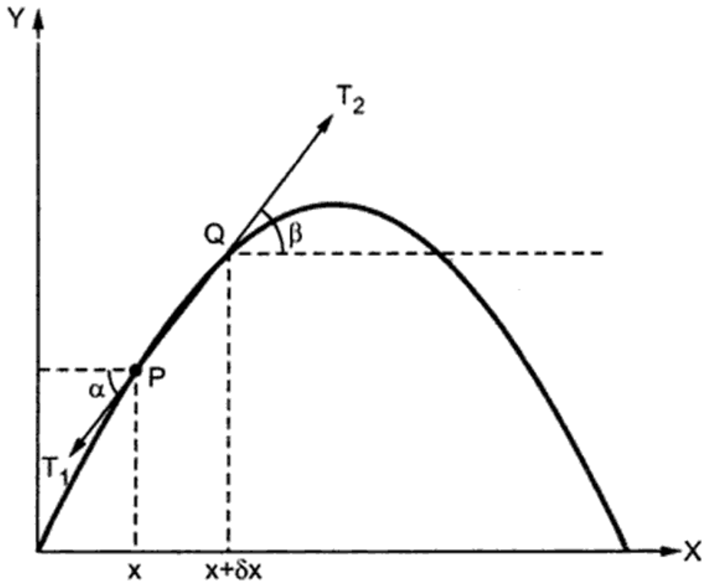


Fig. 4.3

Step 3 : Let m be the mass per unit length of the string. For the motion of an element PQ of length δs, as the string does not offer any resistance to bending, the tension T_1 and P and T_2 at Q are tangential to the curve.

As there is no motion in horizontal direction, we have $T_1 \cos\alpha = T_2 \cos\beta = T$ i.e. a constant α, β are the angle as shown in Figure. Mass of the element PQ is $m\,\delta s$.

Step 4 : By Newton's second law of motion, the equation of motion in vertical direction is

$$m\delta s \frac{\partial^2 y}{\partial t^2} = T_2 \sin\beta - T_1 \sin\alpha$$

or $$\frac{m\delta s}{T}\frac{\partial^2 y}{\partial t^2} = \frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha}$$

or $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\delta s}(\tan\beta - \tan\alpha)$$

or $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\delta s}\left\{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right\}$$

Step 5 : As $\delta s = \delta x$ to a first approximation and $\tan \alpha$, $\tan \beta$ are the slopes at x and $x + \delta x$

$$\text{or} \qquad \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x}\right]$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m} \cdot \frac{\partial^2 y}{\partial t^2}$$

$$\text{or} \qquad \frac{\partial^2 y}{\partial t^2} = c^2 \cdot \frac{\partial^2 y}{\partial t^2} \qquad \text{where } c^2 = \frac{T}{m}$$

which gives transverse vibration of the string. Also known as one dimensional wave equation.

4.7 Solution of Wave Equation

The wave equation is,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \cdot \frac{\partial^2 y}{\partial t^2} \qquad \text{... (1)}$$

Step 1 : Here y is a function of x and t both. Therefore we assume the solution of the form

$$y = XT \qquad \text{... (2)}$$

where X is a function of x alone and T is a function of t alone.

$$\therefore \qquad \frac{\partial^2 y}{\partial t^2} = XT''$$

$$\text{and} \qquad \frac{\partial^2 y}{\partial t^2} = X''T$$

Step 2 : Substituting in equation (1) we have

$$XT'' = c^2 X''T$$

$$\therefore \qquad \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} \qquad \text{... (3)}$$

Step 3 : Here L.H.S. is a function of x alone and R.H.S. is a function of 't' alone. Therefore equation (3) is trues only if both sides are equal to a constant.

$$\therefore \qquad \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k$$

we get $\qquad X'' - kX = 0$ and $T'' - kc^2 T = 0 \qquad$ ... (4)

Step 4 : Case 1 : Now for $k = +m^2$,

$$X'' - m^2 X = 0 \text{ and } T'' - m^2c^2T = 0$$

$$\therefore \quad X = c_1e^{mx} + c_2e^{-mx}$$

and $$T = c_3e^{cmt} + c_4e^{-cmt}$$

$$\therefore \quad Y = \left(c_1e^{mx} + c_2e^{-mx}\right)\left(c_3e^{cmt} + c_4e^{-cmt}\right) \quad \text{... (5)}$$

Case 2 : Now for $k = -m^2$

$$X'' + m^2X = 0 \text{ and } T'' + m^2c^2T = 0$$

$$\therefore \quad X = c_1 \cos mx + c_2 \sin mx$$

$$T = c_3 \cos cmt + c_4 \sin cmt$$

$$\therefore \quad Y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos cmt + c_4 \sin cmt) \quad \text{... (6)}$$

Case 3 : Now for $k = 0$,

$$X'' = 0$$

and $$T'' = 0$$

$$\therefore \quad X = c_1x + c_2 \text{ and } T = c_3t + c_4$$

$$\therefore \quad Y = (c_1x + c_2)(c_3t + c_4) \quad \text{... (7)}$$

Step 5 : We have three possible solutions of equation (1)

$$\therefore \quad Y = \left(c_1e^{mx} + c_2e^{-mx}\right)\left(c_3e^{cmt} + c_4e^{-cmt}\right) \quad \text{... (8)}$$

$$Y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos cmt + c_4 \sin cmt) \quad \text{... (9)}$$

$$\therefore \quad Y = (c_1x + c_2)(c_3t + c_4) \quad \text{... (10)}$$

Step 6 : As we are solving the problem on vibrations, therefore y must be periodic function of x and t. Thus the most suitable solution is

$$Y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos cmt + c_4 \sin cmt) \quad \text{... (11)}$$

4.8 Illustrations

Example 4.10 : *A string is stretched tightly between $x = 0$, $x = L$ and both ends are given displacement $y = a \sin pt$ perpendicular to the string. If the string satisfies the differential equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$ prove that the oscillations of the string are given $y = a \sec \frac{pL}{2c} \cos\left(\frac{px}{c} - \frac{pL}{2c}\right) \sin pt$.*

Solution : **Step 1 :** The solution for above equation is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos cmt + c_4 \sin cmt) \qquad \text{... (1)}$$

Step 2 : Condition (i) is that y (0, t) = a sin pt

Substituting x = 0 in (1)

we get $\quad a \sin pt = (c_1 + 0)(c_3 \cos cmt + c_4 \sin cmt)$

i.e. $\quad a \sin pt = c_1 (c_3 \cos cmt + c_4 \sin cmt)$

Comparing we get

$\Rightarrow c_1 c_3 = 0,\ c_1 c_4 = a$ and $cm = p$ i.e. $m = \dfrac{p}{c}$

$\therefore$ The ratio $c_1 c_3 = 0$ and $c_1 \neq 0 \Rightarrow c_3 = 0$

Step 3 : Substituting in (1) we have

$$y = \left(c_1 \cos \frac{px}{c} + c_2 \sin \frac{px}{c}\right) \cdot c_4 \sin pt$$

$$= \left(a \cos \frac{px}{c} + c_2 c_4 \sin \frac{px}{c}\right) \cdot \sin pt \qquad \text{... (2)}$$

Step 4 : Condition (ii) is y (L, t) = a sin pt gives

$$a \sin pt = \left(a \cos \frac{pL}{c} + c_2 c_4 \sin \frac{pL}{c}\right) \sin pt$$

$$a = a \cos \frac{pL}{c} + c_2 c_4 \sin \frac{pL}{c}$$

$$c_2 c_4 = \frac{a\left(1 - \cos \frac{pL}{c}\right)}{\sin \frac{pL}{c}} = \frac{2a \sin^2 \frac{pL}{2c}}{2 \sin \frac{pL}{2c} \cos \frac{pL}{2c}}$$

$$c_2 c_4 = a \frac{\sin \frac{pL}{c}}{\cos \frac{pL}{c}}$$

Step 5 : Substituting in (2)

$$y(x, t) = \left[a \cos \frac{px}{c} + a \frac{\sin \frac{pL}{c}}{\cos \frac{pL}{c}} \sin \frac{px}{c}\right] \sin pt$$

$$= \frac{a\left[\cos \frac{px}{c} \cos \frac{pL}{2c} + \sin \frac{px}{c} \sin \frac{pL}{2c}\right] \sin pt}{\cos \frac{pL}{2c}}$$

$\Rightarrow \qquad y(x, t) = a \sec \dfrac{pL}{2c} \cos\left(\dfrac{px}{c} - \dfrac{pL}{2c}\right) \sin pt$ is the required solution.

Example 4.11 : *If* $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ *represents the vibrations of a string of length L fixed at both ends find the solution with boundary conditions,*

i) $y(0, t) = 0$, *ii)* $y(L, t) = 0$, *iii)* $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$, *iv)* $y(x,0) = k(Lx - x^2)$, $0 \le x \le L$.

Solution : Step 1 : The solution for above equation is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos amt + c_4 \sin amt) \qquad \text{... (1)}$$

Step 2 : Condition (i) is that $y(0, t) = 0$ substituting $x = 0$ in (1) we get

$$0 = (c_1 + 0)(c_3 \cos amt + c_4 \sin amt)$$

$$0 = c_1 (c_3 \cos amt + c_4 \sin amt) \Rightarrow c_1 = 0$$

Step 3 : Substituting in (1) we get

$$y = c_2 \sin mx (c_3 \cos amt + c_4 \sin amt)$$

$$\Rightarrow \qquad y = \sin mx (c_5 \cos amt + c_6 \sin amt) \qquad \text{...(2)}$$

Step 4 : Differentiating w.r.t. t we get

$$\frac{\partial y}{\partial t} = \sin mx [-c_5\, am \sin amt + c_6\, am \cos amt] \qquad \text{... (3)}$$

Step 5 : Condition (iii) is $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$ gives

$$0 = \sin mx (a\, mc_6) \text{ which requires } c_6 = 0$$

Step 6 : Substituting $c_6 = 0$ in (2) we get

$$y = c_5 \sin mx \cos amt \qquad \text{... (4)}$$

Step 7 : Condition (ii) is $y(L, t) = 0$ gives

$$0 = c_5 \sin mL \cos amt$$

Condition (ii) is satisfied only if $\sin ML = 0$

$\Rightarrow mL = n\pi$

$$\therefore \qquad m = \frac{n\pi}{L},\ n = 1, 2, 3 \ldots$$

Step 8 : Substituting in (4)

$$y = c_5 \sin \frac{n\pi x}{L} \cos \frac{an\pi t}{L} \qquad \text{... (5)}$$

Step 9 : Taking n = 1, 2, 3, ... and varying c_5 as b_1, b_2, b_3, ... the general solution of the given equation is given by

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \cos\frac{an\pi t}{L} \quad \text{... (6)}$$

Step 10 : Finally condition (iv) is

$$y(x, 0) = k\left(Lx - x^2\right), \quad 0 \le x \le L$$

Substituting t = 0 in (6) we get

$$k\left(Lx - x^2\right) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \text{ in } 0 \le x \le L$$

which represents Fourier half range sine series for y (x, 0) in interval (0, l)

Step 11 : The formula for b_n is

$$b_n = \frac{2}{L}\int_0^L y(x, 0) \sin\frac{n\pi x}{L}\, dx$$

$$\therefore \quad b_n = \frac{2}{L}\int_0^L k\left(Lx - x^2\right) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{2k}{L}\left\{\left(Lx - x^2\right)\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (L-2x)\left(-\frac{\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{L}}{\frac{n^3\pi^3}{L^3}}\right)\right\}_0^L$$

$$= \frac{2k}{L}\left[-\frac{2L^3}{n^3\pi^3}(\cos n\pi - 1)\right]$$

$$\Rightarrow \quad b_n = \frac{4kL^3}{n^3\pi^3}\left[1 - (-1)^n\right]$$

Step 12 : Substituting in (6) we have

$$y(x, t) = \frac{4kL^2}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{n^3}\left[1 - (-1)^n\right] \sin\frac{n\pi x}{L} \cos\frac{an\pi t}{L}$$

$$= \frac{8L^3k}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{L}\cos\frac{a\pi\pi}{L} + \frac{1}{3^3}\sin\frac{3\pi\pi}{L}\sin\frac{3a\pi t}{L} + \ldots\right]$$

$$= \frac{8L^3k}{\pi^3}\sum_{r=0}^{\infty} \frac{1}{(2r+1)^3}\sin\frac{(2r+1)\pi x}{L}\cos\frac{(2r+1)a\pi t}{L}$$

Example 4.12 : *An elastic string stretched between two fixed points at a distance L apart one end is taken at the origin and at a distance $\frac{2L}{3}$ from this end the string displaced a distance "a" transversely and is released from rest when in this position. Find y (x, t) the vertical displacement, if y satisfies the equation* $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$.

Solution :

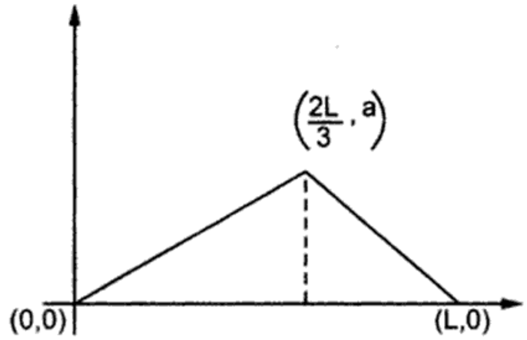


Fig. 4.4

To solve $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ with boundary conditions,

i) y (0, t) = 0, ii) y (L, t) = 0 iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ iv) $y(x, 0) = \begin{cases} \frac{3a}{L} x & 0 < x < \frac{3L}{3} \\ \frac{3a}{L}(L-x) & \frac{2L}{3} < x < L \end{cases}$

As the first three conditions are same the solution is same till equation (6) of problem no (4.11).

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \cos\frac{an\pi t}{L} \qquad \dots (7)$$

Step 10 : Finally condition (iv) is

$$y(x, 0) = \begin{cases} \frac{3a}{L} x & 0 < x < \frac{3L}{3} \\ \frac{3a}{L}(L-x) & \frac{2L}{3} < x < L \end{cases}$$

Substituting t = 0 in (6) we get

$$y(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

which represents Fourier half range sine series for y (x, 0) in interval (0, *l*)

Step 11 : The formula for b_n is

$$b_n = \frac{2}{L}\int_0^L y(x,0)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left[\int_0^{2L/3}\frac{3a}{2L}x\sin\frac{n\pi x}{L}\,dx+\int_{2L/3}^{L}\frac{3a}{L}(L-x)\sin\frac{n\pi x}{L}\,dx\right]$$

$$= \frac{6a}{L^2}\left\{\frac{x}{2}\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right)-\left(\frac{1}{2}\right)\left(-\frac{\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right\}_0^{2L/3}$$

$$+\left\{(L-x)\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right)-(-1)\left(-\frac{\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right\}_{2L/3}^{L}$$

$$= \frac{6a}{L^2}\left[-\frac{L}{2n\pi}\left\{\frac{2L}{3}\cos\frac{2n\pi}{3}+\frac{L^2}{2n^2\pi^2}\sin\frac{2n\pi}{3}\right.\right.$$

$$\left.\left.+\frac{L}{n\pi}\frac{1}{3}\cos\frac{2n\pi}{3}+\frac{L^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right\}\right]$$

$$= \frac{6a}{L^2}\cdot\frac{3}{2}\frac{L^2}{n^2\pi^2}\sin\frac{2n\pi}{3}$$

$$\Rightarrow \qquad b_n = \frac{9a}{n^2\pi^2}\sin\frac{2n\pi}{3}$$

Step 12 : Substituting in (7) we have

$$y(x,t) = \frac{9a}{n^2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{2n\pi}{3}\sin\frac{n\pi x}{L}\cos\frac{cn\pi t}{L}$$

Example 4.13 : *A string is stretched and fastened to two points L apart. Motion is started by displacing the string in the form $u = a\sin\frac{\pi x}{L}$ from which it is released at time $t = 0$. Find the displacement $u(x, t)$ from one end.*

Solution :

To solve $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ with boundary conditions,

i) $y(0, t) = 0$ ii) $y(L, t) = 0$ iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ iv) $y(x, 0) = a\sin\frac{\pi x}{L}$

As the first three conditions are same the solution is same till equation (6) of problem no (2).

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{an\pi t}{L} \qquad \ldots (6)$$

Step 10 : Finally condition (iv) is $y(x, 0) = a\sin\frac{\pi x}{L}$

Substituting t = 0 in (6) we get

$$y(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

i.e.

$$a\sin\frac{\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$= b_1\sin\frac{\pi x}{L} + b_2\sin\frac{2\pi x}{L} + b_3\sin\frac{3\pi x}{L} + \ldots$$

Comparing coefficients we get

$b_1 = a,\ b_2 = 0,\ b_3 = 0,\ b_4 = b_5 = \ldots = 0$

Step 11 : Substituting in (6) we get

$$y = b_1\sin\frac{n\pi}{L}\cos\frac{a\pi t}{L}$$

$$y(x, t) = a\sin\frac{\pi x}{L}\cos\frac{a\pi t}{L}$$

Example 4.14 : *A tightly stretched string with fixed end points x = 0 and x = L is initially in a position given by* $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{L}\right)$ *If it is released from rest from this position find the displacement y at any distance x from one end and at any time t.*

Solution :

To solve $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ with boundary conditions,

i) $y(0, t) = 0$, ii) $y(L, t) = 0$, iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ iv) $y(x, 0) = y\sin^3\frac{x\pi}{L}$

As the first three conditions are same the solution is same till equation (6) of problem no (4.11).

$$y = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{an\pi t}{L} \qquad \ldots (7)$$

Step 10 : Finally condition (iv) is $y(x, 0) = y_0 \sin^3 \frac{\pi x}{L}$

Substituting t = 0 in (6) we get

$$y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$y_0 \sin^3 \frac{\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Using $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\sin^3 \frac{\pi x}{L} = \frac{3}{4}\sin\frac{\pi x}{L} - \frac{1}{4}\sin\frac{3\pi x}{L}$$

$$\frac{3y_0}{4}\sin\frac{\pi x}{L} - \frac{y_0}{4}\sin\frac{3\pi \pi}{L} = \sum_{n=L}^{\infty} b_n \sin\frac{n\pi \pi}{L}$$

$$= b_1 \sin\frac{\pi x}{L} + b_2 \sin\frac{2\pi x}{L} + b_3 \sin\frac{3\pi x}{L} + \ldots$$

Comparing coefficients we get

$b_1 = \frac{3y_0}{4}$, $b_2 = 0$, $b_3 = -\frac{y_0}{4}$, $b_4 = b_5 = \ldots = 0$

Step 11 : Substituting in (6) we get

$$y(x, t) = \frac{3y_0}{4}\sin\frac{\pi x}{L}\cos\frac{c\pi t}{L} - \frac{y_0}{4}\sin\frac{3\pi x}{L}\cos\frac{c\pi t}{L}$$

Example 4.15 : *A string of length l and is initially at rest in its equilibrium position and each of its points is given a velocity v (x) such that* $v(x) = \begin{cases} cx & 0 \le x \le l/2 \\ c(l-x) & l/2 \le x \le l \end{cases}$ *obtain the displacement y at any time t.*

Solution : We have to solve

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Subject to the conditions

i) $y(0, t) = 0$, ii) $y(L, t) = 0$, iii) $y(x, 0) = 0$, iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \begin{cases} cx & 0 \le x \le l/2 \\ c(l-x) & l/2 \le x \le l \end{cases}$

where y is the displacement.

Step 1 : The solution of above equation is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos amt + c_4 \sin amt) \quad \ldots (1)$$

Step 2 : Condition (i) is that y (0, t) = 0 substituting x = 0 in (1) we get.

$$0 = (c_1 + 0)(c_3 \cos amt + c_4 \sin amt)$$

$$0 = c_1 (c_3 \cos amt + c_4 \sin amt) \Rightarrow c_1 = 0$$

Step 3 : Substituting $c_1 = 0$ in (1) we get

$$y = c_2 \sin mx (c_3 \cos amt + c_4 \sin amt)$$

$$\Rightarrow \quad y = \sin mx (c_5 \cos amt + c_6 \sin amt) \quad \text{... (2)}$$

Step 4 : Condition (iii) i.e. y (x, 0) = 0, substituting t = 0 in (2) we get.

$$0 = (\sin mx)(c_5 + 0)$$

$$0 = c_5 = 0$$

Step 5 : Substituting in (2) we get

$$y = c_6 \sin mx \sin amt \quad \text{... (3)}$$

Step 6 : Using **condition** (ii) we have y (L, t) = 0, substituting x = L in (3) we get.

$$0 = c_6 \sin mL \sin amt \Rightarrow \sin mL = 0, \therefore mL = n\pi$$

$$m = \frac{n\pi}{L}$$

– Substituting in (3) we get

$$y = c_6 \sin \frac{n\pi x}{L} \sin \frac{an\pi t}{L}$$

Step 7 : This is the solution of given equation with different values of n thus the general solution is given by

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sin \frac{an\pi t}{L} \quad \text{... (4)}$$

Step 8 : Differentiating w.r.t. t.

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \frac{an\pi}{L} \sin \frac{n\pi\pi}{L} \cos \frac{an\pi t}{L} \quad \text{... (5)}$$

Step 9 : Condition (iv) is

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \begin{cases} cx & 0 \le x \le l/2 \\ c\,(l - x) & l/2 \le x \le l \end{cases}$$

Substituting t = 0 in (5) we get.

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} b_n \frac{an\pi}{L} \sin\frac{n\pi\pi}{L}$$

$$v(x) = \sum_{n}^{\infty} B_n \sin\frac{n\pi x}{L} \qquad \text{where } B_n = b_n \cdot \frac{an\pi}{l}$$

Step 10 : This is half range Fourier sine series.

$$B_n = \frac{2}{L}\int_0^L v(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} cx\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} c(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{L}\left[\left\{cx\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (1)\left(-\frac{\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right\}_0^{L/2}\right.$$

$$\left. + \left\{c(L-x)\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - c(-1)\left(-\frac{\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right\}_{L/2}^{L}\right]$$

$$= \frac{2}{L}\left[\left\{cx\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - c(1)\left(-\frac{\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right\}_0^{L/2}\right.$$

$$\left. + \left\{c(L-x)\left(-\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - c(-1)\left(-\frac{\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right\}_{L/2}^{L}\right]$$

$$= \frac{2c}{l}\left\{\left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} - 0 - 0\right]\right.$$

$$\left.\left[0 - 0 - \frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\right\}$$

$$= \frac{2c}{l}\,\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$B_n = \frac{4cl}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$b_n \cdot \frac{an\pi}{l} = \frac{4cl}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$b_n = \frac{4cl}{an^3\pi^3}\sin\frac{n\pi}{2}$$

Step 11 : Substituting in (4) we get

$$y = \sum_{n=1}^{\infty} \frac{4cl^2}{an^3\pi^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\sin\frac{an\pi t}{L}$$

Example 4.16 : *A tightly stretched string with fixed end point $x = 0$ and $x = L$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its point a velocity $\lambda x(L-x)$. Find the displacement of the string at any distance x from one end at any time t.*

Solution : We have to solve

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

Subject to the conditions

i) $y(0, t) = 0$ ii) $y(L, t) = 0$ iii) $y(x, 0) = 0$ iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l-x)$ for $0 < x < l$

As the first three conditions are same the solution is the same till equation (4) of problem no (4.15).

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\sin\frac{an\pi t}{L} \quad \text{... (4)}$$

Step 8 : Differentiating w.r.t. t

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \frac{an\pi}{L}\sin\frac{n\pi\pi}{L}\cos\frac{an\pi t}{L} \quad \text{... (5)}$$

Step 9 : Condition (iv) is

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l-x) \text{ for } 0 < x < l$$

Substituting $t = 0$ in (5) we get.

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} b_n \frac{an\pi}{L}\sin\frac{n\pi\pi}{L}$$

$$v(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \qquad \text{where } B_n = b_n \cdot \frac{an\pi}{l}$$

Step 10 : This is half range Fourier sine series.

$$B_n = \frac{2}{L}\int_0^L v(x)\sin\frac{n\pi x}{L}\,dx$$

$$\therefore \quad B_n = \frac{2}{L}\int_0^L \lambda x(L-x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2\lambda}{L}\left[x(L-x)\left\{\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\right\} - (l-2x)\left\{\frac{-L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right\}\right.$$

$$\left. + (-2)\left\{+\frac{L^3}{n^3\pi^3}\cos\frac{n\pi x}{L}\right\}\right]_0^L$$

$$B_n = \frac{4\lambda L^2}{n^3\pi^3}(1-\cos n\pi)$$

$$\frac{\pi a n}{L} b_n = \begin{cases} 0 & \text{if n is even} \\ \dfrac{8\lambda L^2}{n^3\pi^3} & \text{if n is odd} \end{cases}$$

replacing n by (2n – 1) for n being odd

$$\therefore \quad \frac{\pi a(2n-1)}{L} b_n = \frac{8\lambda L^2}{(2n-1)^3\pi^3}$$

$$\therefore \quad b_n = \frac{8\lambda L^2}{c\pi^4}\cdot\frac{1}{(2n-1)^4}$$

Step 11 : Substituting in (4) we get

$\therefore$ required solution is y (x, t)

$$= \frac{8\lambda L^2}{c\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi ct}{L}\sin\frac{(2n-1)\pi x}{L}$$

Example 4.17 : *Solve the boundary values problem* $\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}$, *y (0, t) = 0 = y (5, t) and* $y(x, 0) = 0, \left(\frac{\partial y}{\partial t}\right)_{t=0} = 5\sin \pi x.$

Solution : We have to solve

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \qquad \text{(where a = 2 and L = 5)}$$

Subject to the conditions

i) y (0, t) = 0 ii) y (5, t) = 0 iii) y (x, 0) = 0 iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 5\sin\pi x$

As the first three conditions are same the solution is same till equation (4) of problem no (4.15).

$$\therefore \quad y(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi 2t}{5} \sin\frac{n\pi x}{5} \quad \text{... (4)}$$

Step 8 : Differentiating w.r.t. t

$$\therefore \quad \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi 2}{5} b_n \cos\frac{n\pi 2t}{5} \sin\frac{n\pi x}{5} \quad \text{... (5)}$$

Step 9 : Condition (iv) is

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 5 \sin \pi x \quad \text{for } 0 < x < 5$$

Substituting t = 0 in (5) we get.

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} b_n \frac{2n\pi}{5} \sin\frac{n\pi\pi}{5}$$

$$5 \sin \pi x = \sum_{n=1}^{\infty} \frac{2n\pi}{5} b_n \sin\frac{n\pi x}{5}$$

$$v(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{5} \qquad \text{where } B_n = b_n \cdot \frac{2n\pi}{5}$$

Step 10 : This is half range Fourier sine series.

$$B_n = \frac{2}{L}\int_0^L v(x) \sin\frac{n\pi x}{L}\, dx$$

$$\therefore \quad B_n = \frac{2}{5}\int_0^5 5 \sin \pi x \sin\frac{n\pi x}{5}\, dx$$

$$\therefore \quad \frac{2n\pi}{5} b_n = \frac{2}{5}\int_0^5 5 \sin \pi x \sin\frac{n\pi x}{5}\, dx$$

$$\therefore \quad \frac{2n\pi}{5} b_n = \int_0^5 \cos(n-5)\frac{\pi x}{5} - \cos(n+5)\frac{\pi x}{5}\, dx$$

$$= \left\{ \frac{\sin(n-5)\frac{\pi x}{5}}{\frac{(n-5)\pi}{5}} - \frac{\sin(n+5)\frac{\pi x}{5}}{\frac{(n+5)\pi}{5}} \right\}_0^5$$

$$= 0 \text{ for all n except n} = 5$$

If n = 5 then

$$\therefore \quad \frac{n\pi 2}{5} b_n = \frac{2}{5}\int_0^5 5 \sin \pi x \sin \frac{n\pi x}{5} dx \text{ becomes}$$

$$\therefore \quad 2\pi b_5 = \frac{2}{5}\int_0^5 5 \sin^2 \pi x \, dx$$

$$\therefore \quad 2\pi b_5 = \int_0^5 1 - \cos 2\pi x \, dx$$

$$\therefore \quad 2\pi b_5 = \left[x - \frac{\sin 2\pi x}{2\pi}\right]_0^5$$

$$\therefore \quad 2\pi b_5 = 5$$

$$\therefore \quad b_5 = \frac{5}{2\pi}$$

Step 11 : Substituting in (4) we get,

Thus the required solution is

$$y(x, t) = 5 \sin \pi x \sin 2\pi t$$

Exercise 4.2

1. *A string is stretched and fastened to two point distance L apart is displaced into the form $y(x,0) = k\left(Lx - x^2\right)$ from which it is released at $t = 0$. Find the displacement of the string at a distance x from one end.* **[Ans. : $\frac{8kL^2}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{L} \cos \frac{(2n-1)\pi at}{L}$]**

2. *An elastic string is stretched between two points at a distance L apart. One end is taken as origin and point $x = \frac{3L}{4}$ is displaced through distance d perpendicular to x-axis and released from rest from these position. Obtain the displacement.* **[Ans. : $\frac{32d}{3\pi^2}\sum_{n}^{\infty} \frac{1}{n^2} \sin \frac{3n\pi}{4} \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L}$]**

3. *A string of length L fixed as its ends satisfies the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$. Find the solution if the string has initial triangular deflection given by :* $y(x, 0) = \begin{cases} \frac{2k}{L} x & 0 \le x \le \frac{L}{2} \\ \frac{2k}{L}(L-x) & \frac{L}{2} \le x \le L \end{cases}$ *and initial velocity zero.* **[Ans. : $Y(x, t) = \frac{8k}{\pi^2}\sum_{1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{L} \cos \frac{(2n-1)\pi ct}{L}$]**

4. *A taut string of length 2L is fastened at both ends. The mid point of the string is taken to a height b and then released from rest in that position. Obtain the displacement.*

Hint : *Use wave equation* $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

$Y(0, t) = 0;\ \ y(2L, t) = 0, \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

$Y(x, 0) = \frac{bx}{L},\ 0 \le x \le L$

$= \frac{b}{L}(2L - x),\ L \le x \le 2L$

Use formula for $b_n = \frac{2}{2L}\int_0^{2L} f(x) \sin \frac{n\pi x}{2L} dx$ **[Ans. : $y(x,t) = \frac{8b}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{2} \sin\frac{n\pi x}{2L} \cos\left(\frac{n\pi ct}{2L}\right)$]**

5. *A tightly stretched string with fixed ends* $x = 0$ *and* $x = L$ *is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity* $v(x) = 3x(L - x)$ *for* $0 < x < L$, *find the displacement.* **[Ans. : $y(x, t) = \frac{24L^3}{\pi^4 a}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\frac{(2n-1)\pi x}{L} \sin\frac{(2n-1)\pi at}{L}$]**

6. *Tightly stretched string with fixed ends* $x = 0$ *and* $x = L$ *is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity* $v_0 \sin^3 \frac{\pi x}{L}$, $0 \le x \le L$, *find the displacement.* **[Ans. : $y(x, t) = \sin^3 \frac{v_0 L}{4\pi a}\left[\sin\frac{\pi x}{L}\sin\frac{\pi at}{L} - \frac{1}{9}\sin\frac{3\pi x}{L}\sin\frac{3\pi at}{L}\right]$]**

7. *A uniform string stretched between the point* $x = 0$ *and* $x = L$ *is given the initial displacement*

$$y(x,0) = \sin\frac{\pi x}{L},\ 0 < x < L \text{ and initial velocity } v(x) = \begin{cases} 0, & 0 < x < \frac{L}{4} \\ a, & \frac{L}{4} < x < \frac{3L}{4} \\ 0, & \frac{3L}{4} < x < L \end{cases}$$

Find subsequent displacement.

[Ans. : $y(x, t) = \sin\frac{\pi x}{L}\cos\frac{\pi at}{L} + \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{\sin\frac{n\pi}{2}\sin\frac{n\pi}{4}\sin\frac{n\pi x}{L}\sin\frac{n\pi at}{L}}{n^2}\right)$]

8. *Find the deflection* $u(x, t)$ *of a vibrating string (length* $L = \pi$, *ends fixed and* $c^2 = \frac{T}{\rho} = 1$) *corresponding to zero velocity and initial deflection* $0.01(\pi - x)$.

[Ans. : $u(x, t) = \frac{0.04}{\pi}\sum_{1}^{\infty} \frac{1 - \cos n\pi}{n^3} \sin nx \cos nt$]

9. *A flexible string of length* π *is tightly stretched between* $x = 0$, $x = \pi$, *on x-axis, its ends being fixed at these points. When set into small transverse vibration, the displacement* $y(x, t)$ *from x-axis of any point x at time t is given by* $\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}$ *Find the solution of equation which satisfies* $y(0, t) = 0,\ y(\pi, t) = 0, \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ *and* $y(x, 0) = 0.1 \sin x + 0.01 \sin 4x$ *for* $0 < x \le \pi$.

[Ans. : $y(x, t) = 0.1 \sin x \cos 2t + 0.01 \sin 4x \cos 8t$]

Type 3 Laplace's Equations

4.9 Formation of Two Dimensional Heat Flow Equation

Consider the flow of heat in a metal plate in XOY plate. If the temperature at a point does not depend upon z co-ordinate and it depends only on x, y and t, then the flow is called two dimensional and the heat flow lies in XOY plane only and is zero along the z axis.

Consider a rectangular element of the plate with sides δx and δy and thickness 'h'. As studied in one dimensional heat flow along a bar,

Step 1 : The quantity of heat that enters the plate per second from the sides AB is given by $-kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_y$ and

Step 2 : The quantity of heat that enters the plate per second from the sides AD is given by $-kh\delta y\left(\frac{\partial u}{\partial x}\right)_x$

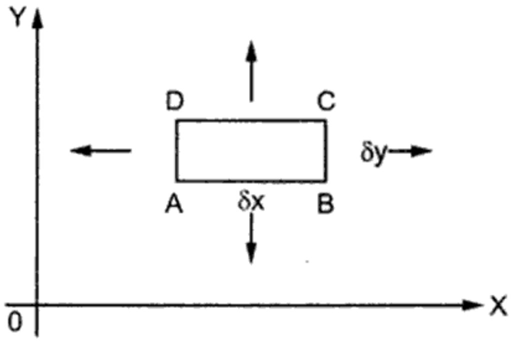


Fig. 4.5

Step 3 : Heat flowing out through sides CD and BC per second is $-kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}$ and

Step 4 : Heat flowing out through sides and BC per second is $-kh\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$

Step 5 : Therefore, the total gain of heat by rectangular plate ABCD per second,

$$= -kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_y - kh\delta y\left(\frac{\partial u}{\partial x}\right)_x + kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + kh\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= kh\,\delta x\,\delta y\left\{\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_x}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right\}$$

Step 6 : The rate of gain of heat by the plate is also given by

$S\rho\, h\,\delta x\, \delta y\left(\frac{\partial u}{\partial t}\right)$ where S = specific heat, ρ = density of the plate.

Step 7 : From **step 5** and **step 6**

$$S\rho\, h\,\delta x\,\delta y\left(\frac{\partial u}{\partial t}\right) = kh\,\delta x\,\delta y\left\{\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_{y}}{\delta y}\right\}$$

i.e.
$$\frac{\partial u}{\partial t} = \frac{k}{S\rho}\left\{\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_{y}}{\delta y}\right\}$$

Step 8 : Taking the limit as $\delta x \to 0$ and $\delta y \to 0$ we get

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

This equation gives the temperature distribution of the plate in the transition state.

Where $\frac{k}{S\rho} = c^2$ is the diffusity.

Step 9 : For steady state $\left(\frac{\partial u}{\partial t}\right) = 0$ therefore we get

$$\boxed{\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = 0}$$

is called Laplace's equation in two dimensions.

4.10 Solution of Two Dimensional Heat Flow Equation

Two dimensional Laplace's equation i.e. heat flow equation in steady state is given by

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{... (1)}$$

Solution :

Step 1 : Let u = X Y be the solution where X, Y are functions of x, y respectively.

Then
$$\frac{\partial^2 u}{\partial x^2} = X''Y \text{ and } \frac{\partial^2 u}{\partial y^2} = XY''$$

Step 2 : Substituting in these in (1)

$$X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y}$$

As x and y are independent variables and L.H.S. is a function of x alone and R.H.S. is a function of y along therefore we must have,

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{constant k}$$

$$\Rightarrow \qquad \frac{X''}{X} = k, -\frac{Y''}{Y} = -k \qquad \text{... (2)}$$

Step 3 : The constant 'k' may be positive or negative or zero.

In **case 1 :** $k = -m^2$

Hence equation (2) becomes

$$X'' + m^2X = 0 \text{ and } Y'' - m^2y = 0$$

G.S. for X is $X = c_1 \cos mx + c_2 \sin mx$

$\therefore$ G.S. for Y is $Y = c_3e^{my} + c_4e^{-my}$

Case 2 : $k = m^2$

Hence equation (2) becomes

$$X'' - m^2X = 0 \text{ and } Y'' + m^2y = 0$$

The G.S. of these are

$$X = c_1e^{mx} + c_2e^{-mx}$$

$$Y = c_3\cos my + c_4 \sin my$$

Case 3 : $k = 0$

Hence equation (2) becomes

$$X'' = 0 \text{ and } Y'' = 0$$

$$\therefore \qquad X = (c_1 + c_2x), Y = (c_3 + c_4y)$$

Step 4 : Therefore three possible solutions are,

$$\boxed{\begin{aligned} u &= (c_1\cos mx + c_2\sin mx)\,(c_3e^{my} + c_4e^{-my}) \\ \text{or } u &= (c_1e^{mx} + c_2e^{-mx})\,(c_3\cos my + c_4\sin my) \\ \text{or } u &= (c_1 + c_2x)\,(c_3 + c_4y) \end{aligned}}$$

Note : If the plate is infinite i.e. it is too long as compared to its breadth then temperature at infinity is always to be taken as zero i.e.

i) If the plate is infinite along Y axis then we write $u(x,\infty) = 0$ for $\forall$ x and

ii) If the plate is infinite along X axis then we write $u(\infty, y) = 0$ for $\forall$ y.

Note : Procedure for selecting the suitable solution :

i) When one of the condition is $u = 0$ for $y \to \infty$ for all x between (0, L) i.e. $u(\infty, y) = 0 \ \forall$ x in (0, L). Then we should take $k = -m^2$ and the most suitable solution is

$$\boxed{u = (c_1 \cos mx + c_2 \sin mx)\left(c_3 e^{my} + c_4 e^{-my}\right)}$$

ii) When one of the condition is $u = 0$ for $x \to \infty$ for all y between (0, L) i.e. $u(x,\infty) = 0 \ \forall$ y in (0, L). Then we should take $k = m^2$ and the most suitable solution is

$$\boxed{u = \left(c_1 e^{mx} + c_2 e^{-mx}\right)(c_3 \cos my + c_4 \sin my)}$$

iii) The case $k = 0$ will never occur.

iv) When the constant $k = -m^2$ the solution can be written as

$$\boxed{u = (c_1 \cos mx + c_2 \sin mx)(c_3 \cosh my + c_4 \sinh my)}$$

v) When the constant $k = m^2$ the solution can be written as

$$\boxed{u = (c_1 \cosh mx + c_2 \sinh mx)(c_3 \cos my + c_4 \sin my)}$$

4.11 Illustrations

Example 4.18 : *An infinitely long uniform metal plate is enclosed between lines $y = 0$ and $y = L$ for $x > 0$. The temperature is zero along the edges $y = 0$, $y = L$ and at infinity. If the edge $x = 0$ is kept at a constant temperature u_0 find the temperature distribution $u(x,y)$.*

Solution :

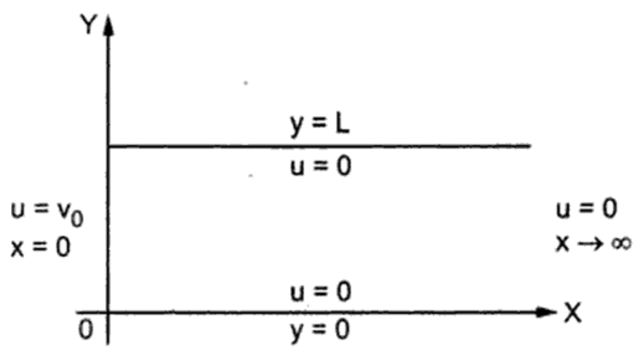


Fig. 4.6

Step 1 : We have to solve

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{... (1)}$$

Subject to boundary conditions

i) $u(x, 0) = 0$ ii) $u(x, L) = 0$ iii) $u(\infty, y) = 0$ iv) $u(0, y) = u_0$, $0 < y < L$

Step 2 : Since the plate extends to infinity along X-axis, the solution of equation (1) is given by

$$u = \left(c_1 e^{mx} + c_2 e^{-mx}\right)\left(c_3 \cos my + c_4 \sin my\right) \qquad \text{... (1 (a))}$$

Step 3 : Now the condition (1) is $u = 0$ when $y = 0$.

Substituting in (1) we get

$$0 = \left(c_1 e^{mx} + c_2 e^{-mx}\right)\left(c_3 \cos 0 + c_4 \sin 0\right)$$

$$0 = c_3\left(c_1 e^{mx} + c_2 e^{-mx}\right)$$

which is possible only if $c_3 = 0$

Step 4 : Substituting in (1 (a)) we get

$$u = c_4 \sin my\left(c_1 e^{mx} + c_2 e^{-mx}\right)$$

$$u = \sin my\left(c_5 e^{mx} + c_6 e^{-mx}\right) \qquad \text{... (2)}$$

Step 5 : Now applying condition (ii) $u = 0$, $y = L$ we get

$$u = \sin mL\left(c_5 e^{mx} + c_6 e^{-mx}\right)$$

which is possible only if $\sin mL = 0 \Rightarrow mL = n\pi$

where n is any positive integer.

Substituting $m = \frac{n\pi}{L}$ in (2) we get

$$u = \sin\frac{n\pi y}{L}\left(c_5 e^{\frac{n\pi x}{L}} + c_6 e^{-\frac{n\pi x}{L}}\right) \qquad \text{... (3)}$$

Step 6 : Now applying condition (iii) $u = 0$ at $x = \infty$ gives

$$0 = \sin\frac{n\pi y}{L}\left(c_5 e^{\infty} + c_6 e^{-\infty}\right)$$

Which is possible only if $c_5 = 0$

Substituting $c_5 = 0$ in (3) we get

$$u = \sin\frac{n\pi y}{L}\left(c_6 e^{-\frac{n\pi x}{L}}\right)$$

Step 7 : Taking n = 1, 2, 3, ... and varying c_6 as b_1, b_2, b_3, ... its general solution is

$$u = \sum_{n=1}^{\infty} b_n e^{\frac{-n\pi x}{L}} \sin\frac{n\pi y}{L} \qquad \ldots (4)$$

Step 8 : Finally using last condition (iv) $u(0, y) = u_0$ equation (5) becomes

$$u_0 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{L},\ 0 < y < L \qquad \ldots (5)$$

where b_n = coefficient of half range Fourier sine series for $f(y) = u_0$ in interval (0, L) therefore we find half range

Fourier sine series for $f(y) = u_0$ in (0, L) i.e.

Step 9 :

where $$b_n = \frac{2}{L}\int_0^L f(y)\sin\frac{n\pi y}{L}\,dy$$

$$b_n = \frac{2}{L}\int_0^L u_0 \sin\frac{n\pi y}{L}\,dy = \frac{2u_0}{L}\left[-\frac{\cos\frac{n\pi y}{L}}{\frac{n\pi}{L}}\right]_0^L$$

$$b_n = \frac{2u_0}{n\pi}\left[1-(-1)^n\right]$$

$$b_n = \begin{cases} \dfrac{4u_0}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Step 10 : Substituting in (4) we have

$$u = \sum_{n=1}^{\infty} \frac{4u_0}{n\pi} e^{\frac{-n\pi x}{L}} \sin\frac{n\pi y}{L} \quad \text{if } n \text{ is odd}$$

Step 11 : Replacing n by (2n + 1) we get

$$u = \frac{4u_0}{\pi}\sum_{n=0}^{\infty}\left[\frac{e^{\frac{-(2n+1)\pi x}{L}}}{(2n+1)} \sin\frac{(2n+1)\pi y}{L}\right]$$

Example 4.19 : *Solve the equation* $\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}=0$ *with conditions*

i) V = 0 when $y \to +\infty$ for all X.

ii) V = 0 when x = 0 for all values of y.

iii) V = 0 when x = 1 for all y.

iv) V = x (1 − x) when y = 0 for 0 < x < 1.

Solution : Step 1 : The boundary conditions are

i) $V(0, y) = 0 \ \forall \ y$

ii) $V(1, y) = 0 \ \forall \ y$

iii) $V(x, \infty) = 0$

iv) $V = x(1-x)$ at $y = 0 \quad 0 < x < 1$

Step 2 : Since the plate is infinitely long along Y axis, the G.S. of given equation is,

$$V = (c_1 \cos mx + c_2 \sin mx)(c_3 e^{my} + c_4 e^{-my}) \quad \text{... (1)}$$

Step 3 : Now the condition (1) is V = 0 when x = 0

Substituting in (1) we get

$$0 = (c_1 \cos 0 + c_2 \sin 0)(c_3 e^{my} + c_4 e^{-my})$$

$$0 = c_1 (c_3 e^{my} + c_4 e^{-my})$$

which is possible only if $c_1 = 0$

Step 4 : Substituting in (1) we get

$$V = c_2 \sin mx (c_3 e^{my} + c_4 e^{-my})$$

$$V = \sin mx (c_5 e^{my} + c_6 e^{-my}) \quad \text{... (2)}$$

Step 5 : Now applying condition (ii) V = 0, x = 1 we get

$$0 = \sin m (c_5 e^{my} + c_6 e^{-my})$$

which is possible only if $\sin m = 0 \Rightarrow m = n\pi$

where n is any positive integer.

Substituting in (2) we get

$$v = \sin n\pi x (c_5 e^{n\pi y} + c_6 e^{-n\pi y}) \quad \text{... (3)}$$

Step 6 : Now applying condition (iii) $V = 0$ at $y = \infty$ gives

$$0 = \sin n\pi x (c_5 e^{\infty} + c_6 e^{-\infty})$$

which is possible only if $c_5 = 0$

Substituting $c_5 = 0$ in (3) we get

$$V = c_6 \sin n\pi x \, e^{-n\pi y}$$

Step 7 : Taking n = 1, 2, 3, ... and varying c_6 as b_1, b_2, b_3 ... Its general solution is

$$V = \sum_{n=1}^{\infty} b_n \sin n\pi x \, e^{-n\pi y} \quad \text{... (4)}$$

Step 8 : Now applying condition (iv) at y = 0 V = x (1 − x), for 0 < x < 1 equation (4) reduces to,

$$V(x, 0) = \sum_{n=1}^{\infty} c_n \sin n\pi x$$

which represents half range sine series in interval (0, 1) where

Step 9 :

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx \quad \text{here } L = 1$$

$$= 2\int_0^1 (x - x^2) \sin n\pi x\, dx$$

$$= 2\left[(x-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1-2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) - 2\left(-\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$b_n = \frac{4}{n^3\pi^3}(1 - \cos n\pi)$$

$$= \frac{8}{n^3\pi^3}, \text{ if n odd}$$

$$= 0, \text{ if n is even.}$$

Step 10 : Substituting the value of b_n in (4) we get

$$V = \frac{8}{\pi^3}\sum \frac{1}{n^3} \sin n\pi x\, e^{-n\pi y}$$

Step 11 : Replacing n by (2r − 1) we get

$$V = \frac{8}{\pi^3}\sum_{r=1}^{\infty} \frac{1}{(2r-1)^3} \sin (2r-1)\pi x\, e^{-(2r-1)\pi y}$$

Example 4.20 : *A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along short edge y = 0 is given* $u(x, 0) = 100 \sin\left(\frac{\pi x}{10}\right)$, *$0 \le x \le 10$, while the two long edges x = 0 and x = 10 as well as the other short edge are kept 0 °C Find steady state temperature u (x, y).*

Solution : Please refer Fig. 4.7 on next page.

Step 1 : We have to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

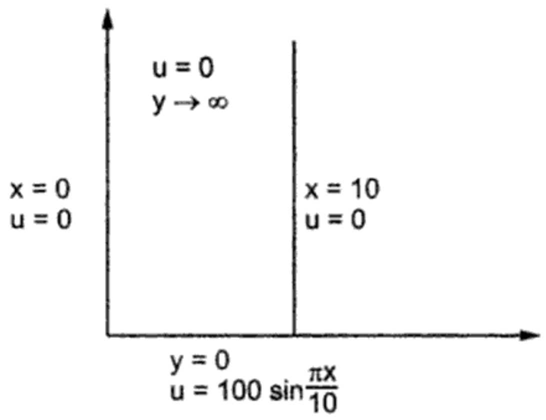


Fig. 4.7

Subject to conditions :

i) $u(0, y) = 0$

ii) $u(10, y) = 0$

iii) $u(x, \infty) = 0$

iv) $u(x, 0) = 100 \sin\left(\frac{\pi x}{10}\right)$, $0 \leq x \leq 10$

Step 2 : Since the plate is infinitely long along Y axis, the G.S. of given equation is,

$$u = (c_1 \cos mx + c_2 \sin mx)\left(c_3 e^{my} + c_4 e^{-my}\right) \quad \text{... (1)}$$

Step 3 : Now the condition (1) is $u = 0$ when $x = 0$

Substituting in (1) we get

$$0 = (c_1 \cos 0 + c_2 \sin 0)\left(c_3 e^{my} + c_4 e^{-my}\right)$$

$$0 = c_1\left(c_3 e^{my} + c_4 e^{-my}\right)$$

which is possible only if $c_1 = 0$

Step 4 : Substituting in (1) we get

$$u = c_2 \sin mx\left(c_3 e^{my} + c_4 e^{-my}\right)$$

$$u = \sin mx\left(c_5 e^{my} + c_6 e^{-my}\right) \quad \text{... (2)}$$

Step 5 : Now applying condition (ii) $u = 0$, $x = 10$ we get

$$0 = \sin 10m\left(c_5 e^{my} + c_6 e^{-my}\right)$$

which is possible only if $\sin 10m = 0 \Rightarrow 10m = n\pi$

where n is any positive integer.

Substituting $m = \frac{n\pi}{10}$ in (2) we get

$$u = \sin\frac{n\pi x}{10}\left(c_5 e^{\frac{n\pi y}{10}} + c_6 e^{-\frac{n\pi y}{10}}\right) \quad \text{... (3)}$$

Step 6 : Now applying condition (iii) $u = 0$ at $y = \infty$ gives

$$0 = \sin\frac{n\pi x}{10}\left(c_5 e^{\infty} + c_6 e^{-\infty}\right)$$

which is possible only if $c_5 = 0$

Substituting $c_5 = 0$ in (3) we get

$$u = c_6 \sin\frac{n\pi x}{10} e^{-\frac{n\pi y}{10}}$$

Step 7 : Taking n = 1, 2, 3, ... And varying c_6 as b_1, b_2, b_3 .... Its general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} e^{-\frac{n\pi y}{10}} \quad \text{... (4)}$$

Step 8 : Applying condition (iv), we have

$$100 \sin\left(\frac{\pi x}{10}\right) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}$$

$$100 \sin\left(\frac{\pi x}{10}\right) = b_1 \sin\frac{\pi x}{10} + b_2 \sin\frac{2\pi x}{10} + \ldots$$

Comparing the coefficients we get

$b_1 = 100,\ b_2 = 0 = b_3 = b_4 = \ldots.\ b_n = \ldots$

Step 9 : Substituting the value of b_n in (4) we get

The complete solution as

$$u(x, y) = b_1 \sin\frac{\pi x}{10} e^{-\frac{\pi y}{10}}$$

i.e.
$$u(x, y) = 100 \sin\frac{\pi x}{10} e^{-\frac{\pi y}{10}}$$

Example 4.21 : *A rectangular plate with insulated surface is 10 cm wide and so long to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge y = 0 is given by,*

$u = 20\,x \qquad \text{for } 0 \le x \le 5$

$\quad = 20\,(10 - x) \qquad \text{for } 5 \le x \le 10$

and the two long edges $x = 0$, $x = 10$ as well as the other short edge are kept at $0°C$, then prove that the temperature u at any point (x, y) is given by

$$u = \frac{800}{\pi^2}\sum_{1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{10}\cdot e^{\frac{-(2n-1)\pi y}{10}}$$

Solution :

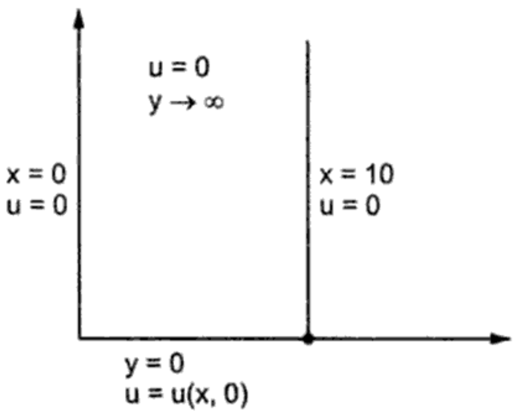


Fig. 4.8

Step 1 : We have to solve

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$$

Subject to conditions :

i) $u(0, y) = 0$

ii) $u(10, y) = 0$

iii) $u(x, \infty) = 0$

iv) $u(x, 0) = \begin{cases} 20\,x & , \text{ for } 0 \le x \le 5 \\ 20\,(10-x) & , \text{ for } 5 \le x \le 10 \end{cases}$

Step 2 : Since the plate is infinitely long along Y axis, the G.S. of given equation is,

$$u = (c_1 \cos mx + c_2 \sin mx)\left(c_3 e^{my} + c_4 e^{-my}\right) \quad \dots (1)$$

Step 3 : Now the condition (1) is $u = 0$ when $x = 0$

Substituting in (1) we get

$$0 = (c_1 \cos 0 + c_2 \sin 0)\left(c_3 e^{my} + c_4 e^{-my}\right)$$

$$0 = c_1\left(c_3 e^{my} + c_4 e^{-my}\right)$$

Which is possible if $c_1 = 0$

Step 4 : Substituting in (1) we get

$$u = c_1 \sin mx\left(c_3 e^{my} + c_4 e^{-my}\right)$$

$$u = \sin mx\left(c_5 e^{my} + c_6 e^{-my}\right) \quad \text{... (2)}$$

Step 5 : Now applying condition (ii) u = 0, x = 10 we get

$$0 = \sin 10m\left(c_5 e^{my} + c_6 e^{-my}\right)$$

which is possible only if sin 10m = 0 $\Rightarrow$ 10m = nπ

where n is any positive integer.

Substituting in $m = \frac{n\pi}{10}$ (2) we get

$$u = \sin\frac{n\pi x}{10}\left(c_5 e^{\frac{n\pi y}{10}} + c_6 e^{-\frac{n\pi y}{10}}\right) \quad \text{... (3)}$$

Step 6 : Now applying condition (iii) u = 0 at y = ∞ gives

$$0 = \sin\frac{n\pi x}{10}\left(c_5 e^{\infty} + c_6 e^{-\infty}\right)$$

which is possible only if $c_5 = 0$

Substituting $c_5 = 0$ in (3) we get

$$u = c_6 \sin\frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}}$$

Step 7 : Taking n = 1, 2, 3, ... And varying c_6 as b_1, b_2, b_3, ... It general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}} \quad \text{... (4)}$$

Step 8 : Applying condition (iv), we have

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}$$

Step 9 : Where b_n is coefficient of half range Fourier sine series for f(x) = u (x, 0) in interval (0, 10) therefore using the half range sine series formula, we have

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{Here } L = 10$$

$$b_n = \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}\,dx$$

$$= \frac{1}{5}\int_0^5 20\,x \sin\frac{n\pi x}{10}\,dx + \frac{1}{5}\int_5^{10} 20(10-x)\sin\frac{n\pi x}{10}\,dx$$

$$= \frac{20}{5}\left[\left\{x\left(-\frac{\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)-(1)\left(-\frac{\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right\}_0^5\right.$$

$$\left.+\left[\left\{(10-x)\,x\left(-\frac{\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)-(1)\left(-\frac{\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right\}_5^{10}\right]\right]$$

$$= 4\left[-\frac{50}{n\pi}\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{50}{n\pi}\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Step 10 : Substituting this in equation (4), we get

$$u = \sum \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}e^{-\frac{n\pi y}{10}}$$

Example 4.22 : *An infinitely long plane uniform plate is bounded by two parallel edges $x = 0$ and $x = \pi$ and an end at right angles to them. The breadth of the plate is π. This end is maintained at temperature u_0 at all points and other edges at zero temperature. Find the steady state temperature function $u(x, y)$.*

Solution :

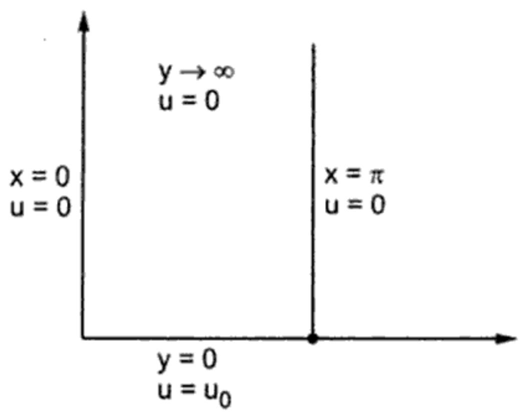


Fig. 4.9

Step 1 : To solve

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$$

Subject to

i) $u = 0$ when $x = 0$ for all y.

ii) $u = 0$ when $x = \pi$ for all y.

iii) $u = 0$ when $y \to \infty$ for all x.

iv) $u = u_0$ when $y = 0, 0 < x < \pi$

Step 2 : Since the plate is infinitely long along Y axis, the G.S. of given equation is,

$$u = (c_1 \cos mx + c_2 \sin mx)\left(c_3 e^{my} + c_4 e^{-my}\right) \quad \text{... (1)}$$

Step 3 : Now the condition (1) is $u = 0$ when $x = 0$

Substituting in (1) we get

$$0 = (c_1 \cos 0 + c_2 \sin 0)\left(c_3\ e^{my} + c_4\ e^{-my}\right)$$

$$0 = c_1\left(c_3\ e^{my} + c_4\ e^{-my}\right)$$

Which is possible only if $c_1 = 0$

Step 4 : Substituting in (1) we get

$$u = c_2 \sin mx\left(c_3\ e^{my} + c_4\ e^{-my}\right)$$

$$u = \sin mx\ \left(c_5\ e^{my} + c_6\ e^{-my}\right) \quad \text{... (2)}$$

Step 5 : Now applying condition (ii) $u = 0$, $x = \pi$ we get

$$0 = \sin m\pi\left(c_5\ e^{my} + c_6\ e^{-my}\right)$$

Which is possible only if $\sin m\pi = 0 \Rightarrow\ m\pi = n\pi$

where n is any positive integer.

Substituting in $m = n$ (2) we get

$$u = \sin nx\left(c_5\ e^{my} + c_6\ e^{-my}\right) \quad \text{... (3)}$$

Step 6 : Now applying condition (iii) $u = 0$ when $y \to \infty$ for all x.

$$0 = \sin nx\left(c_5\ e^{\infty} + c_6\ e^{-\infty}\right)$$

Which is possible only if $c_5 = 0$

Substituting $c_5 = 0$ in (3) we get

$$u = c_6 \sin mx\, e^{-my}$$

Step 7 : Taking $n = 1, 2, 3$ ..... And varying c_6 as b_1, b_2, b_3 ... Its general solution is

$$u = \sum_{n=1}^{\infty} b_n \sin nx\ e^{-ny} \quad \text{... (4)}$$

Step 8 : Now using condition (iv)

$$u = u_0 \text{ when } y = 0,\ 0 < x < \pi \text{ we get}$$

$$u_0 = \sum_{n=1}^{\infty} b_n \sin nx \qquad 0 < x < \pi$$

Step 9 : Which is half range Fourier sine series for $f(x) = u_0$ in interval $0 < x < \pi$.

Where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx\,dx$$

$$= \frac{2u_0}{\pi}\left(-\frac{\cos nx}{n}\right)_0^{\pi} = \frac{2u_0}{\pi n}(1-\cos n\pi)$$

$$b_n = \frac{2u_0}{\pi n}(1-\cos n\pi)$$

$$b_n = \begin{cases} \dfrac{4u_0}{n\pi} & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$$

Step 10 : Substituting in (4) we have

$$u = \sum_{n=1}^{\infty} \frac{4u_0}{n\pi} e^{-ny} \sin nx \quad \text{if n is odd}$$

Step 11 : Replacing n by (2n + 1) we get

$$u = \frac{4u_0}{\pi}\sum_{n=0}^{\infty}\left[\frac{e^{-(2n+1)y}}{(2n+1)}\sin(2n+1)x\right]$$

Example 4.23 : *A rectangular plate is bounded by $x = 0$, $x = a$, $y = 0$, $y = b$. Its surface are insulated and temperature along three edges $x = 0$, $x = a$, $y = 0$ is maintained at $0\,°C$ while the fourth edge $y = b$ is maintained at constant temperature u_0, until steady state is reached. Find $u(x, y)$.*

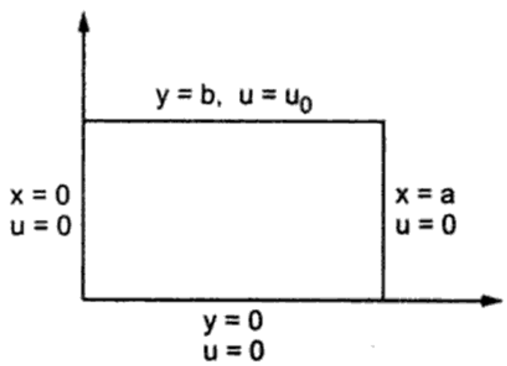


Fig. 4.10

Solution : We have to solve the P.D.E.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{... (1)}$$

i) $u(0, y) = 0$

ii) $u(x, 0) = 0$

iii) $u(a, y) = 0$

iv) $u(x, b) = u_0$

Step 1 : Consider $u(x, y) = (c_1 \cosh mx + c_2 \sinh mx)(c_3 \cos my + c_4 \sin my)$

then by applying (i), we get

$$0 = (c_1 + 0)(c_3 \cos my + c_4 \sin my) \Rightarrow c_1 = 0,$$

then $u(x, y)$ becomes

$$u(x, y) = (c_2 \sinh mx)(c_3 \cos my + c_4 \sin my)$$

then by applying (iii), we get

$$u(a, y) = 0 = (c_2 \sinh ma)(c_3 \cos my + c_4 \sin my)$$

which is not possible because sinh am cannot be zero, for any non-zero value of m. Hence we can't take the above equation as the possible solution.

Step 2 : Thus the possible solution may be given by

$$u(x, y) = (c_1 \cosh mx + c_2 \sinh mx)(c_3 \cos my + c_4 \sin my) \qquad \text{... (2)}$$

Step 3 : Then by applying (i), we get

$$0 = (c_1 + 0)(c_3 \cos my + c_4 \sin my)$$

$$\Rightarrow \quad c_1 = 0$$

Step 4 : Then by applying (ii), we get

$$0 = (c_1 \cosh mx + c_2 \sinh mx)(c_3 + 0)$$

$$\Rightarrow \quad c_3 = 0$$

Step 5 : Substituting in (2) we get

$$\therefore \quad u(x, y) = c_2\, c_4 \sin mx \sinh my$$

$$\therefore \quad u(x, y) = c_5 \sin mx \sinh my \qquad \text{... (3)}$$

Step 6 : Then by applying (iii), we get $0 = c_5 \sin ma \sinh my$

$c_5 \neq 0, \sinh my \neq 0 \ \therefore \ \sin ma = 0$

$$ma = n\pi, \quad m = \frac{n\pi}{a}, \quad n = 1, 2 \ldots$$

$$\therefore \quad u(x, y) = c_5 \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}, \quad n = 1, 2 \ldots$$

Step 7 : Taking n = 1, 2, 3 .... And varying c_5 as b_1, b_2, b_3 .... Its general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \qquad \ldots (4)$$

Step 8 : Applying condition (iv), we get

$$u_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a}$$

$$u_0 = \sum_{n=1}^{\infty} \left(b_n \sinh \frac{n\pi b}{a}\right) \sin \frac{n\pi x}{a}$$

$$u_0 = \sum_{n=1}^{\infty} (B_n) \sin \frac{n\pi x}{a}$$

Step 9 : Which is Fourier sine series, where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \quad \text{here } L = a$$

$$\therefore \quad b_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a u_0 \sin \frac{n\pi x}{a} dx$$

$$= \frac{2u_0}{a} \left[-\frac{a}{n\pi} \cos \frac{n\pi x}{a}\right]_0^a$$

$$B_n = \frac{2u_0}{\pi} \left(\frac{1-(-1)^n}{n}\right)$$

$$b_n = \frac{2u_0}{\pi \sinh \frac{n\pi b}{a}} \left(\frac{1-(-1)^n}{n}\right)$$

Step 10 : Substituting in (4) we get,

$$\therefore \quad u(x, y) = \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n \sinh \frac{n\pi b}{a}} \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

Example 4.24 : *Solve the equation* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

subject to the conditions

i) $u(0, y) = 0$

ii) $u(x, 0) = 0$

iii) $u(L, y) = 0$

iv) $u(x, a) = \sin \dfrac{n\pi x}{L}$

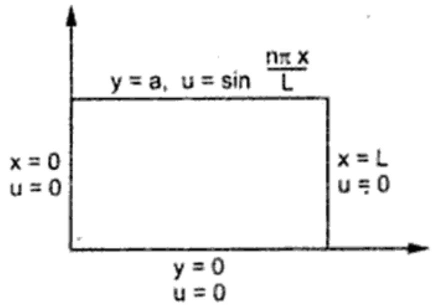


Fig. 4.11

Solution : Step 1 : Consider $u(x, y) = (c_1 \cosh mx + c_2 \sinh mx)(c_3 \cos my + c_4 \sin my)$ ... (1)

then by applying (i), we get

$0 = (c_1 + 0)(c_3 \cos my + c_4 \sin my) \Rightarrow c_1 = 0,$

then $u(x, y)$ becomes

$u(x, y) = (c_2 \sinh mx)(c_3 \cos my + c_4 \sin my)$

then by applying (iii), we get

$u(L, y) = 0 = (c_2 \sinh mL)(c_3 \cos my + c_4 \sin my)$

which is not possible because sinh Lm cannot be zero, for any non-zero value of m. Hence we can't take the above equation as the possible solution.

Step 2 : Thus the possible solution may be given by

$$u(x, y) = (c_1 \cosh mx + c_2 \sinh mx)(c_3 \cos my + c_4 \sin my) \quad \text{... (2)}$$

Step 3 : Then by applying (i), we get

$$0 = (c_1 + 0)(c_3 \cos my + c_4 \sin my)$$

$$\Rightarrow \quad c_1 = 0$$

Step 4 : Then by applying (ii), we get

$$0 = (c_1 \cosh mx + c_2 \sinh mx)(c_3 + 0)$$

$$\Rightarrow \quad c_3 = 0$$

Step 5 : Substituting in (2) we get

$$\therefore \quad u(x, y) = c_2\, c_4 \sin mx \sinh my$$

$$\therefore \quad u(x, y) = c_5 \sin mx \sinh my \quad \text{... (3)}$$

Step 6 : Then by applying (iii), we get

$$0 = c_5 \sin mL \sinh my$$

$$c_5 \neq 0, \sinh my \neq 0 \quad \therefore \quad \sin mL = 0$$

$$mL = n\pi, \quad m = \frac{n\pi}{L}, \quad n = 1, 2 \ldots.$$

$$\therefore \qquad u(x, y) = c_5 \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}, \quad n = 1, 2 \ldots.$$

Step 7 : Taking n = 1, 2, 3, ... And varying c_5 as $b_1, b_2, b_3, \ldots$ Its general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L} \qquad \ldots (4)$$

Step 8 : Applying condition (iv), we get

$$\sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi a}{L}$$

$$\sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} \left(b_n \sinh \frac{n\pi a}{L}\right) \sin \frac{n\pi x}{L}$$

$$\sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} (B_n) \sin \frac{n\pi x}{L}$$

Step 9 : Comparing we get $B_n = 1$

i.e. $\left(b_n \sinh \frac{n\pi a}{L}\right) = 1$

$$\therefore \qquad b_n = \frac{1}{\sinh \frac{n\pi a}{L}}$$

Thus there only one value of b_n

$\therefore$ equation (4) reduces to

$$u(x, y) = \frac{1}{\sinh \frac{n\pi a}{L}} \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}$$

Example 4.25 : *A square metal plate of side a has edges represented by lines $x = 0$, $x = a$, $y = 0$, $y = a$. The edges $x = a$ and $y = a$ are insulated. The edge $x = 0$ is kept at $0\,°C$ and $y = 0$ at $u_0\,°C$, where u_0 is a constant. Obtain the temperature distribution $u(x, y)$. Under steady state condition.*

Solution : Please refer Fig. 4.12 on next page.

We have to solve the P.D.E. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Subject to the boundary condition.

Step 4 : Substituting in (2) we get

$$u = (c_2 \sin mx)(c_3 \cosh my + c_4 \sinh my)$$

$$u = (\sin mx)(c_5 \cosh my + c_6 \sinh my) \quad \text{... (3)}$$

Step 5 : Differentiating w.r.t. y we get

$$\frac{\partial u}{\partial y} = (\sin mx)[m\,c_5 \sinh my + mc_6 \cosh my]$$

Step 6 : Then by applying (ii), we get

$$0 = (\sin mx)[mc_5 \sinh ma + mc_6 \cosh ma]$$

$\therefore \quad c_5 \sinh ma + c_6 \cosh ma = 0$

$$c_6 = -\frac{c_5 \sinh ma}{\cosh ma}$$

Step 7 : Substituting in (3) we get

$$\therefore \quad u(x, y) = \sin mx \left[c_5 \cosh my - \frac{c_5 \sinh ma}{\cosh ma} \sinh my\right]$$

$$= \frac{c_5 \sin mx}{\cosh ma}(\cosh my \cosh ma - \sinh ma \sinh my)$$

$$u = \frac{c_5 \sin mx}{\cosh ma} \cosh m(y-a) \quad \text{... (4)}$$

Step 8 : Differentiating w.r.t. x we get

$$\frac{\partial u}{\partial x} = \frac{c_5\, m \cos mx}{\cosh ma} \cosh(a-y)\, m$$

Step 9 : Then by applying (iii), we get

$$0 = \frac{c_5\, m \cos ma}{\cosh ma} \cosh(a-y)\, m$$

Thus $\cos ma = 0, \quad ma = \frac{(2n-1)\pi}{2}$

$$m = \frac{(2n-1)\pi}{2a}, \quad n = 1, 2, 3, \ldots.$$

Step 10 : Substituting (4) we get

$$\therefore \quad u(x, y) = \frac{c_5 \sin\frac{(2n-1)\pi x}{2a} \cosh\left(\frac{(2n-1)\pi(a-y)}{2a}\right)}{\cosh\frac{(2n-1)\pi}{2}}$$

Step 11 : Taking n = 1, 2, 3, ... and varying c_5, as b_1, b_2, b_3, ... its general solution is

$$u(x, y) = \sum_{n=1}^{\infty} \frac{b_n \sin\frac{(2n-1)\pi x}{2a}}{\cosh\frac{(2n-1)\pi}{2}} \cosh\left(\frac{(2n-1)\pi(a-y)}{2a}\right) \quad \dots (5)$$

Step 12 : Applying condition (iv), we get

$$u_0 = \sum_{n=1}^{\infty} \frac{b_n \sin\frac{(2n-1)\pi x}{2a}}{\cosh\frac{(2n-1)\pi}{2}} \cosh\frac{(2n-1)\pi}{2}$$

$$u_0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)\pi x}{2a}\right)$$

Step 13 : Which is Fourier sine series, where

$$\therefore \quad b_n = \frac{2}{a}\int_0^a u_0 \sin\frac{(2n-1)\pi x}{2a}\, dx$$

$$= \frac{2u_0}{a}\left[\frac{-2a}{(2n-1)\pi} \cos\frac{(2n-1)\pi x}{2a}\, dx\right]_0^a$$

$$= \frac{-4u_0}{\pi}\frac{1}{2n-1}[0-1]$$

$$= \frac{4u_0}{\pi}\frac{1}{2n-1}$$

Step 14 : Substituting in (5) we get

$$u(x, y) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty} \frac{\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{2a}}{\cosh\frac{(2n-1)\pi}{2}} \cosh\left(\frac{(2n-1)\pi(a-y)}{2a}\right)$$

Exercise 4.3

1. *A thin metal plate bounded by the x-axis and the lines x = 0 and x = 1 and stretching to infinity in the y-direction has its upper and lower faces perfectly insulated and its vertical edges and edge at infinity are maintained at the constant temperature 0°C, while over the base temperature of 50°C is maintained. Find steady state temperature $u(x,t)$.*

[Ans. : $u(x, y) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x}{2n-1} e^{-(2n-1)\pi y}$]

2. *A thin sheet of metal bounded by the x-axis and the lines x = 0 and x = 1 and stretching to infinity in the y-direction and its vertical edges and edge at infinity are maintained at the constant*

temperature 0°C, *if the temperature along short edge* $y = 0$ *is* $u(x,0) = (x - x^2)$ *degrees. Find steady state temperature* $u(x,t)$.

$$\left[\text{Ans.}: y(x, y) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\, nx}{(2n-1)^3}\, e^{-(2n-1)\pi y}\right]$$

3. *A thin metal plate bounded the lines* $y = 0$, $x = 0$ *and* $x = 1$ *and stretching to infinity in the y-direction. The vertical edges and edge at infinity are maintained at the constant temperature* 0°C. *If the temperature along short edge* $y = 0$ *is* $u(x,o) = \begin{cases} x & 0 < x \le 0.5 \\ 1 - x & 0.5 \le x \le 1 \end{cases}$ *Find steady state temperature* $u(x,t)$.

$$\left[\text{Ans.}: u(x, y) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin(2n-1)\,\pi x}{(2n-1)^2}\, e^{-(2n-1)\pi y}\right]$$

4. *A square plate with insulated upper and lower faces is bounded by* $x = 0$, $x = \pi$, $y = 0$, $y = \pi$ *has its edges* $x = 0$, $x = \pi$ *are insulated. Its edges* $y = 0$ *and* $y = \pi$ *are kept at temperature* 0°C *and* $f(x)$ *respectively. Show that*

$$u(x, y) = \frac{1}{2\pi} a_0\, y + \sum_{n=1}^{\infty} a_n \frac{\sinh ny}{\sinh n\pi} \cos nx \quad \textit{where } a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx \text{ , } n = 0, 1, 2, \ldots$$

5. *A rectangular plate with insulated upper and lower surface is bounded by* $x = 0$, $x = a$, $y = 0$, $y = b$. *The lower side* $(y = 0)$ *of a metal plate is kept at* 100°C, *while upper side* $y = b$ *is insulated. If the other two sides are kept at* 0°C, *find the steady state temperature distribution of the plate.*

$$\left[\text{Ans.}: u(x, y) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{\left(1 - (-1)^n\right)}{n} \sin \frac{n\pi x}{a} \left(\cos \frac{n\pi y}{a} - \sin \frac{n\pi y}{a} \tanh \frac{n\pi b}{a}\right)\right]$$

6. *A square plate has its faces insulated. The edges* $x = 0$ *and* $x = \pi$ *are also insulated while the edge* $y = \pi$ *is kept as* 0°C. *The edge* $y = 0$ *has temperature distribution* $u(x,0) = x^2$. *Find the steady state temperature distribution.*

$$\left[\text{Ans.}: u(x, y) = \frac{\pi}{3}(\pi - y) + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \sinh n(\pi - y) \cos nx}{n^2 \sinh n\pi}\right]$$

7. *A rectangular plate having insulated faces is bounded by lines* $x = 0$, $x = a$, $y = 0$, $y = b$ *and the edges* $x = 0$, $x = a$ *and* $y = b$ *are kept at zero degree and lower edge* $(y = 0)$ *is having temperature distribution.* $5 \sin \frac{4\pi x}{a} + 3 \sin \frac{3\pi x}{a}$. *Find the steady state temperature distribution.*

$$\left[\text{Ans.}: u(x, y) = 3 \sin \frac{3\pi x}{a} \sinh \frac{3\pi(b - y)}{a} \operatorname{cosech} \frac{3\pi b}{a} + 5 \sin \frac{4\pi x}{a} \sinh \frac{4\pi(b - y)}{a} \operatorname{cosech} \frac{4\pi b}{a}\right]$$

8. *A rectangular metal plate is bounded by* $x = 0$, $x = a$, $y = 0$, $y = b$. *The edges* $x = 0$, $x = a$, $y = b$ *are insulated and the edge* $y = 0$ *is kept at temperature* $u_0 \cos \frac{\pi x}{a}$. *Find the temperature distribution* $u(x, y)$ *in steady state condition.*

University Questions

Dec. - 98

1. *If* $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

   represents the vibrations of a string of length L fixed at both ends, find the displacement y at any distance 'x' from one end at any time t, provided

   $y(0, t) = 0, \ y(L, t) = 0, \ \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ *and*

   $y(x, 0) = \frac{3}{4} \sin\left(\frac{\pi x}{L}\right) - \frac{1}{4} \sin\left(\frac{3\pi x}{L}\right)$ **[8 Marks]**

May - 99

1. *A string is stretched tightly between x = 0 and x = l and both ends are given the displacement y = asin pt perpendicular to the string. If the string satisfies the differential equation*

   $\frac{\partial^2 y}{\partial x^2} = \frac{l}{c^2} \cdot \frac{\partial^2 y}{\partial t^2}$

   Show that the oscillations of the string are given by :

   $y = a \sec \frac{pl}{2c} \cos\left(\frac{px}{c} - \frac{pl}{2c}\right) \sin pt.$ **[8 Marks]**

Dec. - 99

1. *Solve :* $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ *if*

   i) u is finite for all t

   ii) $u(0, t) = 0$, *for all t*

   iii) $u(l, t) = 0$, *for all t*

   iv) $u(x, 0) = u_0$ *for* $0 \le x \le l$

   where l being the length of the bar. **[9 Marks]**

May - 2000

1. *A tightly stretched string with fixed end points x = 0 and x = l is initially in a position given by :*

   $y(x, 0) = y_0 \sin^3\left(\frac{\pi x}{l}\right).$

   If it is released from rest from this position. Find the displacement y at any distance x from one end and at any time t :

   $\left(\text{Use the equation } \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}\right).$ **[6 Marks]**

Dec. - 2000

1. *The temperature at any point of the insulated metal rod of one meter length is governed by the differential equation :*

$$\frac{\partial u}{\partial t} = c^2 \frac{d^2 u}{dx^2}$$

Find $u(x, t)$ subjected to the following conditions :

i) $u(0, t) = 0°C$

ii) $u(1, t) = 0°C$

iii) $u(x, 0) = 50°C$

and hence find the temperature in the middle of the rod any subsequent time. **[8 Marks]**

May - 2001

1. *A rectangular plate is bounded by $x = 0$, $x = a$, $y = 0$, $y = b$. Its surfaces are insulated and temperature along three edges $x = 0$, $x = a$, $y = 0$ is maintained at 0°C, while the fourth edge $y = b$ is maintained at constant temperature u_0. Until steady state is reached. Find $u(x, y)$.* **[7 Marks]**

Dec. - 2001

1. *Solve* $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ *if*

   i) u is finite for all t *ii) $u(0, t) = 0$, for all t*

   iii) $u(l, t) = 0$, for all t *iv) $u(x,0) = u_0$, for $0 \le x \le l$*

   where l being the length of bar. **[7 Marks]**

May - 2002

1. *A string is stretched and fastened to two points L apart. Motion is started by displacing the string in the form of*

   $u = a \sin\left(\frac{\pi x}{L}\right)$ *from which it is released at time $t = 0$. Find the displacement $u(x,t)$ one end.*

   [8 Marks]

Dec. - 2002

1. *Solve* $\dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2}$ *for the conduction of heat along a rod without radiation, subject to the following conditions :*

   i) u is not infinite as $t \to \infty$

   ii) $\dfrac{\partial u}{\partial x} = 0$ *for $x = 0$, $x = l$*

   (i.e. no heat flow through the ends.)

   iii) $u = lx - x^2$ for $t = 0$ between $x = 0$, $x = l$. **[8 Marks]**

May - 2003

1. *The temperature at any point of the insulated metal rod of one metre length is governed by the differential equation :*

   $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

   Find $u(x,t)$ subject to the following conditions :

i) $u(0, t) = 0°C$

ii) $u(1, t) = 0°C$

iii) $u(x, 0) = 50°C$ and hence find the temperature in the middle of the rod at any subsequent time. **[8 Marks]**

Dec. - 2003

1. *A string is stretched and fastened to two points λ apart. Motion is started by displacing the string in the form $U = a\sin\frac{\pi x}{\lambda}$ from which it is released at time $t = 0$. Find the displacement $u(x, t)$ from one end when wave equation is* **[8 Marks]**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

May - 2004

1. *An insulated rod AB of length L has its ends maintained at 0°C and 100°C respectively until steady state prevail. If B is reduced to 0°C and maintained at the same temperature, find the temperature at a distance x from 0 at time t by solving*

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

using appropriate boundary conditions. **[7 Marks]**

Dec. - 2004

1. *A tightly streched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y(x, 0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement y at any distance x from one end and at any time t. The displacement $y(x, t)$ satisfies the differential equation* **[8 Marks]**

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

May - 2005

1. *Solve :* **[8 Marks]**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x}, \text{ if}$$

i) $u(0, t) = 0$

ii) $u_x(l, t) = 0$

iii) $u(x, t)$ is bounded

iv) $u(x, 0) = \frac{u_0 x}{l}$ for $0 \le x \le \frac{l}{2}$

2. *A string is stretched tightly between $x = 0$ and $x = l$ and both ends are given the displacement : $y = a \sin pt$ perpendicular to the string. If the string satisfies the differential equation :*

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{C^2} \cdot \frac{\partial^2 y}{\partial t^2}$$

Show that the oscillation of the string are given by :

$$y = \alpha \sec \frac{Pl}{2C} \cos\left(\frac{Px}{C} - \frac{Pl}{2C}\right) \sin pt.$$

[8 Marks]

Dec. - 2005

1. *Solve one dimensional heat flow equation :*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions

$u(0, t) = 0, \qquad u(1, t) = 0$

$y(x, 0) = 2x \qquad 0 \le x \le \frac{1}{2}$

$\qquad = 2(1 - x) \quad \frac{1}{2} \le x \le 1.$

[9 Marks]

2. *A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along short edge $y = 0$ is given*

$$u(x, 0) = 100 \sin\left(\frac{\pi x}{10}\right), \; 0 \le x \le 10,$$

while the two long edges $x = 0$ and $x = 10$ as well as the other short edge are kept at 0°C. Find steady state temperature $u(x, y)$. **[9 Marks]**

May - 2006

1. *Find the solution of :*

$$\frac{\partial u}{\partial t} = l^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the condition :

i) $u = 0$ when $x = 0$, $x = a$ for all t.

ii) $u = 2\sin\frac{P\pi x}{a}$, P is an integer, for all x when $t = 0$.

[5 Marks]

2. *Solve :*

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

subject to the conditions :

i) $v = 0$, when $y \to \infty$

ii) $v = 0$, when $x = 0$ for all y

iii) $v = 0$, when $x = 1$

iv) $v = x(1 - x)$ when $y = 0$, $0 < x < 1$. **[9 Marks]**

5

Matrices

5.1 Introduction to Eigen Values and Eigen Vectors

If A $[a_{ij}]$ be a square matrix of order $n \times n$ and there exists a scalar λ and vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ such that they satisfy the equation.

$$AX = \lambda X$$

or $\quad (A - \lambda I_n) X = 0 \quad$ (known as matrix equation) ... (1)

then λ is **characteristic value** or **eigen value** and X is called corresponding **eigen vector** or **characteristic vector.**

From (1), the matrix $[A - \lambda I_n]$ is called **characteristic matrix** and the equation $|A - \lambda I_n| = 0$ is called **characteristic equation** in terms of λ which has n roots $\lambda_1, \lambda_2, \lambda_2$ .... λ_n are called **Characteristic or eigen values**

Note :

1. For each characteristic value there corresponds a characteristic vector.
2. For $A_{2\times 2}$ the characteristic equation $|A - \lambda I_2| = 0$ can be obtained as

$$\boxed{\lambda^2 - S_1\lambda + |A| = 0}$$

where S_1 = sum of the diagonal elements of the matrix A

$|A|$ = determinant of A.

3. For $A_{3\times 3}$ the characteristic equation $|A - \lambda I_3| = 0$ can be obtained as

$$\boxed{\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0}$$

where

S_1 = sum of the diagonal elements of the matrix A

S_2 = sum of the minors of the diagonal elements of 'A'.

$$|A| = \text{del of A}$$

4. Eigen values may be zero but eigen vector may not be zero vector.
5. **Orthogonal vectors :** Two eigen vectors X_1 and X_2 are said to be orthogonal if $X_1' X_2 = 0$. ($\because X_1'$ is transpose of X_1)
6. **Modal matrix :** For a matrix $A_{3\times3}$, if $\lambda_1, \lambda_2, \lambda_3$ are eigen values and X_1, X_2, X_3 are corresponding vectors as

$$X_1 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, X_2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, X_3 = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

then the matrix

$$U = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$ is called modal matrix which satisfies the relation

$$U^{-1} A U = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Illustrative Examples

Example 5.1 : *Find eigen values and eigen vectors. Also find the modal matrix U and the resulting diagonal matrix D of A, where* $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Solution :

Step 1 : The characteristic equation of the matrix A is $|A - \lambda I| = 0$

$$\text{or} \begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)(-2\lambda + \lambda^2 + 1) - 2(-\lambda + 1) - 2(-1 + 2 - \lambda) = 0$$

$$\Rightarrow (-\lambda - 1)(\lambda^2 - 2\lambda + 1) + 2\lambda - 2 - 2 + 2\lambda = 0$$

$$\Rightarrow (\lambda - 1)(\lambda^2 - 5) = 0 \text{ or } \lambda = 1, \lambda = \pm\sqrt{5}$$

OR $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

where

S_1 = sum of the diagonal elements of A

$= -1 + 2 + 0 = 1$

S_2 = sum of the minors of diagonal elements

$$= \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -1 & -2 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix}$$

$= 1 - 2 - 4 = -5$

$$A = \begin{vmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{vmatrix} = -5$$

$$\therefore \quad \lambda^3 - \lambda^2 - 5\lambda + 5 = 0$$

$$\text{or} \quad (\lambda - 1)(\lambda^2 - 5) = 0$$

$\Rightarrow \lambda = 1, \lambda = \pm\sqrt{5}$ are eigen values for the matrix A.

Step 2 : Let $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be eigen vector corresponding to eigen value $\lambda = 1$

Consider $(A - \lambda I) X_1 = 0$

i.e. $$\begin{bmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Put $\lambda = 1$

$$\therefore \begin{bmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\lambda = 1)$$

$$\Rightarrow \left.\begin{aligned} -2x_1 + 2x_2 - 2x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}\right]$$

Solving above equations by Crammer's rule, we have

$$\frac{x_1}{\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{4} = \frac{-x_2}{0} = \frac{x_3}{-4}$$

Hence $$X_1 = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{\lambda=1}$$

Step 3 : When $\lambda = -\sqrt{5}$, we have

$$\begin{bmatrix} -1+\sqrt{5} & 2 & -2 \\ 1 & 2+\sqrt{5} & 1 \\ -1 & -1 & 0+\sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} \left(-1+\sqrt{5}\right) x_1 + 2x_2 - 2x_3 &= 0 \\ x_1 + \left(2+\sqrt{5}\right) x_2 + x_3 &= 0 \end{aligned}\right]$$

Solving above equations by Crammer's Rule, we get

$$\frac{x_1}{6+2\sqrt{5}} = \frac{x_2}{-1-\sqrt{5}} = \frac{x_3}{1+\sqrt{5}}$$

Hence $$X_2 = \begin{bmatrix} 1+\sqrt{5} \\ -1 \\ 1 \end{bmatrix}_{\lambda = -\sqrt{5}}$$

Step 4 : Where $\lambda = \sqrt{5}$, we have

$$\begin{bmatrix} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & 0-\sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} \left(-1-\sqrt{5}\right) x_1 + 2x_2 - 2x_3 &= 0 \\ x_1 + \left(2-\sqrt{5}\right) x_2 + x_3 &= 0 \end{aligned}\right]$$

Solving above equations by Crammer's Rule

$$\frac{x_1}{6-2\sqrt{5}} = \frac{x_2}{-1+\sqrt{5}} = \frac{x_3}{1-\sqrt{5}}$$

or $$\frac{x_1}{1-\sqrt{5}} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Hence, $$X_3 = \begin{bmatrix} 1-\sqrt{5} \\ -1 \\ 1 \end{bmatrix}$$

Step 5 : Now,

$$\text{Modal matrix} = U = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} -1 & 1+\sqrt{5} & 1-\sqrt{5} \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$U^{-1}\ A\ U = \frac{1}{2\sqrt{5}}\begin{bmatrix} 0 & 2\sqrt{5} & 2\sqrt{5} \\ -1 & -2-\sqrt{5} & -1 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1-\sqrt{5} & 1+\sqrt{5} \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}}\begin{bmatrix} 2\sqrt{5} & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix} = D$$

Example 5.2 : *Find eigen values and eigen vectors of the matrix*

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Solution : Step 1 : Given matrix is

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Step 2 : The characteristic equation for A is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

where $S_1 = 2 + 3 + 2$

$= 7$

S_2 = Sum of the minors of the diagonals

$= 11$

and $|A| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 5$

$$\therefore \quad \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

1	1	− 7	11	− 5
		1	− 6	5
	1	− 6	5	0

$\therefore\ (\lambda-1)(\lambda^2-6\lambda+5) = 0$

$\Rightarrow \lambda = 1, 1, 5$ are the characteristic values for 'A'.

Step 3 : To find characteristic vectors

consider $(A-\lambda I)X=0$

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{... (1)}$$

Put $\lambda = 5$ then (1) becomes

$$\therefore \quad \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} -3x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \end{aligned}\right] \text{Solving by Crammer's Rule}$$

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{4} = \frac{-x_4}{-4} = \frac{x_3}{4}$$

$$\therefore \quad X_1 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{The eigen vector for } \lambda = 5$$

Step 4 : To find eigen vectors for $\lambda = 1, 1$ from (1), put $\lambda = 1$ in (1)

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + 2x_2 + x_3 = 0$$

Let $x_3 = 0$

$$\therefore \quad x_1 + 2x_2 = 0, \quad \text{put } x_2 = t$$

$$\therefore \quad x_1 = -2t$$

$$\therefore \quad X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Similarly, let $x_2 = 0$,

$$\therefore \quad x_1 + x_3 = 0 \quad \text{put } x_3 = 1$$

$$\Rightarrow \quad x_1 = -1$$

$$\therefore \quad X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Step 5 : The eigen vectors are

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\lambda=5} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}_{\lambda=1} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}_{\lambda=1}$$

Example 5.3 : *Find eigen values and eigen vectors for* $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

Solution : Step 1 : The characteristic equations of A is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

where $S_1 = 6 + 3 + 3 = 12$

S_2 = Sum of the minors of diagonals

$= 36$

and $|A| = 32$

$$\therefore \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

Synthetic division

$\lambda = 2$	1	− 12	36	− 32
		2	− 20	32
	1	− 10	16	0

$$\therefore (\lambda-2)(\lambda^2 - 10\lambda + 16) = 0$$

$$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-8) = 0 \Rightarrow \lambda = 2, 2, 8 \text{ are eigen value for the matrix A.}$$

Step 2 : i) Eigen vector for $\lambda = 8$ is X_1

$$\therefore \quad (A - \lambda I) X_1 = 0 \Rightarrow \begin{bmatrix} 4-\lambda & -2 & 2 \\ -2 & 1-\lambda & -1 \\ 2 & -1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{... (1)}$$

Step 3 : We find X_3 for $\lambda = 8$

$\therefore$ From (1)

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} -x_1 - x_2 + x_3 &= 0 \\ 2x_1 + 5x_2 + x_3 &= 0 \end{aligned}\right] \text{Solving by Crammer's Rule}$$

$$\Rightarrow \quad \frac{x_1}{-6} = \frac{-x_2}{-3} = \frac{x_3}{-3}$$

$$\Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$\therefore \quad X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}_{\lambda=8}$$

Step 4 : Put $\lambda = 2$ then (1) becomes

$$\Rightarrow \quad \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2x_1 - x_2 + x_3 = 0 \quad \text{put } x_1 = 0$$

$$\therefore \quad x_2 = x_3$$

$$\therefore \quad X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Step 5 : To find X_2 for $\lambda = 2$, Let $X_2 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

For a symmetric matrix X_1, X_2, X_3 are mutually orthogonal.

i.e. $X_2' X_3 = 0$ and $X_2' X_1 = 0$ ($\because X_2' \Rightarrow$ transpose of X_2)

$$\Rightarrow \quad [l, m, n] \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 0 \quad \text{and} \quad [l, m, n] \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \quad 2l - m + n = 0$$

$$0l + m + n = 0$$

$$\Rightarrow \quad \frac{l}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{-m}{\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{l}{-2} = \frac{-m}{2} = \frac{n}{2}$$

$$\therefore \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

5.2 Mechanical Vibrations

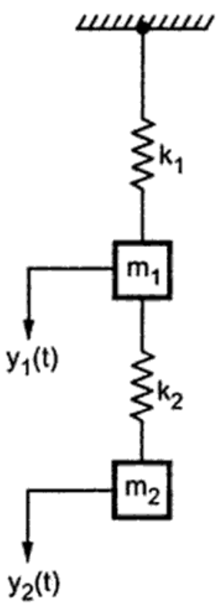


Fig. 5.1

The system of differential equations have their applications in various Engineering problems. For large system consisting number of equations in several variables, solutions can be obtained by matrix method. In what follows, we discuss the role of matrices and Eigen values to the solutions of vibrations of multi degree systems.

I) Vertical System :

Two masses m_1 and m_2 are connected to the springs as shown in figure 5.1.

Let $y_1(t)$ and $y_2(t)$ are vertical displacements from their initial positions in equilibrium i.e. when $y_1 = 0$ and $y_2 = 0$.

The equations of motion are (by Newton's Law)

$$\boxed{\begin{aligned} m_1\ddot{y}_1 &= -k_1 y_1 + k_2(y_2 - y_1) \\ m_2\ddot{y}_2 &= -k_2(y_2 - y_1) \end{aligned}}$$

which can be expressed as

$$\ddot{Y} = AY \qquad \text{... (1)}$$

Similarly, for vertical system consisting of masses m_1, m_2, m_3 connected to the spring in (figure 5.2).

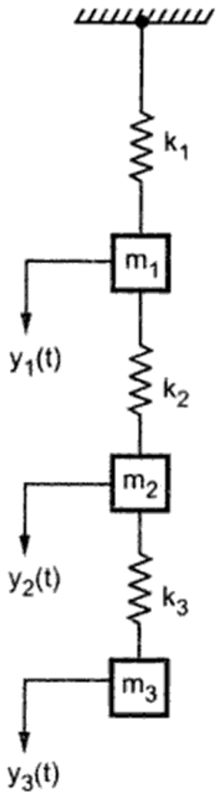


Fig. 5.2

The equations of motion are

$$\boxed{\begin{aligned} m_1\ddot{y}_1 &= -k_1y_1+k_2(y_2-y_1) \\ m_2\ddot{y}_2 &= -k_2(y_2-y_1)+k_3(y_3-y_2) \\ m_3\ddot{y}_3 &= -k_3(y_3-y_2) \end{aligned}}$$

which can be written in the matrix form

$$\ddot{Y} = AY$$

Here k_1, k_2 are stiffness constants known as spring modulli or modulus of elasticity. At every stage tension is acting upward as under

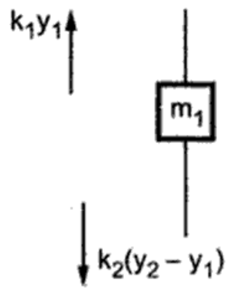


Fig. 5.3

Similarly, if m_1, m_2 ... m_n (n masses) connected by springs and y_1, y_2, ... y_n be small displacements at any time 't' then the equations of motion are

$$\begin{aligned}
m_1\ddot{y}_1 &= -k_1y_1+k_2(y_2-y_1)\\
m_2\ddot{y}_2 &= -k_2(y_2-y_1)+k_3(y_3-y_2)\\
m_3\ddot{y}_3 &= -k_3(y_3-y_2)+k_4(y_4-y_3)\\
&\vdots\\
m_n\ddot{y}_n &= -k_n(y_n-y_{n-1})
\end{aligned}$$

II) Horizontal System :

If $x_1(t)$ and $x_2(t)$ be the linear displacements of masses m_1 and m_2 connected to springs as in Fig. 5.4, from their initial position.

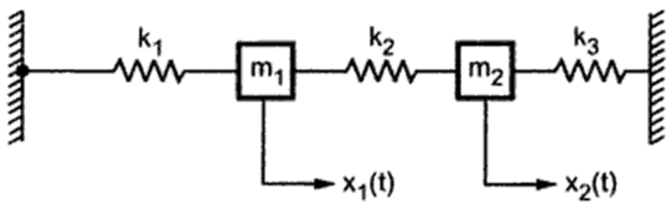


Fig. 5.4

The equations of motion are

$$\begin{aligned}
m_1\ddot{x}_1 &= -k_1x_1+k_2(x_2-x_1)\\
m_2\ddot{x}_2 &= -k_2(x_2-x_1)-k_3x_2
\end{aligned}$$

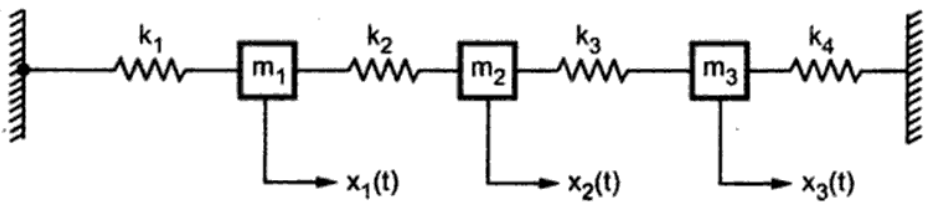


Fig. 5.5

Similarly, the equations of motion for three masses m_1, m_2, m_3 connected by springs as shown in Fig. 5.5 are

$$\boxed{\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1+k_2(x_2-x_1) \\ m_2\ddot{x}_2 &= -k_2(x_2-x_1)+k_3(x_3-x_2) \\ m_3\ddot{x}_3 &= -k_3(x_3-x_2)-k_4x_3 \end{aligned}}$$

Note :

1) Any vertical system can be represented as $\ddot{Y}=AY$

The solution of such a system is in the form $Y = X\,e^{\omega t}$

then $\dot{Y} = X\omega e^{\omega t}$ and

$$\ddot{Y} = X\omega^2 e^{\omega t}$$

Thus $\ddot{Y} = AY$ becomes

$$X\omega^2 e^{\omega t} = A\,X\,e^{\omega t}$$

$$\therefore \quad AX = \omega^2 X$$

which indicates that ω^2 is the eigen value of A and X is its eigen vector.

Thus the solution $\ddot{Y} = AY$ is given by $Y = X\,e^{\omega t}$ where ω^2 is the eigen value of A and X is its eigen vector.

2) Any horizontal system can be represented as $\ddot{X} = AX$. The solution of such a system is in the form $X = Y\,e^{\omega t}$ where ω^2 is the eigen value of A and Y is its eigen vector.

5.3 Illustrated Examples

Example 5.4 : *The system shown in the figure begins to vibrate with initial displacements* $Y_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\dot{Y} = \begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$. *Assuming that there is no friction, determine the subsequent motion.*

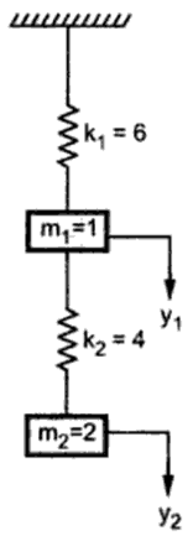


Fig. 5.6

Solution : Step 1 : The equations of motion are

$$\left.\begin{aligned} m_1\ddot{y}_1 &= -k_1 y_1 + k_2(y_2 - y_1) \\ m_2\ddot{y}_2 &= -k_2(y_2 - y_1) \end{aligned}\right\} \quad \ldots (1)$$

Step 2 : Substituting the values of m_1, m_2, k_1, k_2 we get

$$\ddot{y}_1 = -6y_1 + 4(y_2 - y_1)$$

$$\ddot{y}_2 = -4(y_2 - y_1)$$

or

$$\ddot{y}_1 = -10\,y_1 + 4y_2$$

$$\ddot{y}_2 = 4(y_1 - y_2)$$

Step 3 : $\therefore$ The system takes the form $\ddot{Y} = AY$

i.e.

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} -10 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \ldots (1)$$

where

$$A = \begin{bmatrix} -10 & 4 \\ 4 & -4 \end{bmatrix}$$

Step 4 : The characteristic equation for A is $|A - \lambda I| = 0$ i.e. $\begin{vmatrix} -10-\lambda & 4 \\ 4 & -4-\lambda \end{vmatrix} = 0$

$$\therefore \quad \lambda^2 + 14\lambda + 24 = 0$$

$$\Rightarrow \quad (\lambda+2)(\lambda+12) = 0$$

$$\Rightarrow \quad \lambda = -2,\ \lambda = -12 \text{ are eigen values of A}$$

Step 5 : To find eigen vectors consider the matrix equation $[A - \lambda I]X = 0$

i.e.

$$\begin{bmatrix} -10-\lambda & 4 \\ 4 & -4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda = -2$ we get

$$\begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From R_2 $\quad 4x_1 - 2x_2 = 0$

i.e. $\quad 2x_1 - x_2 = 0$

Put $x_1 = 1$, we get $x_2 = 2$

$$\therefore \quad X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is the eigen vector corresponding to $\lambda = -2$

Similarly we can find

$$X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ for } \lambda = -12$$

Thus $$X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\lambda=-2}, X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}_{\lambda=-12}$$

Step 6 : As our system is $\ddot{Y} = AY$ its solution is in the form $Y = X e^{\omega t}$ where ω^2 is the eigen value and X is its eigen vector.

As we have two eigen values and two eigen vectors.

i.e. $$\omega_1^2 = \lambda_1 = -2 \text{ and } \omega_2^2 = \lambda_2 = -12$$

$\therefore$ $$\omega_1 = \sqrt{2}\, i, \quad \omega_2 = +2\sqrt{3}\, i$$

Step 7 : $\therefore$ General solution is

$$Y = X_1 e^{\omega_1 t} + X_2 e^{\omega_2 t}$$

$$Y = X_1 e^{\sqrt{2} i t} + X_2 e^{2\sqrt{3} i t}$$

rearranging the constants we get

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}\left(a_1 \cos\sqrt{2}\, t + b_1 \sin\sqrt{2}\, t\right) + \begin{bmatrix} -2 \\ 1 \end{bmatrix}\left(a_2 \cos 2\sqrt{3}t + b_2 \sin 2\sqrt{3}t\right)$$

where a_1, b_1, a_2, b_2 are arbitrary constants.

Step 8 : then $$y_1(t) = a_1 \cos\sqrt{2}t + b_1 \sin\sqrt{2}t - 2a_2 \cos\left(2\sqrt{3}\right)t - 2b_2 \sin\left(2\sqrt{3}\right)t \quad \text{... (2)}$$

$$y_2(t) = 2a_1 \cos\sqrt{2}t + 2b_1 \sin\sqrt{2}t + a_2 \cos\left(2\sqrt{3}\right)t + b_2 \sin\left(2\sqrt{3}\right)t \quad \text{... (3)}$$

Using initial conditions;

$$Y_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ we get}$$

$$0 = a_1 - 2a_2$$

$$0 = 2a_1 + a_2$$

$\Rightarrow$ $$a_1 = 0, a_2 = 0$$

Step 9 : Now differentiating (2) and (3) w.r.t. 't'.

$$\dot{y}_1(t) = -\sqrt{2}\, a_1 \sin\sqrt{2}t + \sqrt{2}\, b_1 \cos\sqrt{2}\, t$$
$$+4\sqrt{3}\, a_2 \sin\left(2\sqrt{3}\right)t - 4\sqrt{3}\, b_2 \cos\left(2\sqrt{3}\right) t \quad \text{... (4)}$$

$$\dot{y}_2(t) = -2\sqrt{2}a_1 \sin\sqrt{2}\,t + 2\sqrt{2}\,b_1 \cos\sqrt{2}\,t$$
$$-2\sqrt{3}\,a_2 \sin 2\sqrt{3}\,t + 2\sqrt{3}\,b_2 \cos 2\sqrt{3}\,t \quad \ldots (5)$$

Using the condition :

$$\dot{Y} = \begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \end{bmatrix} \Rightarrow \begin{array}{l} \dot{y}_1(t) = \sqrt{2} \quad \text{at } t=0 \\ \dot{y}_2(t) = 2\sqrt{2} \quad \text{at } t=0 \end{array}$$

From (4) and (5)

$$\sqrt{2} = \sqrt{2}\,b_1 - 4\sqrt{3}\,b_2$$
$$2\sqrt{2} = 2\sqrt{2}\,b_1 + 2\sqrt{3}\,b_2$$

$\Rightarrow b_2 = 0,\ b_1 = 1$

Step 10 : Substitute values of a_1, a_2, b_1, b_2 then equation of (3) and (4) becomes

$$\left.\begin{array}{l} y_1(t) = \sin\sqrt{2}\,t \\ y_2(t) = 2\sin\sqrt{2}t \end{array}\right] \text{required solution}$$

Example 5.5 : *For the system shown in Fig. if $m_1 = 1$, $m_2 = 3$, $k_1 = 1$, $k_2 = 3$, $k_3 = 3$. Assuming that there is no friction, find the natural frequencies of the system and corresponding normal modes of vibrations using matrix method.*

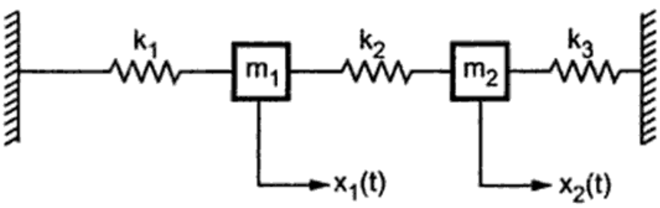


Fig. 5.7

Solution : Step 1 : The equation of motion are

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1)$$
$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - k_3x_2$$

Step 2 : Given $m_1 = 1$, $m_2 = 3$, $k_1 = 1$, $k_2 = 3$, $k_3 = 3$

$$\Rightarrow \quad \ddot{x}_1 = -x_1 + 3(x_2 - x_1)$$
$$3\ddot{x}_2 = -3(x_2 - x_1) - 3x_2$$

Step 3 : The given system takes the form

or $$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or $$\ddot{X} = AX \qquad \text{... (1)}$$

Step 4 : Now, the characteristic equation for the matrix A is

$$|A - \lambda I| = 0$$

or $$\begin{vmatrix} -4-\lambda & 3 \\ 1 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \qquad \lambda^2 + 6\lambda + 5 = 0$$

or $\lambda = -1$, $\lambda = -5$ are the eigen values of A.

Step 5 : To find eigen vectors consider the matrix equation $[A - \lambda I]Y = 0$

i.e. $$\begin{bmatrix} -4-\lambda & 3 \\ 1 & -2-\lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda = -1$, the matrix equation is $[A - \lambda I] Y_1 = 0$

$$\Rightarrow \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

by R_2 we get $y_1 - y_2 = 0$ i.e. $y_1 = y_2$

Put $y_1 = 1$ we get $y_2 = 1$

$\therefore$ The first eigen vector is

$$Y_1 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similarly we can obtain the 2nd eigen vector as

$$Y_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \text{ for } \lambda = -5$$

Step 6 : As our system is $\ddot{X} = AX$ the solution is in the form $X = Y e^{\omega t}$ where ω^2 is the eigen value and Y is its eigen vector.

As we have two eigen values and two eigen vectors we have

$\omega^2 = \lambda = -1, -5$ corresponds to $\omega_1 = i$ and $\omega_2 = i\sqrt{5}$

Step 7 : The general solution is

$$\therefore \qquad X = Y_1 e^{\omega_1 t} + Y_2 e^{\omega_2 t}$$

or $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Y_1 e^{it} + Y_2 e^{\sqrt{5}\,it}$$

rearranging the constants we get

or $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(a_1 \cos t + b_1 \sin t) + \begin{bmatrix} -3 \\ 1 \end{bmatrix}\left(a_2 \cos\sqrt{5}\, t + b_2 \sin\sqrt{5}\, t\right)$$

Step 8 : $\therefore$ $$x_1(t) = a_1 \cos t + b_1 \sin t - 3a_2 \cos\left(\sqrt{5}\, t\right) - 3b_2 \sin\left(\sqrt{5}\, t\right)$$

$$x_2(t) = a_1 \cos t + b_1 \sin t + a_2 \cos\left(\sqrt{5}\, t\right) + b_2 \sin\left(\sqrt{5}\, t\right)$$

Example 5.6 : *The system shown in figure begins to move with initial displacement $y_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and initial velocities $\dot{y}_0 = \begin{bmatrix} -2\sqrt{6} \\ \sqrt{6} \end{bmatrix}$, assuming that there is no friction in the system, determine subsequent motion using eigen values.*

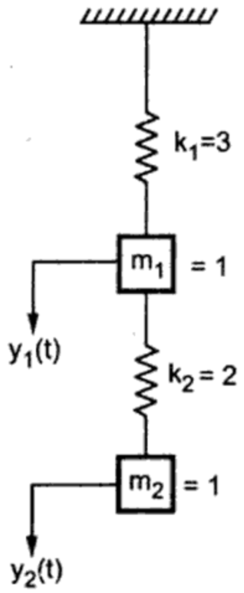


Fig. 5.8

Solution : Step 1 : The equations of motion are

$$m_1 \ddot{y}_1 = -k_1 y_1 + k_2 (y_2 - y_1)$$

$$m_2 \ddot{y}_2 = -k_2 (y_2 - y_1)$$

Step 2 : As $m_1 = 1$, $m_2 = 1$, $k_1 = 3$, $k_2 = 2$

$\therefore$ $$\ddot{y}_1 = -3y_1 + 2(y_2 - y_1)$$

$$\ddot{y}_2 = -2(y_2 - y_1)$$

Step 3 : $\therefore$ The system takes the form $\ddot{Y} = AY$

or $$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

which is of the form

$$\ddot{Y} = AY \quad \text{... (i) where } A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Step 4 : Now the characteristic equation for A is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 + 7\lambda + 6 = 0$$

$$\Rightarrow \quad \lambda = -1, -6 \text{ are the eigen values.}$$

To find eigen vectors consider the matrix equation.

Step 5 : For $\lambda = 1$, the matrix equation is

$$[A - \lambda I] X_1 = 0$$

$$\Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

by R_2 $\quad 2x_1 - x_2 = 0 \Rightarrow 2x_1 = x_2$

$$\therefore \quad X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{for } \lambda = -1$$

For $\lambda = -6$, the matrix equation is

$$|A - \lambda I| X_2 = 0$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad x_1 + 2x_2 = 0$$

$$\Rightarrow \quad x_1 = -2x_2$$

$$\therefore \quad X_2 = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} \text{ or } \begin{bmatrix} +2 \\ -1 \end{bmatrix}$$

Step 6 : Our system is in the form $\ddot{Y} = AY$. The solution of such a system is in the form $Y = X e^{\omega t}$ where ω^2 is the eigen value of A and X is its eigen vector. Here we have two eigen values and two eigen vectors.

$\therefore \omega_1^2 = \lambda_1 = -1$, $\omega_2^2 = \lambda_2 = -6$ corresponds to $\omega_1 = \pm i$, $\omega_2 = \pm i\sqrt{6}$

Step 7 : $\therefore$ General solution is

$$\text{or} \quad Y = X_1 e^{\omega_1 t} + X_2 e^{\omega_2 t}$$

$$= X_1 e^{it} + X_2 e^{i\sqrt{6}\,t}$$

Rearranging the constants we get

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}(a_1 \cos t + b_1 \sin t) + \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}\left[a_2 \cos\left(\sqrt{6}\, t\right) + b_2 \sin\left(\sqrt{6}\, t\right)\right]$$

where a_1, b_1, a_2, b_2 are arbitrary constants.

Step 8 : $\therefore$

$$y_1(t) = a_1 \cos t + b_1 \sin t + a_2 \cos\left(\sqrt{6}\, t\right) + b_2 \sin\left(\sqrt{6}\, t\right)$$

$$y_2(t) = 2a_1 \cos t + 2b_1 \sin t - \frac{a_2}{2} \cos\left(\sqrt{6}\, t\right) - \frac{1}{2} b_2 \sin\left(\sqrt{6}\, t\right)$$

Step 9 : Given $y_1(0) = 1$, $y_2(0) = 2$

$$\therefore \quad a_1 + a_2 = 1 \quad \text{and} \quad 2a_1 - \frac{1}{2} a_2 = 2$$

Solving we get $a_1 = 1$, $a_2 = 0$

$$\therefore \quad \left.\begin{aligned} y_1(t) &= \cos t + b_1 \sin t + b_2 \sin\sqrt{6}\, t \\ y_2(t) &= 2\cos t + 2b_1 \sin t - \frac{b_2}{2} \sin\sqrt{6}\, t \end{aligned}\right] \qquad \text{... (2)}$$

Step 10 : Differentiating (2) w.r.t t.

$$\dot{y}_1(t) = -\sin t + b_1 \cos t + \sqrt{6}\, b_2 \cos\left(\sqrt{6}\, t\right)$$

$$\dot{y}_2(t) = -2 \sin t + 2 b_1 \cos t - \frac{\sqrt{6}}{2} b_2 \cos t$$

Step 11 : Using $\dot{y}_1(0) = -2\sqrt{6}$, $\dot{y}_2(0) = \sqrt{6}$, we get

$$-2\sqrt{6} = b_1 + \sqrt{6}\, b_2 \qquad \text{... (3)}$$

and

$$\sqrt{6} = 2 b_1 - \frac{\sqrt{6}}{2} b_2 \qquad \text{... (4)}$$

Solving (3) and (4), we get

$b_1 = 0$, $b_2 = -2$

Step 12 : $\therefore$ The required solution is

$$y_1(t) = \cos t - 2 \sin\left(\sqrt{6}\, t\right)$$

$$y_2(t) = 2 \cos t + \sin\left(\sqrt{6}\, t\right)$$

Example 5.7 : *The equation of motion of a certain mass spring system is given by* $\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_1 \\ -k_2 & k_1+k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *Determine the normal mode of vibration for* $m_1 = m_2 = m$ *and* $k_1 = k_2 = k$ *with initial conditions* $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *and the displacement vector* $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution : Step 1 : The given equation can be written as

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

i.e. $$\ddot{X} = AX \qquad \dots (1)$$

Step 2 : Let the solution be of the form $X = Y e^{\omega t}$

Where ω^2 is the eigen value of A and Y is corresponding eigen vector.

Step 3 : Now, the characteristic equations for A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} -\frac{2k}{m} - \lambda & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 + \frac{4k}{m}\lambda + \frac{3k^2}{m^2} = 0$$

$$\Rightarrow \quad \left(\lambda + \frac{3k}{m}\right)\left(\lambda + \frac{k}{m}\right) = 0$$

$$\therefore \quad \omega_1^2 = \lambda = -\frac{3k}{m}, \quad \omega_2^2 = \lambda = -\frac{k}{m}$$

which corresponds to

$$\omega_1 = i\sqrt{\frac{3k}{m}}, \quad \omega_2 = i\sqrt{\frac{k}{m}}$$

Step 4 : To find the eigen vectors for the eigen values consider the matrix equation $[A - \lambda I]Y = 0$

$$\begin{bmatrix} -\frac{2k}{m} - \lambda & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} - \lambda \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Put $\lambda = -\frac{3k}{m}$

$$\therefore \quad \begin{bmatrix} \frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & \frac{k}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e. $$\frac{k}{m}(y_1 - y_2) = 0$$

Put $y_2 = 1 \Rightarrow y_1 = -1$

$\therefore\ Y_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the first eigen vector

$$Y_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_1 = -\frac{3k}{m}$ and $\lambda_2 = -\frac{k}{m}$

Thus the eigen vectors are

$$Y_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } Y_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ respectively}$$

Step 5 : General solution is

or $$X = Y_1\, e^{\omega_1 t} + Y_2\, e^{\omega_2 t}$$

or $$X = Y_1\, e^{\sqrt{\frac{3k}{m}}\, i\, t} + Y_2\, e^{\sqrt{\frac{k}{m}}\, i\, t}$$

Rearranging the constants we get

$$\therefore \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\left(a_1 \cos\sqrt{\frac{3k}{m}}\, t + b_2 \sin\sqrt{\frac{3k}{m}}\, t\right)$$

$$+ \begin{bmatrix} 1 \\ 1 \end{bmatrix}\left(a_2 \cos\sqrt{\frac{k}{m}}\, t + b_2 \sin\sqrt{\frac{k}{m}}\, t\right) \qquad \text{... (2)}$$

Step 6 : Differentiating (2), w.r.t. t we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sqrt{\frac{3k}{m}} \left(-a_1 \sin\sqrt{\frac{3k}{m}}\, t + b_1 \cos\sqrt{\frac{3k}{m}}\, t\right)$$

$$+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sqrt{\frac{k}{m}} \left(-a_2 \sin\sqrt{\frac{k}{m}}\, t + b_2 \cos\sqrt{\frac{k}{m}}\, t\right)$$

Step 7 : Using the given conditions at t = 0

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we get, $-a_1 + a_2 = 1$ and $a_1 + a_2 = 0$

$$\Rightarrow \quad \boxed{a_1 = -\frac{1}{2}, \ a_2 = \frac{1}{2}}$$

Step 8 : Using $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at $t = 0$

gives $-\sqrt{\frac{3k}{m}}\, b_1 + \sqrt{\frac{k}{m}}\, b_2 = 0$ and $\sqrt{\frac{3k}{m}}\, b_1 + \sqrt{\frac{k}{m}}\, b_2 = 0$

$$\Rightarrow \quad \boxed{b_1 = 0, \ b_2 = 0}$$

Step 9 : $\therefore$ From (2), we get

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \left(-\frac{1}{2}\right) \cos\sqrt{\frac{3k}{m}}\, t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{1}{2}\right) \cos\sqrt{\frac{k}{m}}\, t$$

Example 5.8 : *A system shown in the figure begins to move with initial displacement $x_1 = 1$, $x_2 = 1$, $x_3 = -1$ and initial velocities $V_1 = 1$, $V_2 = 0$, $V_3 = 2$. Assuming that there is no friction. Determine subsequent motion.*

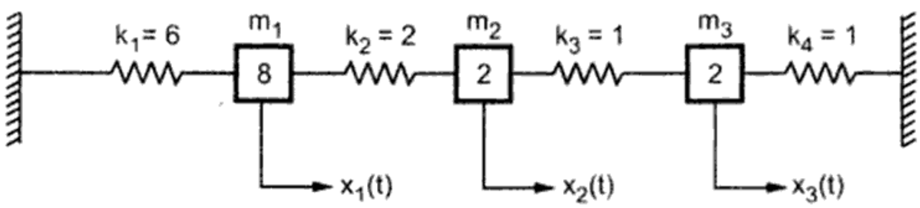


Fig. 5.9

Solution : Step 1 : The equation of motion are

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1) + k_3 (x_3 - x_2)$$

$$m_3 \ddot{x}_3 = -k_3 (x_3 - x_2) - k_4 x_3$$

Step 2 : Substituting the values m_1, m_2, m_3, k_1, k_2, k_3, k_3 we get

or $$8\ddot{x}_1 = -6x_1 + 2(x_2 - x_1)$$

$$2\ddot{x}_2 = -2(x_2 - x_1) + (x_3 - x_2)$$

$$2\ddot{x}_3 = -(x_3 - x_2) - x_3$$

or
$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{4} & 0 \\ 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Step 3 : The system takes the form

$$\ddot{X} = AX \qquad \dots (1)$$

Where
$$A = \begin{bmatrix} -1 & \frac{1}{4} & 0 \\ 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{bmatrix}$$

Step 4 : Now, the characteristic equation for 'A' is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} -1-\lambda & \frac{1}{4} & 0 \\ 1 & -\frac{3}{2}-\lambda & \frac{1}{2} \\ 0 & \frac{1}{2} & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 2\lambda^3 + 7\lambda^2 + 7\lambda + 2 = 0$$

$$(\lambda+1)(\lambda+2)(2\lambda+1) = 0$$

$\Rightarrow \lambda = -1, -2, -\frac{1}{2}$ which corresponds to $\omega_1 = i$, $\omega_2 = \sqrt{2}\, i$, $\omega_3 = \frac{1}{\sqrt{2}} i$

Step 5 : To find eigen vectors consider the matrix equation $[A - \lambda I] X = 0$

$$\begin{bmatrix} -1-\lambda & \frac{1}{4} & 0 \\ 1 & -\frac{3}{2}-\lambda & \frac{1}{2} \\ 0 & \frac{1}{2} & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = -1$

$$\begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by R_2 $$x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = 0$$

by R_3 $$0x_1 + \frac{1}{2}x_2 + 0x_3 = 0$$

i.e. $$2x_1 + x_2 + x_3 = 0$$

$$0x_1 + x_2 + 0x_3 = 0$$

Using Cramers rule

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}}$$

$\therefore$ $$\frac{x_1}{-1} = \frac{-x_2}{0} = \frac{x_3}{2}$$

i.e. $$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{2}$$

$\therefore$ $$X_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

is the first eigen vector corresponding to $\lambda_1 = -1$

Similarly we can obtain eigen vectors corresponding to $\lambda_2 = -2$ and $\lambda_3 = -\frac{1}{2}$

$$X_2 = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Thus the eigen vectors

$X_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ corresponding to $\lambda = -1, -2, -\frac{1}{2}$ respectively.

Step 6 : The general solution is

$\therefore$ $$X = Y_1 e^{\omega_1 t} + Y_2 e^{\omega_2 t} + Y_3 e^{\omega_3 t}$$

or $$X = Y_1 e^{it} + Y_2 e^{\sqrt{2}\, it} + Y_3 e^{\frac{1}{\sqrt{2}} it}$$

Rearranging the constants we get

$$\therefore \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} (a_1 \cos t + b_1 \sin t) + \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \left(a_2 \cos\sqrt{2}\, t + b_2 \sin\sqrt{2}\, t\right)$$

$$+ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \left(a_3 \cos\frac{1}{\sqrt{2}} t + b_3 \sin\frac{1}{\sqrt{2}} t\right) \qquad \ldots (2)$$

Step 7 : Now, given at $t = 0$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Substituting in (2), we get

$$a_1 + a_2 + a_3 = 1$$

$$-4a_2 + 2a_3 = 1$$

$$-2a_1 + 2a_2 + 2a_3 = -1$$

Solving, we get $a_1 = -\frac{3}{2}, a_2 = -\frac{1}{6}, a_3 = \frac{2}{3}$

Step 8 : Also differentiating (2), we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} (-a_1 \sin t + b_1 \cos t) + \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \sqrt{2}\left(-a_2 \sin\sqrt{2}\, t + b_2 \cos\sqrt{2}\, t\right)$$

$$+ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \frac{1}{\sqrt{2}} \left(-a_3 \sin\frac{1}{\sqrt{2}} t + b_3 \cos\frac{1}{\sqrt{2}} t\right) \qquad \ldots (3)$$

Given at $t = 0$, the velocities are

$$V_1 = \dot{x}_1 = 1$$

$$V_2 = \dot{x}_2 = 0$$

$$V_3 = \dot{x}_3 = 2$$

Step 9 : $\therefore$ Substituting in (3)

$$b_1 + \sqrt{2}\, b_2 + \frac{b_3}{\sqrt{2}} = 1$$

$$-4\sqrt{2}\, b_2 + \sqrt{2}\, b_3 = 0$$

$$-2b_1 + 2\sqrt{2}\, b_2 + \sqrt{2}\, b_3 = 2$$

Solving, we get $b_1 = 0, b_2 = \frac{\sqrt{2}}{3}, b_3 = \frac{4\sqrt{2}}{3}$

Step 10 : Substituting $a_1, a_2, a_3, b_1, b_2, b_3$ in equation (2), we get

$$x_1(t) = -\frac{3}{2}\cos t - \frac{1}{6}\cos\sqrt{2}\,t + \frac{\sqrt{2}}{3}\sin\sqrt{2}\,t + \frac{2}{3}\cos\frac{1}{\sqrt{2}}t + \frac{4\sqrt{2}}{3}\sin\frac{1}{\sqrt{2}}t$$

$$x_2(t) = \frac{4}{6}\cos\sqrt{2}\,t - \frac{4\sqrt{2}}{3}\sin\sqrt{2}\,t + \frac{4}{3}\cos\frac{1}{\sqrt{2}}t + \frac{8\sqrt{2}}{3}\sin\frac{1}{\sqrt{2}}t$$

and

$$x_3(t) = 3\cos t - \frac{1}{3}\cos\sqrt{2}\,t + \frac{2\sqrt{2}}{3}\sin\sqrt{2}\,t + \frac{4}{3}\cos\frac{1}{\sqrt{2}}t + \frac{8\sqrt{2}}{3}\sin\frac{1}{\sqrt{2}}t$$

is the required solution.

Exercise

1. *Solve the following equation of motion of a certain damped mass spring system, using matrix method.*

   i) $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *with initial conditions :*

   $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and* $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

   ii) $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 24 & -4 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *with initial conditions*

   $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$

2. *The system shown in figure begins to move with initial displacements* $\bar{y}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and initial velocities* $\dot{\bar{y}}_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. *Assuming that there is no friction in the system, determine it's subsequent motion.*

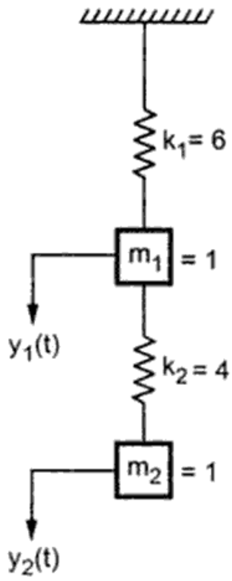


Fig. 5.10

3. *The mass spring system shown in figure is given an initial displacement so that* $x_1 = 2$, $x_2 = -1$, $x_3 = 1$

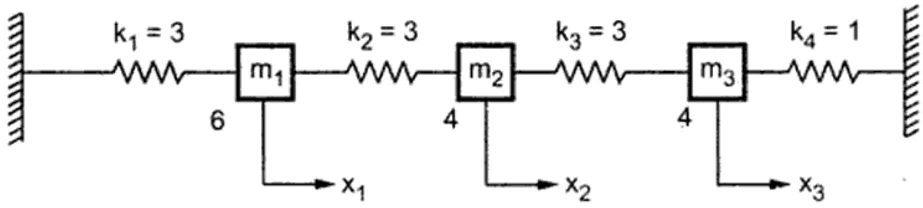


Fig. 5.11

From these positions the system begins to move with initial velocities $V_1 = 1$, $V_2 = 2$, $V_3 = 0$. *Assuming there is no friction in the system, determine the subsequent motion of the system.*

[Ans. : $x_1 = \frac{1}{2}\cos\frac{t}{2} + \frac{4}{5}\cos + \frac{7}{10}\cos\frac{3t}{2} + \frac{3}{2}\sin\frac{t}{2} + \frac{3}{5}\sin t - \frac{7}{20}\sin\frac{3}{2}t$

$x_2 = \frac{3}{4}\cos\frac{t}{2} - \frac{7}{4}\cos\frac{3}{2}t + \frac{9}{4}\sin\frac{t}{2} + \frac{7}{8}\sin\frac{3}{2}t$

$x_3 = \frac{3}{4}\cos\frac{t}{2} - \frac{4}{5}\cos t + \frac{21}{20}\cos\frac{3}{2}t + \frac{9}{4}\sin\frac{t}{2} - \frac{3}{5}\sin t - \frac{21}{40}\sin\frac{3}{2}t$ **]**

4. *Find the natural frequencies and normal modes of vibrations of the system shown in figure*

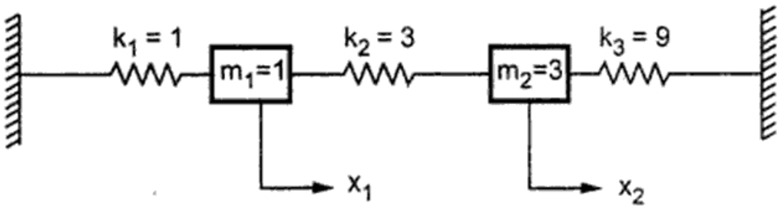


Fig. 5.12

[Ans. : $\omega_1 = \sqrt{4+\sqrt{3}}$, $\omega_2 = \sqrt{4-\sqrt{3}}$

$x_1 = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \sin(\omega_1 t + \alpha_1)$, $x_2 = \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix} \sin(\omega_2 t + \alpha_2)$ **]**

University Questions

Dec. - 98

1. *For the system shown in Fig. 5.13 if $m_1 = 1$, $m_2 = 3$, $k_1 = 1$, $k_2 = 3$, $k_3 = 3$, assuming that there is no friction, find the natural frequencies of the system and corresponding normal modes of vibration, using matrix-method.* **[8 Marks]**

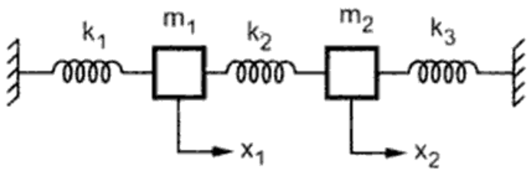


Fig. 5.13

May - 99

1. 1. *For the system shown in the following Fig. 5.13 (a) if $m_1 = 1$, $m_2 = 3$, $k_1 = 1$, $k_2 = 3$, $k_3 = 3$ assuming that there is no friction, find the natural frequencies of the system and corresponding normal modes of vibration using matrix method.* **[9 Marks]**

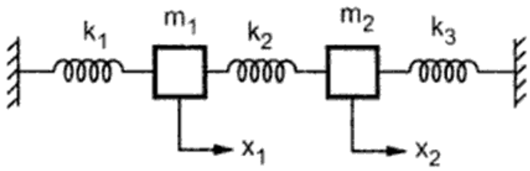


Fig. 5.13 (a)

Dec. - 99

1. *For the system shown in the following Fig. 5.14 if $m_1 = 1$, $m_2 = 3$, $k_1 = 1$, $k_2 = 3$, $k_3 = 3$ assuming that there is no friction, find the natural frequencies of the system and corresponding normal modes of vibration using matrix method.* **[8 Marks]**

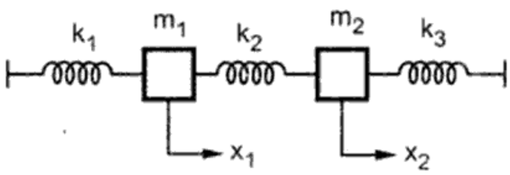


Fig. 5.14

May - 2000

1. *For the system shown in the Fig. 5.15 if :*

   $m_1 = 1$, $m_2 = 3$, $k_1 = 1$, $k_2 = 3$, $k_3 = 3$,

   Assuming that there is no friction, find the natural frequencies of the system and corresponding normal modes of vibration using matrix method. **[8 Marks]**

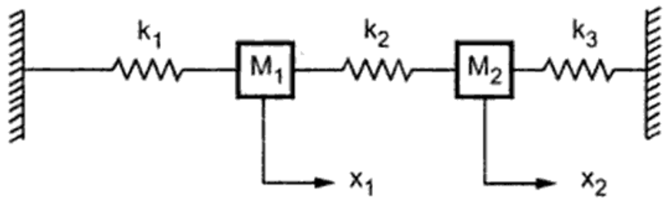


Fig. 5.15

Dec. - 2000

1. *System of differential equations of an undamped mechanical system is given by*

   $\ddot{x}_1 = x_1 - 3(x_1 - x_2)$

   $3\ddot{x}_2 = 3(x_1 - x_2) - 3x_2$

   Assuming that there is no friction, find the natural frequencies of the system and corresponding normal modes of vibration using matrix method.

   Initial conditions are :

   $x_1(0) = 1, \quad x_2(0) = 2, \quad x_1(0) = 2, \quad \dot{x}_2(0) = -1$ **[8 Marks]**

May - 2001

1. *System of different equations of an undamped mechanical system is given by*

   $m_1\ddot{y}_1 = -k_1y_1 - k_2(y_1 - y_2)$

   $m_2\ddot{y}_2 = k_2(y_1 - y_2)$

   with initial conditions

   $y_1(0) = 1, \quad y_2(0) = 2, \quad \dot{y}_1(0) = -2\sqrt{6}, \quad \dot{y}_2(0) = \sqrt{6}, \quad k_1 = 3, \quad k_2 = 2$

   $m_1 = m_2 = 1$

   Assuming that there is no friction in the system, find the solution of the system, using eigen values. **[8 Marks]**

Dec. - 2001

1. *The system shown in the Fig. 5.16 below begins to move with initial displacement* $Y_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and initial velocity* $\dot{Y}_0 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

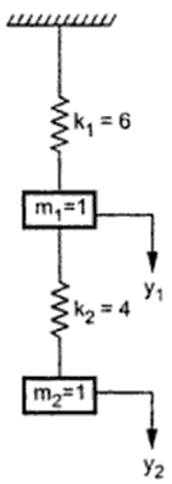


Fig. 5.16

May - 2002

1. *For the system shown in Fig. 5.17 if $m_1 = 1$, $m_2 = 3$, $k_1 = 1$, $k_2 = 3$, assuming that there is no friction, find the natural frequencies of the system and corresponding normal modes of vibration, using matrix method.* **[8 Marks]**

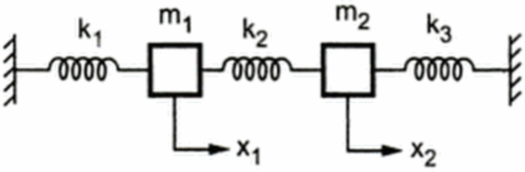


Fig. 5.17

Dec. - 2002

1. *System of differential equations of an undamped mechanical system is given by*

   $\ddot{x}_1 = -x_1 - 3(x_1 - x_2)$, $3\ddot{x}_2 = 3(x_1 - x_2) - 3x_2$.

   Assuming that there is no friction, find the natural frequencies of the system and the corresponding normal modes of vibrations using matrix method. **[8 Marks]**

May - 2003

1. *For the horizontal mass spring system if $m_1 = 1$, $m_2 = 3$, $k_1 = 1$, $k_2 = 3$, $k_3 = 3$, assuming that there is no friction, find the natural frequencies of the system and corresponding normal modes of vibration by using matrix method.* **[8 Marks]**

Dec. - 2003

1. *For the system shown below if $M_1 = 1$, $M_2 = 3$, $k_1 = 1$, $k_2 = 3$, $k_3 = 3$, assuming that there is no friction find the natural frequencies of the system and corresponding normal modes of vibration, using matrix method.* **[8 Marks]**

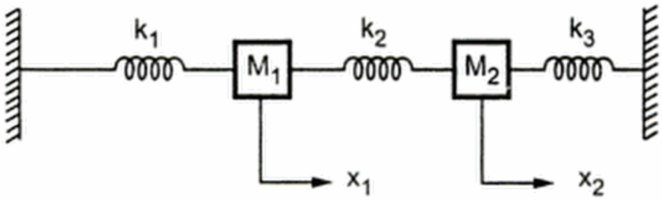


Fig. 5.18

May - 2004

1. *Use Matrix method to find normal model of vibration and natural frequencies for the following system :*

   Assuming that there is no friction in the system. **[8 Marks]**

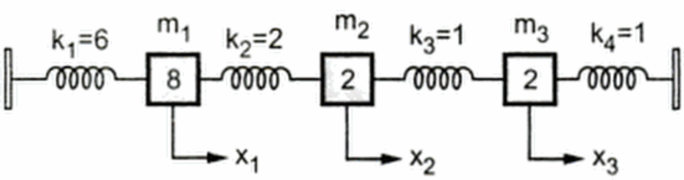


Fig. 5.19

Dec. - 2004

1. *For the system shown in figure if $m_1 = 1$, $m_2 = 3$, $k_1 = 1$, $k_2 = 3$, $k_3 = 3$ assuming that there is no friction find the natural frequencies of the system and corresponding normal modes of vibration using matrix method.* **[8 Marks]**

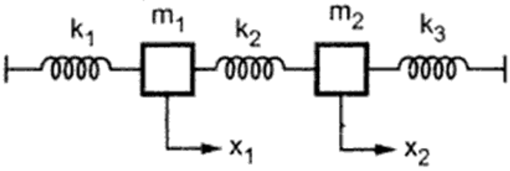


Fig. 5.20

Dec. - 2005

1. *For the system shown in the figure below :*

   where $m_1 = 1$, $m_2 = 3$, $k_1 = 1$, $k_2 = 3$, $k_3 = 3$.

   Assuming that there is no friction, find the natural frequencies of the system and corresponding normal modes of vibration, using matrix method. **[8 Marks]**

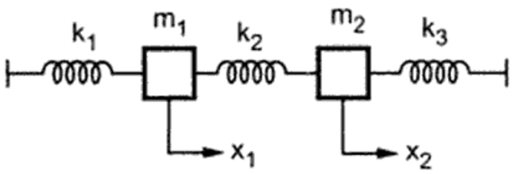


Fig. 5.21

May - 2006

1. *The system shown in Fig. 5.22 begins to move with an initial conditions*

   $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *and* $\dot{x}(0) = \begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$

   Assuming there is not friction, determine subsequent motion. **[7 Marks]**

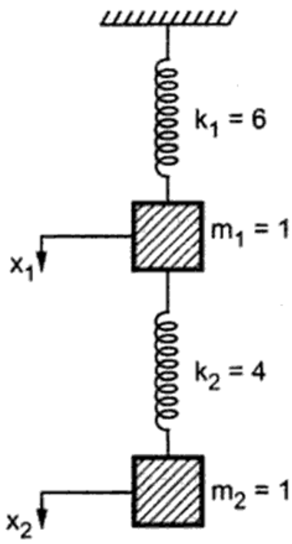


Fig. 5.22

2. *The mass system in Fig. 5.23 begins to oscillate with initial conditions :*

$x(0) = [1, \ 1, \ 1]^T$ *and* $\dot{x}(0) = [1, \ 0, \ 2]^T$.

Assuming that there is no friction find normal modes. **[8 Marks]**

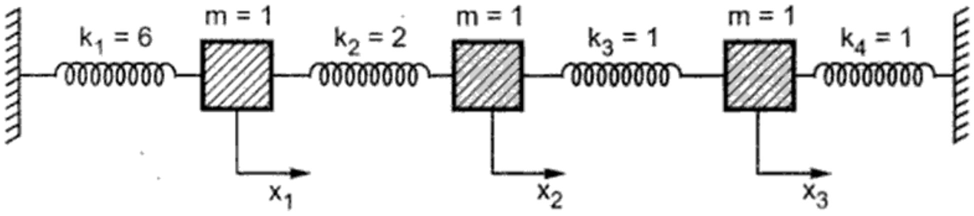


Fig. 5.23

❑❑❑

6

Laplace Transform

6.1 Introduction

While doing circuit analysis we need to solve linear differential equations with constant coefficients. Solving these differential equations mathematically is difficult. Laplace transform converts a problem of solving differential equations into one or more algebraic equations with few initial conditions. Finally the solution of a differential equation is obtained by taking inverse Laplace Transform.

Thus Laplace Transforms is a very essential versatile tool used by engineers, whose application is considerably easier than other traditional available techniques.

6.2 Definition

The Laplace transformation is an operation denoted by 'L' which associates to each function f(t) ($t > 0$), a unique function $\phi(s)$ called the Laplace Transform of f(t) defined as

$$L\ f(t) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \phi(s)$$

where L is called as Laplace transform operator and s is a parameter, real or complex.

Note : As Laplace transform is a definite integral, therefore all the properties of definite integrals are true for Laplace transforms.

6.3 Condition for Existance of Laplace Transform

A function f(t) is said to have its Laplace transform if

i) f(t) is piece wise continuous.

ii) f(t) is a function of exponential order.

a) Piece wise continuous function

A function f(t) is said to be piece wise continuous in the given interval, if f(t) is continuous in finite number of subintervals which are obtained by subdividing the given interval.

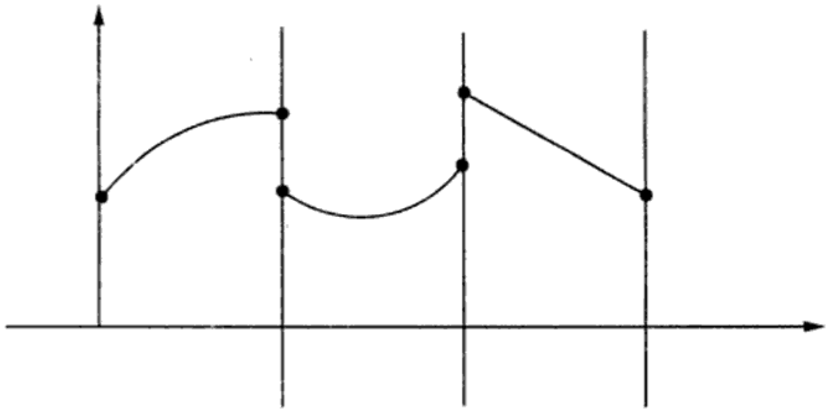


Fig. 6.1

b) Function of exponential order

If there is a constant β with the property that $e^{-\beta t}|f(t)|$ remains bounded as $t \to \infty$ then $f(t)$ is said to be of exponential order.

i.e. $f(t)$ is of exponential order if $e^{-\beta t}|f(t)| < M$ for $t > N$

where M, N, β are all constants.

Note : The above two conditions are sufficient for existance of Laplace transforms but not necessary. Laplace transform exists even if these conditions are not satisfied.

6.4 Laplace Transform of Some Standard Functions

1) $$f(t) = e^{at}$$

$$L\, f(t) = \int_0^\infty e^{-st} f(t)\, dt$$

$$L(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at} dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= 0 - \frac{1}{-(s-a)} \qquad \text{if } s > a$$

$$L(e^{at}) = \frac{1}{s-a} \qquad \text{if } s > a$$

Similarly $$L(e^{-at}) = \frac{1}{s+a}$$

2) $$f(t) = \sinh at$$

$$L\, f(t) = L \sinh at$$

$$= L\left(\frac{e^{at} - e^{-at}}{2}\right)$$

$$= \frac{1}{2}\left[L\left(e^{at}\right) - L\left(e^{-at}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{a}{s^2 - a^2}$$

3) $$f(t) = \cosh at$$

$$L\, f(t) = L\,(\cosh at)$$

$$= L\left(\frac{e^{at} + e^{-at}}{2}\right)$$

$$= \frac{1}{2} L\left(e^{at} + e^{-at}\right)$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{s}{s^2 - a^2}$$

4) $$f(t) = \sin at$$

$$L\, f(t) = L\,(\sin at)$$

$$= \int_0^\infty e^{-st} \cdot \sin at\, dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2}(-s \sin at - a \cos at)\right]_0^\infty$$

$$= 0 - \frac{1}{s^2 + a^2}\,(0 - a)$$

$$= \frac{a}{s^2 + a^2}$$

{formula used $\int e^{ax} \sin bx\, dx = \dfrac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$ $e^{-\infty} = 0$, $e^0 = 1$, $\cos 0 = 1$, $\sin 0 = 0$}

5) $$f(t) = \cos at$$

$$L\, f(t) = L \cos at$$

$$= \int_0^\infty e^{-st} \cos at\, dt$$

$$= \left\{ \frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \right\}_0^\infty$$

$$= 0 - \frac{1}{s^2 + a^2} (-s + 0)$$

$$= \frac{s}{s^2 + a^2}$$

{formula used $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$}

6) $$f(t) = t^n$$

{Recall formulae $\overline{|n} = \int_0^\infty e^{-u}\, u^{n-1} du$

$$\overline{|n+1} = n \overline{|n}$$

$$\overline{|n+1} = n! \text{ for positive integers}\}$$

$\therefore$ $$L\, f(t) = \int_0^\infty e^{-st}\, t^n\, dt$$

Put $st = u$

$s\, dt = du$

t	0	∞
u	0	∞

$$L(t^n) = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \cdot \frac{du}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u}\, u^n\, du$$

$$= \frac{\overline{|n+1}}{s^{n+1}} = \frac{n!}{s^{n+1}} \quad \text{(if n positive integer)}$$

Substituting $n = 0, 1, 2$ in above result we get

7) $$L(1) = \frac{1}{s}$$

8) $$L(t) = \frac{1}{s^2}$$

9) $$L(t^2) = \frac{2}{s^3} \text{ and so on.}$$

6.5 Properties of Laplace Transforms

1) First shifting property :

If $L\ f(t) = \phi(s)$ then $L\left\{e^{+at} f(t)\right\} = \phi(s-a)$

Proof : We have

$$L\ f(t) = \int_0^\infty e^{-st} f(t)\, dt = \phi(s) \qquad \text{... (1)}$$

Consider

$$\therefore \quad L\left\{e^{at} f(t)\right\} = \int_0^\infty e^{-st} \cdot e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} \cdot f(t)\, dt$$

$$= \phi(s-a) \quad \text{from (1)}$$

Similarly we can prove that

$$L\left\{e^{-at} f(t)\right\} = \phi(s+a)$$

2) Second shifting property : (Laplace of displaced function)

If $L\ f(t) = \phi(s)$ and a function $g(t)$ is defined as

$$g(t) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$$

then

$$L\ g(t) = e^{-as} \cdot \phi(s)$$

$$= e^{-as} \cdot L\ f(t)$$

Fig. 6.2 Graph of f(t)

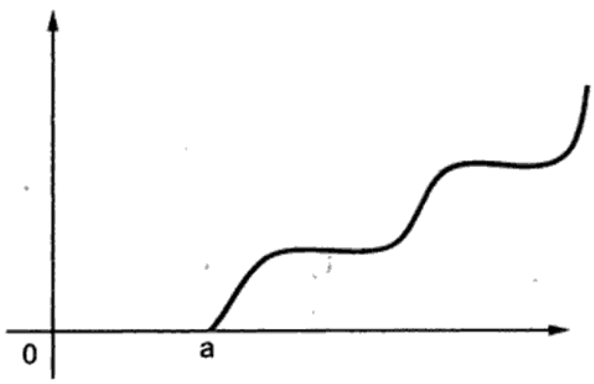


Fig. 6.3 Graph of displaced function g(t)

Proof : We have

$$L\ f(t) = \int_0^\infty e^{-st} f(t) dt = \phi(s) \qquad \text{... (1)}$$

Consider $\quad L\ g(t) = \int_0^\infty e^{-st} g(t)\ dt$

As g(t) involves two parts split the integral

$$= \int_0^a e^{-st} \cdot 0 \cdot dt + \int_a^\infty e^{-st} f(t-a)\ dt$$

Put t - a = u

dt = du

t	a	∞
u	0	∞

$$= \int_0^\infty e^{-s(a+u)} \cdot f(u)\ du$$

$$= e^{-as} \int_0^\infty e^{-su} f(u)\ du$$

$$= e^{-as}\ \phi(s) \qquad \text{from (1)}$$

3) Effect of multiplication by t :

If $L\ f(t) = \phi(s)$ then $L\ [t\ f(t)] = -\phi'(s)$

and $L\ [t^2\ f(t)] = (-1)^2 \dfrac{d^2}{ds^2} \phi(s)$ and so on.

Proof : We have

$$L\ f(t) = \int_0^\infty e^{-st} f(t)\ dt = \phi(s) \qquad \text{... (1)}$$

Differentiating w.r.t s we get

$$\frac{d}{ds}\phi(s) = \frac{d}{ds}\int_0^\infty e^{-st}\ f(t)\ dt$$

Using the rule of differentiation under integral sign (D.U.I.S) we get

$$\phi'(s) = \int_0^\infty \frac{\partial}{\partial s} e^{-st}\ f(t)\ dt$$

$$= \int_0^\infty -t\, e^{-st}\ f(t)\ dt$$

$$\phi'(s) = -L[t\,f(t)]$$

Thus $$L\,t\,f(t) = -\phi'(s)$$

Similarly using the rule of D.U.I.S again, we get

$$L\,t^2 f(t) = (-1)^2 \frac{d^2}{ds^2}\phi(s) \quad \text{and so on.}$$

4) Effect of division by t :

If $L\,f(t) = \phi(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \phi(s)\,ds$

Proof : We have

$$L\,f(t) = \int_0^\infty e^{-st} f(t)\,dt = \phi(s) \quad \dots (1)$$

Integrating both sides within the limits s to ∞ we get

$$\int_s^\infty \phi(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st}\,ds\right] f(t)\,dt$$

$$= \int_0^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)\,dt$$

$$= \int_0^\infty \left(0 - \frac{e^{-st}}{-t}\right) f(t)\,dt$$

$$= \int_0^\infty \frac{e^{-st}}{t} f(t)\,dt$$

$$= L\frac{f(t)}{t}$$

Similarly we can show that

$$L\frac{f(t)}{t^2} = \int_s^\infty \left\{\int_s^\infty \phi(s)\,ds\right\} ds$$

6.6 Convolution Theorem

The convolution of f(t) and g(t) is, denoted by $f(t) * g(t)$ and defined by

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

and now if L [f(t)] = F(s) and L [g(t)] = G(s), then

$$L[f(t) * g(t)] = F(s) \,.\, G(s)$$

and since the convolution is commutative, we have

$$L[f(t) * g(t)] = L\left[\int_0^t g(u) \cdot f(t-u)\, du\right] = F(s) \cdot G(s)$$

Proof : We have,

$$L[f(t) * g(t)] = L\left[\int_0^t f(u) \cdot g(t-u)\, du\right] \quad \text{... by definition of convolution}$$

$$= \int_{t=0}^{\infty} e^{-st}\left[\int_{u=0}^{t} f(u) \cdot g(t-u)\, du\right] dt \quad \text{... by definition of L.T.}$$

$$= \int_{t=0}^{\infty} \int_{u=0}^{t} e^{-st} f(u) \cdot g(t-u)\, du \cdot dt$$

Note that the region of integration for the above integration is the entire region lying between the lines u = 0 and u = t in the (u – t) plane as shown in Fig. 6.4. Thus, changing the order of integration we get Fig. 6.5.

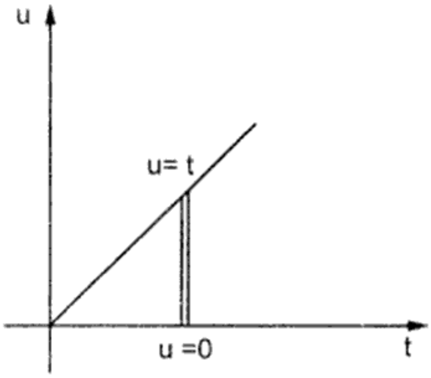


Fig. 6.4 Region from given limits

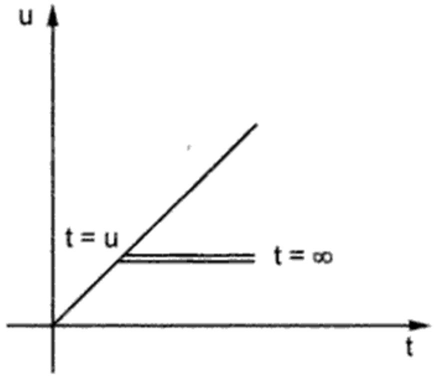


Fig. 6.5 Region after changing the order of integration

$$L[f(t) * g(t)] = \int_{u=0}^{\infty} \left[\int_{t=u}^{\infty} e^{-st} f(u) \cdot g(t-u)\, dt \right] du$$

$$= \int_{u=0}^{\infty} f(u) \left[\int_{t=u}^{\infty} e^{-st} g(t-u)\, dt \right] du$$

As u is independent of t multiply by e^{-su} outside and e^{su} inside the integral.

$$= \int_{u=0}^{\infty} e^{-su} f(u) \left[\int_{t=u}^{\infty} e^{su} \cdot e^{-st} \cdot g(t-u) dt \right] du$$

$$= \int_{u=0}^{\infty} e^{-su} f(u) \left[\int_{t=u}^{\infty} e^{-s(t-u)} \cdot g(t-u)\, dt \right] du$$

Put $t - u = v \quad \therefore \quad dt = dv$

t	u	∞
v	0	∞

$$\therefore \qquad L[f(t) * g(t)] = \int_{u=0}^{\infty} e^{-su} f(u) \left[\int_{v=0}^{\infty} e^{-sv} \cdot g(v) \cdot dv \right] du$$

$$= \int_{u=0}^{\infty} e^{-su} f(u) \cdot G(s)\, du \qquad \text{... by definition of L.T.}$$

$$= G(s) \int_{0}^{\infty} e^{-su} \cdot f(u)\, du \qquad \text{... } \because G(s) \text{ is independent of u}$$

$$= G(s) \cdot F(s) \qquad \text{by definition of L.T.}$$

$$\therefore \qquad \boxed{L[f(t) * g(t)] = G(s) \cdot F(s)}$$

Thus

$$\boxed{L^{-1}[F(s) \cdot g(s)] = f(t) * g(t) = \int_{0}^{t} f(u) \cdot g(t-u)\, du}$$

6.7 Initial Value Theorem

If $L[f(t)] = \phi(s)$ then

$$\boxed{\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\,\phi(s)} \qquad \text{provided the limits exist}$$

Proof :

We know that

$$L\,[f'(t)] = s\,\phi(s) - f(0) \qquad \text{... assuming } f(t) \text{ to be continuous}$$

i.e. $$\int_0^{\infty} e^{-st} f'(t)\, dt = s\,\phi(s) - f(0) \qquad \text{... by definition of L.T.}$$

Taking limits of both the sides as $s \to \infty$ we get

$$\lim_{s\to\infty} \int_0^{\infty} e^{-st} f'(t) dt = \lim_{s\to\infty} \left[s\,\phi(s) - f(0)\right]$$

i.e. $$\int_0^{\infty} \lim_{s\to\infty} e^{-st} f'(t) dt = \lim_{s\to\infty} s\,\phi(s) - f(0) \quad \text{... } \because s \text{ and } t \text{ are independent and } f(0) \text{ is constant}$$

$$\therefore \quad \int_0^{\infty} (0)\, f'(t)\, dt = \lim_{s \to \infty} s\,\phi(s) - f(0)$$

$$0 = \lim_{s \to \infty} s\,\phi(s) - f(0) \qquad \text{... assuming } f'(t) \text{ to be continuous}$$

$$f(0) = \lim_{s \to \infty} s\,\phi(s)$$

i.e. $$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\,\phi(s)$$

6.8 Final Value Theorem

If $$L[f(t)] = \phi(s) \text{ then}$$

$$\boxed{\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,\phi(s)} \qquad \text{provided these limit exists}$$

Proof :

We have,

$$L[f'(t)] = s\,\phi(s) - f(0)$$

i.e. $$\int_0^{\infty} e^{-st} f'(t)\, dt = s\,\phi(s) - f(0)$$

Taking limits of both the sides as $s \to 0$ we get

$$\lim_{s \to 0} \int_0^{\infty} e^{-st} f'(t)\, dt = \lim_{s \to 0} [s\,\phi(s) - f(0)]$$

$$\therefore \int_0^{\infty} \lim_{s \to 0} e^{-st} f'(t)\, dt = \lim_{s \to 0} s\,\phi(s) - f(0) \qquad \text{... } \because s \text{ and } t \text{ are independent}$$

$$\therefore \quad \int_0^\infty (1)\ f'(t)\ dt = \lim_{s \to 0} s\,\phi(s) - f(0)$$

$$\therefore \quad [f(t)]_0^\infty = \lim_{s \to 0} s\,\phi(s) - f(0)$$

$$\text{i.e.} \quad \lim_{t \to \infty} f(t) - f(0) = \lim_{s \to 0} s\,\phi(s) - f(0)$$

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,\phi(s)$$

Illustrations

Example 6.1 : *Verify initial value theorem for*

1) $f(t) = t^2 + \cos 2t$

Solution : Consider $\lim_{t \to 0} f(t)$

$$= \lim_{t \to 0} \left(t^2 + \cos 2t\right)$$

$$= 0 + 1$$

$$= 1 \qquad \text{... (i)}$$

Now $\phi(s) = L\,f(t)$

$$= L\ t^2 + \cos 2t$$

$$= \frac{2\,!}{s^3} + \frac{s}{s^2+4}$$

$$\therefore \quad s\,\phi(s) = \frac{2\,!}{s^2} + \frac{s^2}{s^2+4}$$

Now $\lim_{s \to \infty} s\,\phi(s) = \lim_{s \to \infty} \left(\frac{2}{s^2} + \frac{1}{1+4/s^2}\right)$

$$= 0 + 1$$

$$= 1 \qquad \text{... (ii)}$$

Thus from (i) and (ii)

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\,\phi(s)$$

$\therefore$ Initial value theorem is verified.

Example 6.2 : *Verify the final value theorem for* $f(t) = t^2 e^{-2t}$

Solution :

$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} \frac{t^2}{e^{2t}} = 0 \quad \text{(by L Hospital's rule)} \qquad \text{... (i)}$$

Now $\phi(s) = L\ f(t) = L\left(t^2\ e^{-2t}\right) = \dfrac{2\,!}{(s+2)^3}$ by first shifting

Now $\lim\limits_{s \to 0} s\,\phi(s) = \lim\limits_{s \to 0} \dfrac{2s}{(s+2)^3}$

$$= 0 \qquad \text{... (ii)}$$

Thus from (i) and (ii)

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,\phi(s)$$

∴ Final value theorem is verified.

Example 6.3 : *Verify convolution theorem for $f(t) = t^2$ and $g(t) = e^t$.*

Solution : Here

$$f(t) = t^2 \qquad F(s) = \frac{2}{s^3}$$

$$g(t) = e^t \qquad g(s) = \frac{1}{s-1}$$

$$\therefore \quad F(s)\cdot g(s) = \frac{2}{s^3(s-1)} \qquad \text{... (i)}$$

Now
$$L\big(f(t) * g(t)\big) = L\int_0^t f(u)\ g(t-u)\,du$$

$$= L\int_0^t u^2 \cdot e^{(t-u)}\,du$$

$$= L\ e^t \int_0^t u^2\,e^{-u}\,du$$

$$= L\,e^t\left[u^2\left(\frac{e^{-u}}{-1}\right)-(2u)\left(\frac{e^{-u}}{1}\right)+(2)\left(\frac{e^{-u}}{-1}\right)\right]_0^t$$

$$= L\ e^t\left\{\left[\frac{t^2\,e^{-t}}{-1}-\frac{2t\,e^{-t}}{1}-2\,e^{-t}\right]-[0-0-2]\right\}$$

$$= L\left\{-t^2-2t-2+2\,e^t\right\}$$

$$= -\frac{2}{s^3}-\frac{2}{s^2}-\frac{2}{s}+\frac{2}{s-1}$$

$$= \frac{2}{s^3(s-1)} \qquad \text{... (ii) verified.}$$

Example 6.4 : *Verify Convolution theorem for Laplace transforms, for the function $f(t)=t^2$, $g(t)=\sin at$.* **[May-2001]**

Solution : We have

$$f(t) = t^2 \text{ and } g(t)=\sin at$$

$$F(s) = L[t^2] = \frac{2!}{s^3}$$

$$G(s) = L[\sin at] = \frac{a}{s^2+a^2}$$

We have to prove Convolution Theorem i.e.

$$L[f(t)*g(t)] = F(s)\cdot G(s) \qquad \dots (1)$$

Now, $$\text{RHS} = F(s)\cdot G(s) = \frac{2}{s^3}\cdot\frac{a}{(s^2+a^2)} = \frac{2a}{s^3(s^2+a^2)} \qquad \dots (2)$$

and $$f(t)*g(t) = \int_0^t f(u)\cdot g(t-u)\,du \qquad \text{... by definition of convolution}$$

$$= \int_0^t u^2 \sin a(t-u)\,du \qquad \text{... } \because f(t) = t^2 \text{ and } g(t) = \sin at$$

$$= \left\{u^2\left[\frac{-\cos a(t-u)}{-a}\right]-(2u)\left[\frac{-\sin a(t-u)}{a^2}\right]+2\left[\frac{+\cos a(t-u)}{-a^3}\right]\right\}_0^t$$

... Integration by parts i.e.

$$\left[\int uv\,dx = u\,v_1 - u'\,v_2 + u''\,v_3 \dots \text{etc}\right]$$

$$= \left[\frac{t^2}{a}(1)+0-\frac{2}{a^3}(1)\right]-\left[0+0-\frac{2}{a^3}\cos at\right]$$

$$= \frac{t^2}{a} - \frac{2}{a^3}(1-\cos at)$$

$$\text{LHS} = L\left[f(t) * g(t)\right]$$

$$= \frac{1}{a} L\left[t^2\right] - \frac{2}{a^3} L\left[1 - \cos at\right]$$

$$= \frac{1}{a}\frac{2!}{s^3} - \frac{2}{a^3}\left[\frac{1}{s} - \frac{s}{s^2 + a^2}\right]$$

$$= \frac{2}{a s^3} - \frac{2}{a^3}\left[\frac{\left(s^2 + a^2\right) - s^2}{s\left(s^2 + a^2\right)}\right]$$

$$= \frac{2}{a s^2} - \frac{2}{a}\frac{1}{s\left(s^2 + a^2\right)}$$

$$= \frac{2}{a}\left[\frac{1}{s^3} - \frac{1}{s\left(s^2 + a^2\right)}\right]$$

$$= \frac{2}{a}\left[\frac{\left(s^2 + a^2\right) - s^2}{s^3\left(s^2 + a^2\right)}\right]$$

$$= \frac{2}{a}\frac{a^2}{s^3\left(s^2 + a^2\right)} \qquad \text{... (3)}$$

$$= \frac{2a}{s^3\left(s^2 + a^2\right)}$$

Comparing (2) and (3), LHS = RHS i.e. Convolution Theorem is verified.

Example 6.5 : *Show that* $1 * 1 = t$, *hence prove that*

$$\underbrace{1 * 1 * 1 \ldots\ldots * 1}_{\leftarrow n\text{-ones} \rightarrow} = \frac{t^{n-1}}{(n-1)!}$$

Solution :

$$1 * 1 = \int_0^t 1 \cdot 1 \, du = t$$

$$(1*1)*1 = t * 1$$

$$= \int_0^t u \cdot 1 \, du$$

$$= \frac{t^2}{2}$$

$$(1*1*1)*1 = \frac{t^2}{2} * 1$$

$$= \int_0^t \frac{u^2}{2}\, 1\, du$$

$$= \frac{t^3}{3!}$$

$$1*1*1.....*1 = \frac{t^{n-1}}{(n-1)!}$$

Exercise 6.1

1. *Verify initial value theorem for the following :*

   i) $f(t) = 5 + 2\sin ht$

   ii) $f(t) = (2t-3)^2$

   iii) $f(t) = t + \sin 3t$

   iv) $f(t) = (3t+4)^2$

   v) $f(t) = 3 - 2\cos t$

   vi) $f(t) = 3e^{-2t}$

   vii) $f(t) = 5 + 2\cos 3t$

2. *Verify final value theorem for*

   i) $f(t) = 3e^{-2t}$

   ii) $f(t) = 2 + 3e^{-2t}\sin 4t$

   iii) $f(t) = t^3 e^{-4t}$

   iv) $f(t) = 4e^{-3t}$

   v) $f(t) = 1 + e^{-t}(\sin 2t + \cos 2t)$

   vi) $f(t) = t^2 e^{-t}$

3. *Verify convolution theorem for*

   i) $f(t) = t,\ g(t) = e^{at}$ **[Ans. : $L[f(t) * g(t)] = \frac{1}{s^2(s-a)}$]**

   ii) $f(t) = t,\ g(t) = \cos at$ **[Ans. : $L[f(t) * g(t)] = \frac{1}{s(s^2+a^2)}$]**

   iii) $f(t) = t^2,\ g(t) = e^{-at}$ **[Ans. : $L[f(t) * g(t)] = \frac{2}{s^3(s+a)}$]**

   iv) $f(t) = t,\ g(t) = \cos t$ **[Ans. : $L[f(t) * g(t)] = \frac{1}{s^2(s^2+1)}$]**

Table of theorems of laplace transforms

1.	First shifting theorem	$L\left[e^{-at} f(t)\right] = \phi(s+a)$
2.	Second shifting theorem	If $g(t) = f(t-a)$, $t > a$, $= 0$, $t < a$ then $L\left[g(t)\right] = e^{-as}\phi(s)$
3.	Change of scale theorem	$L\left[f(at)\right] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$
4.	Transforms of Derivatives	$L\left[f'(t)\right] = s\phi(s) - f(0)$, $L\left[f''(t)\right] = s^2\phi(s) - s\, f(0) - f'(0)$, $L\left[f'''(t)\right] = s^3 f(s) - s^2 f(0) - s\, f'(0) - f''(0)$ Thus, derivates $\rightarrow$ Multiplication by powers of s
5.	Transform of Integrals	$L\left[\int_0^t f(t)\, dt\right] = \frac{1}{s}\phi(s)$ $L\left[\int_0^t \int_0^t f(t)\, dt \cdot dt\right] = \frac{1}{s^2}\phi(s)$ Thus, integration $\rightarrow$ division
6.	Multiplication by t^n	$L\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n}\phi(s)$ Thus, multiplication $\rightarrow$ Derivatives
7.	Division by t	$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds$ Thus, division $\rightarrow$ integration
8.	Convolution Theorem	$L\left[f(t) * g(t)\right] = L\left[\int_0^t f(u)\, g(t-u)\, du\right]$ $= F(s) \cdot G(s)$
9.	Initial Value Theorem	$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\phi(s)$
10.	Final Value Theorem	$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\phi(s)$

Table 6.1 : Theorems of Laplace Transforms

Illustrations on Laplace Transform

6.9 Type 1 : Using Basic Definition

Example 6.6 : *Find Laplace transform of f(t) using definition* $f(t) = \begin{cases} t/T & 0 \le t \le T \\ 1 & t > T \end{cases}$

Solution : Step 1 : Given function f(t).

$$f(t) = \begin{cases} t/T & 0 \le t \le T \\ 1 & t > T \end{cases}$$

Step 2 : Use the formula of Laplace Transform.

$$L\,[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt$$

Step 3 : Substitute the value of f(t).

$$\therefore \quad L\,[f(t)] = \int_0^{T} e^{-st}\left(\frac{t}{T}\right) dt + \int_T^{\infty} e^{-st}\ 1\ dt$$

Step 4 : Integrate w.r.t t using integration by parts.

$$= \frac{1}{T}\left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^T + \left[\frac{e^{-st}}{-s}\right]_T^{\infty}$$

Step 5 : Substitute the limits of t.

$$= -\frac{e^{-sT}}{s} - \frac{(e^{-sT}-1)}{Ts^2} + \frac{e^{-sT}}{s}$$

Step 6 : Simplify.

$$= \frac{1-e^{-sT}}{Ts^2}$$

Example 6.7 : $f(t) = \begin{cases} \sin 2t & 0 < t < \pi \\ 0 & t > \pi \end{cases}$

Solution :

Step 1 : Given $f(t) = \begin{cases} \sin 2t & 0 < t < \pi \\ 0 & t > \pi \end{cases}$

Step 2 : Use the formula of L.T.

$$L\,[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt$$

Step 3 : Substitute value of f(t).

$$\therefore \quad L\,[f(t)] = \int_0^{\pi} e^{-st}\sin 2t \cdot dt + \int_{\pi}^{\infty} e^{-st}\cdot 0 \cdot dt$$

Step 4 : Integrate w.r.t. t.

$$= \left[\frac{e^{-st}}{s^2+4}(-s \sin 2t - 2\cos 2t)\right]_0^{\pi}$$

$$= \left[\frac{-e^{-st}}{s^2+4}(s \sin 2t + 2\cos 2t)\right]_0^{\pi}$$

Step 5 : Substitute the limits of t.

$$= \frac{-e^{-\pi s}}{s^2+4}[0+2] + \frac{1}{s^2+4}[0+2]$$

Step 6 : Simplify.

$$= \frac{2}{s^2+4}\left[1-e^{-\pi s}\right]$$

Example 6.8 : $f(t) = \begin{cases} \cos t & 0 < t < 2\pi \\ 0 & t > 2\pi \end{cases}$

Solution :

Step 1 :

$$f(t) = \begin{cases} \cos t & 0 < t < 2\pi \\ 0 & t > 2\pi \end{cases}$$

Step 2 : Use the formula of L.T.

$$L\ f(t) = \int_0^{\infty} e^{-st}\ f(t)\, dt$$

Step 3 : Substitute the value of f(t).

$$\therefore \quad L\ [f(t)] = \int_0^{2\pi} \cos\ t \cdot e^{-st} dt$$

Step 4 : Integrate w.r.t. t.

$$= \left[\frac{e^{-st}}{s^2+1}(-s \cos t + \sin t)\right]_0^{2\pi}$$

Step 5 : Substitute limits of t.

$$= \frac{e^{-2s\pi}}{s^2+1}[-s+0] - \left[\frac{1}{s^2+1}(-s+0)\right]$$

Step 6 : Simplify.

$$= \frac{-se^{-2s\pi}}{s^2+1} + \frac{s}{s^2+1}$$

$$= \frac{s}{s^2+1}\left[1 - e^{-2s\pi}\right]$$

Example 6.9 : $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$

Solution :

Step 1 :

$$f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$$

Step 2 : Use the formula of L.T.

$$L\ f(t) = \int_0^{\infty} e^{-st}\ f(t)\ dt$$

Step 3 : Substitute the value of f (t).

$$\therefore \quad L\ [f(t)] = \int_0^{\pi} e^{-st}\ \cos t\, dt + \int_{\pi}^{\infty} e^{-st}\ \sin t \cdot dt$$

Step 4 : Integrate w.r.t.t.

$$= \left[\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\right]_0^{\pi} + \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_{\pi}^{\infty}$$

Step 5 : Substitute the limits of t.

$$= \frac{e^{-\pi s}}{s^2+1}(s+0) - \frac{1}{s^2+1}(-s+0) + 0 + \frac{-e^{-\pi s}}{s^2+1}(0+1)$$

Step 6 : Simplify.

$$= \frac{se^{-\pi s}}{s^2+1} + \frac{s}{s^2+1} - \frac{e^{-\pi s}}{s^2+1}$$

$$= \frac{e^{-\pi s}\ (s-1)}{s^2+1} + \frac{s}{s^2+1}$$

$$= \frac{s + (s-1)\ e^{-\pi s}}{s^2+1}$$

Example 6.10 : $f(t)=\begin{cases} 0 & 0 \le t < 1 \\ t & 1 < t \le 2 \\ 0 & t > 2 \end{cases}$

Solution :

Step 1 :

$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ t & 1 < t \le 2 \\ 0 & t > 2 \end{cases}$$

Step 2 : Use the formula of L.T.

$$L\ f(t) = \int_0^\infty e^{-st} f(t) \cdot dt$$

Step 3 : Substitute the value of f(t).

$$\therefore \quad L\ [f(t)] = \int_0^1 e^{-st} \cdot 0 \cdot dt + \int_1^2 e^{-st}\ t \cdot dt + \int_2^\infty 0 \cdot dt$$

$$= \int_1^2 e^{-st} t \cdot dt$$

Step 4 : Integrate w.r.t. t. using by parts.

$$= \left[t \frac{e^{-st}}{-s}\right]_1^2 - \int_1^2 \frac{e^{-st}}{-s} \cdot dt$$

$$= -\frac{2e^{-2s}}{s} + \frac{e^{-s}}{s} - \left[\frac{1}{-s}\frac{e^{-st}}{-s}\right]_1^2$$

Step 5 : Substitute the limits of t.

$$= \frac{-2e^{-2s}}{s} + \frac{e^{-s}}{s} - \left[\frac{e^{-st}}{s^2}\right]_1^2$$

$$= \frac{-2e^{-2s}}{s} + \frac{e^{-s}}{s} - \left[\left(\frac{e^{-2s}}{s^2}\right) - \frac{e^{-s}}{s^2}\right]$$

Step 6 : Simplify.

$$= \frac{-2e^{-2s}}{s} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s^2}$$

$$= \left(\frac{1}{s^2} + \frac{1}{s}\right) e^{-s} - \left(\frac{1}{s^2} + \frac{2}{s}\right) e^{-2s}$$

Example 6.11 : $f(t)=\begin{cases} t^2 & 0<t<2 \\ 2t & t>2 \end{cases}$

Solution :

Step 1 :

$$f(t) = \begin{cases} t^2 & 0<t<2 \\ 2t & t>2 \end{cases}$$

Step 2 : Use the formula of L.T.

$$L\ f(t) = \int_0^{\infty} e^{-st}\ f(t)\cdot dt$$

Step 3 : Substitute the value of f(t).

$$\therefore \qquad L\ f(t) = \int_0^2 t^2 e^{-st}\cdot dt + 2\int_2^{\infty} t\cdot e^{-st}\cdot dt$$

Step 4 : Integrate w.r.t. t.

$$L\ f(t) = \int_0^2 t^2\ e^{-st} dt + \int_2^{\infty} 2t\, e^{-st}\ dt$$

Use integration by parts.

$$= \left[t^2\left(\frac{e^{-st}}{-s}\right) - 2t\left(\frac{e^{-st}}{s^2}\right) + 2\left(\frac{e^{-st}}{-s^3}\right)\right]_0^2 + 2\left[t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{s^2}\right)\right]_2^{\infty}$$

Step 5 : Substitute the limits of t.

$$= \left[\left(\frac{4\,e^{-2s}}{-s} - \frac{4\,e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3}\right) - \left(0 - 0 - \frac{2}{s^3}\right)\right] + 2\left[(0-0) - \left(\frac{2e^{-2s}}{-s} - \frac{e^{-2s}}{s^2}\right)\right]$$

Step 6 : Simplify.

$$= \frac{2}{s^3} - \frac{2e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3}$$

Example 6.12 : $f(t)=\begin{cases} a & 0<t<b \\ 0 & t>b \end{cases}$

Solution :

Step 1 :

$$f(t) = \begin{cases} a & 0<t<b \\ 0 & t>b \end{cases}$$

Step 2 : Use the formula given by,

$$L\ f(t) = \int_0^\infty e^{-st}\ f(t) \cdot dt$$

Step 3 : Substitute the value of f(t).

$$= \int_0^b e^{-st}\ (a)\ dt + \int_b^\infty e^{-st}\ (0)\ dt$$

Step 4 : Integrate w.r.t. t.

$$= a\left(\frac{e^{st}}{-s}\right)_0^b + 0$$

Step 5 : Substitute the limits of t.

$$= -\frac{a}{s}\left[e^{-sb} - 1\right]$$

Step 6 : Simplify.

$$= \frac{a}{s}\left(1 - e^{-bs}\right)$$

Example 6.13 : $f(t) = \begin{cases} (t-1)^2 & t > 1 \\ 0 & 0 < t < 1 \end{cases}$

Solution :

Step 1 :

$$f(t) = \begin{cases} (t-1)^2 & t > 1 \\ 0 & 0 < t < 1 \end{cases}$$

Step 2 : Use the formula given by,

$$L\ f(t) = \int_0^\infty e^{-st}\ f(t) \cdot dt$$

Step 3 : Substitute the value of f(t).

$$= \int_0^1 e^{-st}\ (0)\ dt + \int_1^\infty e^{-st} (t-1)^2\ dt$$

Step 4 : Integrate w.r.t. t.

$$= 0 + \left[(t-1)^2\left(\frac{e^{-st}}{-s}\right) - \{2\,(t-1)\}\left(\frac{e^{-st}}{s^2}\right) + (2)\left(\frac{e^{-st}}{-s^3}\right)\right]_1^\infty$$

Step 5 : Substitute the limits of t.

$$= 0-0-\frac{2}{s^3}(0-e^{-s})$$

Step 6 : Simplify.

$$= \frac{2e^{-s}}{s^3}$$

Laplace Transforms of Standard Functions	
i)	$L(1) = \frac{1}{s}$
ii)	$L(e^{at}) = \frac{1}{(s-a)}, \quad s > a$
iii)	$L(t^n) = \frac{\Gamma n+1}{s^{n+1}} = \frac{n!}{s^{n+1}} \leftarrow$ if n is integer.
iv)	$L(\sin at) = \frac{a}{s^2+a^2}$
v)	$L(\cos at) = \frac{s}{s^2+a^2}$
vi)	$L(\sinh at) = \frac{a}{s^2-a^2}$
vii)	$L(\cosh at) = \frac{s}{s^2-a^2}$
viii)	$L\left(\text{erf}\sqrt{t}\right) = \frac{1}{s\sqrt{s+1}}$

6.10 Type 2 : Using First Shifting Property

First Shifting Property

If, $L[f(t)] = \phi(s)$ then, $L\ e^{at}[f(t)] = \phi(s-a)$

Example 6.14 : *Find* $L\left(e^t 4^t\right)$

Solution :

Step 1 :

$$L[4^t] = L\left[e^{t\log 4}\right] = L\left[e^{(\log 4)t}\right]$$

$$\therefore \quad L\left[e^{(\log 4)t}\right] = \frac{1}{s-\log 4} \quad \text{where } s > \log 4$$

Step 2 : Now, using first shifting property.

If, $L[f(t)] = \phi(s)$ then, $L\ e^{at}[f(t)] = \phi(s-a)$

Step 3 : $\therefore\ L\ e^{t}\ f(t) = \phi(s-1)$

$$\therefore \qquad L\left(e^{t}4^{t}\right) = \frac{1}{(s-1)-\log 4}$$

Example 6.15 : *Find L (cosh at sin at)*

Solution :

Step 1 :
$$L\,[\cosh at \sin at] = L\left[\left(\frac{e^{at}+e^{-at}}{2}\right)\sin at\right]$$
$$= \frac{1}{2}\left\{L\left[e^{at}\sin at\right]+L\left[e^{-at}\sin at\right]\right\}$$

Step 2 : Now, using first shifting property.

If, $L\,[f(t)] = \phi(s)$ then, $L\ e^{at}\ [f(t)] = \phi\,(s-a)$

As $L \sin at = \dfrac{a}{s^2+a^2}$

Step 3 : $\therefore\ = \dfrac{1}{2}\left[\dfrac{a}{(s-a)^2+a^2}+\dfrac{a}{(s+a)^2+a^2}\right]$

$$\therefore \qquad L\,[f(t)] = \frac{1}{2}\left\{\frac{a}{s^2-2as+2a^2}+\frac{a}{s^2+2as+2a^2}\right\}$$
$$= \frac{a\,(s^2+2a^2)}{s^4+4a^4}$$

Example 6.16 : $L\left[e^{-at}t^{n}\right]$

Solution :

Step 1 :
$$L\,t^{n} = \frac{n\,!}{s^{n+1}}$$

Step 2 : Now, using first shifting property.

If, $L\,[f(t)] = \phi(s)$ then, $L\ e^{at}\ [f(t)] = \phi\,(s-a)$

Step 3 : $\therefore\ L\,e^{-at}\ f(t) = \phi\,(s+a)$

$$\therefore \qquad L\,e^{-at}\ t^{n} = \frac{n\,!}{(s+a)^{n+1}}$$

Example 6.17 : $L\,e^{-t}\cos(2t+3)$

Solution : Step 1 : Consider

$$L\cos(2t+3) = L(\cos 2t\cos 3 - \sin 2t\sin 3)$$
$$= \cos 3\, L(\cos 2t) - \sin 3\, L[\sin 2t]$$
$$= \cos 3\frac{s}{s^2+4} - \sin 3\frac{2}{s^2+4}$$

Step 2 : Now, using first shifting property.

If, $L[f(t)] = \phi(s)$ then, $L\, e^{at}[f(t)] = \phi(s-a)$

Step 3 :

$$\therefore \quad L\,e^{-t} f(t) = \phi(s+1)$$
$$\therefore \quad L\,e^{-t}\cos(2t+3) = \cos 3\frac{(s+1)}{(s+1)^2+4} - \sin 3\frac{2}{(s+1)^2+4}$$
$$= \frac{(s+1)\cos 3 - 2\sin 3}{s^2+2s+5}$$

Example 6.18 : $L\,e^{2t}\cos t\cos 2t$

Solution :

Step 1 : We know that $2\cos A\cos B = \cos(A-B)+\cos(A+B)$

$$L\cos t\cos 2t = \frac{1}{2}L[\cos t + \cos 3t]$$
$$= \frac{1}{2}\left[\frac{s}{s^2+1}+\frac{s}{s^2+9}\right]$$

Step 2 : Now, using first shifting property.

If, $L[f(t)] = \phi(s)$ then, $L\, e^{at}[f(t)] = \phi(s-a)$

$$\therefore \quad L\,e^{2t} f(t) = \phi(s-2)$$

Step 3 :

$$\therefore \quad L\,e^{2t}\cos t\cos 2t = \frac{1}{2}\left[\frac{s-2}{(s-2)^2+1}+\frac{s-2}{(s+2)^2+9}\right]$$

Example 6.19 : $L\, e^{2t} \sin^3 t$

Solution :

Step 1 : Consider

$$L \sin^3 t = L\left[\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right]$$

$$= \frac{3}{4} L \sin t - \frac{1}{4} L \sin 3t$$

$$= \frac{3}{4}\left[\frac{1}{s^2+1}\right] - \frac{1}{4}\left[\frac{3}{s^2+9}\right]$$

Step 2 : Now, using first shifting property.

If, L $[f(t)] = \phi(s)$ then, L $e^{at}\,[f(t)] = \phi(s-a)$

Step 3 :

$$\therefore \quad L\; e^{2t}\; f(t) = \phi(s-2)$$

L e^{2t} $\sin^3 t$ is given by,

$$= \frac{3}{4}\,\frac{1}{(s-2)^2+1} - \frac{1}{4}\,\frac{3}{(s-2)^2+9}$$

$$= \frac{3}{4}\left[\frac{1}{(s-2)^2+1} - \frac{1}{(s-2)^2+9}\right]$$

Example 6.20 : $L\, e^{-t} \cos^3 t$

Solution :

Step 1 : Consider

$$L \cos^3 t = L\left[\frac{1}{4}\cos 3t + \frac{3}{4}\cos t\right]$$

$$= \frac{1}{4} L\,[\cos 3t] + \frac{3}{4} L\,[\cos t]$$

$$= \frac{1}{4}\left[\frac{s}{s^2+9}\right] + \frac{3}{4}\left[\frac{s}{s^2+1}\right]$$

Step 2 : Now, using first shifting property.

If, L $[f(t)] = \phi(s)$ then, L $e^{at}\,[f(t)] = \phi(s-a)$

Step 3 :

$$\therefore \quad L\; e^{-t}\; f(t) = \phi(s+1)$$

$L\, e^{-t} \cos^3 t$ is given by,

$$L\, e^{-t} \cos^3 t = \frac{1}{4}\frac{(s+1)}{(s+1)^2+9} + \frac{3}{4}\frac{(s+1)}{(s+1)^2+1}$$

$$= \frac{1}{4}\left[\frac{(s+1)}{(s+1)^2+9} + \frac{3\,(s+1)}{(s+1)^2+1}\right]$$

Example 6.21 : $L t^2 \cosh t$

Solution :

Step 1 : $L\, t^2 \cosh t = L\, t^2\left[\frac{e^t + e^{-t}}{2}\right]$

$$= L\left[\frac{e^t t^2 + e^{-t} t^2}{2}\right]$$

$$= \frac{1}{2}\left\{L\, e^t t^2 + L\, e^{-t} t^2\right\}$$

We know that

$$L\, t^2 = \frac{2\,!}{s^3}$$

Step 2 : Now, using first shifting property.

If, $L\,[f(t)] = \phi(s)$ then, $L\; e^{at}\;[f(t)] = \phi\,(s-a)$

Step 3 :

$$\therefore \quad L\; f(t) = \frac{1}{2}\left\{\frac{2}{(s-1)^3} + \frac{2}{(s+1)^3}\right\}$$

$$= \left[\frac{1}{(s-1)^3} + \frac{1}{(s+1)^3}\right]$$

Example 6.22 : $L \cosh t \cos t$

Solution :

Step 1 : $L \cosh t \cos t = L\left(\frac{e^t + e^{-t}}{2}\right)\cos t$

$$= \frac{1}{2}\left\{L\, e^t \cos t + L e^{-t} \cos t\right\}$$

We know that

$$L \cos t = \frac{s}{s^2+1}$$

Step 2 : Now, using first shifting property.

If, L [f(t)] = ϕ(s) then, L e^{at} [f(t)] = ϕ(s − a)

Step 3 :

$$\frac{1}{2}\{Le^{t}\cos t + Le^{-t}\cos t\} = \frac{1}{2}\left[\frac{s-1}{(s-1)^2+1} + \frac{s+1}{(s+1)^2+1}\right]$$

Example 6.23 : *$L\, e^{t} \cos t \cos 2t \cos 3t$*

Solution :

Step 1 : Use 2 cos A cos B = cos(A − B) + cos(A + B)

$$L \cos t \cos 2t \cos 3t = \frac{1}{2} L[(\cos 3t + \cos t) \cos 3t]$$

$$= \frac{1}{2} L (\cos^2 3t + \cos\ t \cos 3t)$$

$$= \frac{1}{2} L\left(\frac{1-\cos 6t}{2}\right) + \frac{1}{4} L (\cos 4t + \cos 2t)$$

$$= \frac{1}{4}\{L (1) - L \cos 6t + L \cos 4t + L \cos 2t\}$$

Step 2 : Now, using first shifting property.

If, L [f(t)] = ϕ(s) then, L e^{at} [f(t)] = ϕ(s − a)

Step 3 :

$$\therefore \quad L\ e^{-t}\ f(t) = \phi(s-1)$$

L e^{t} cost cos2t cos3t is given by,

$$= \frac{1}{4}\left[\frac{1}{(s-1)} + \frac{(s-1)}{(s-1)^2+36} + \frac{(s-1)}{(s-1)^2+16} + \frac{(s-1)}{(s-1)^2+4}\right]$$

Exercise 6.2

1) *Find Laplace transform of*

i) $f(t) = \begin{cases} t^2 - 2t + 2 & t \geq 1 \\ 0 & t < 1 \end{cases}$ **[Ans. : L f(t) = $e^{-s}\left(\frac{1}{s} + \frac{2}{s^3}\right)$]**

ii) $f(t) = \begin{cases} 1 & 0 < t < 2 \\ 0 & t > 2 \end{cases}$ **[Ans. : $\frac{1}{s}(1 - e^{-2s})$]**

iii) $f(t) = \begin{cases} e^{-2(t-4)} \sin 2(t-4) & t \geq 4 \\ 0 & t < 4 \end{cases}$ **[Ans. : L f(t) = $\frac{e^{-4s}}{(s+2)^2 + 2^2}$]**

2) *Find Laplace transform of f(t).*

i) $f(t) = \left(e^{-2t} + e^{3t}\right)^2$ [Ans. : $L\,f(t) = \frac{1}{s+4} + \frac{2}{s-1} + \frac{1}{s-6}$, $s > 6$]

ii) $f(t) = e^{2t+3}$ [Ans. : $L\,f(t) = \frac{e^3}{s-2}$, $s > 2$]

iii) $f(t) = a^t$ [Ans. : $L\,f(t) = \frac{1}{(s-\log a)}$, $s > \log a$]

iv) $f(t) = (2t+3)^2$ [Ans. : $L\,f(t) = \frac{48}{s^4} + \frac{72}{s^3} + \frac{54}{s^2} + \frac{27}{s}$, $s > 0$]

v) $f(t) = \sin 2t \cos 3t$ [Ans. : $L\,f(t) = \frac{1}{2}\left\{\frac{5}{s^2+25} - \frac{1}{s^2+1}\right\}$]

vi) $f(t) = \cos t \cos 2t$ [Ans. : $L\,f(t) = \frac{1}{2}\left\{\frac{s}{s^2+9} + \frac{s}{s^2+1}\right\}$]

vii) $f(t) = \cosh at - \cos bt$ [Ans. : $L\,f(t) = \frac{s}{s^2-a^2} - \frac{s}{s^2+b^2}$, $s > |a|$]

viii) $f(t) = \sin^2 at$ [Ans. : $L\,f(t) = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4a^2}\right]$]

ix) $f(t) = \cos^3 2t$ [Ans. : $L\,f(t) = \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{3s}{s^2+4}\right]$]

x) $f(t) = \cosh^3 2t$ [Ans. : $L\,f(t) = \frac{1}{4}\left[\frac{s}{s^2-36} + \frac{3s}{s^2-4}\right]$]

xi) $f(t) = \sin^3 2t$ [Ans. : $L\,f(t) = \frac{48}{(s^2+4)(s^2+36)}$]

xii) $f(t) = \sinh^3 2t$ [Ans. : $L\,f(t) = \frac{48}{(s^2-4)(s^2-36)}$]

xiii) $f(t) = (\sin t - \cos t)^2$ [Ans. : $L\,f(t) = \frac{s^2-2s+4}{s\left(s^2+4\right)}$]

3) *Find Laplace transform of following functions.*

i) $f(t) = (t+2)^2 e^{4t}$ [Ans. : $L\,f(t) = \frac{4s^2-28s+50}{(s-4)^3}$]

ii) $f(t) = e^{-3t}\sin^2 t$ [Ans. : $L\,f(t) = \frac{2}{(s+3)\left(s^2+6s+13\right)}$]

iii) $f(t) = e^{at}(2\cos bt - 3\sin bt)$ [Ans. : $L\,f(t) = \frac{2s-2a-2b}{(s-a)^2+b^2}$]

iv) $f(t) = \left(t e^{-t} + 1\right)^3$ [Ans. : $L\,f(t) = \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$]

v) $f(t) = e^{-t}\cos(4t+7)$ [Ans. : $L\,f(t) = \frac{(s+1)\cos 7 - 4\sin 7}{s^2+2s+17}$]

vi) $f(t) = \sinh\frac{t}{2}\sin\frac{\sqrt{3}t}{2}$ [Ans. : $L\,f(t) = \frac{\sqrt{3}}{2}\,\frac{s}{s^4+s^2+1}$]

vii) $f(t) = \sinh at \cos at$ [Ans. : $L\,f(t) = \frac{a(s^2-2a^2)}{s^4+4a^4}$]

viii) $f(t) = e^{4t} t^{3/2}$ [Ans. : $L\,f(t) = \frac{3}{4}\frac{\sqrt{\pi}}{(s-4)^{5/2}}$]

ix) $f(t) = e^{-t}\sin^3 t$ [Ans. : $L\,f(t) = \frac{6}{(s^2+2s+2)(s^2+2s+10)}$]

x) $f(t) = \frac{\cosh at}{\sqrt{t}}$ [Ans. : $L\,f(t) = \frac{1}{2}\left[\sqrt{\frac{\pi}{s-a}}+\sqrt{\frac{\pi}{s+a}}\right]$]

6.11 Type 3 : Problems on Second Shifting Property

Second Shifting property (Laplace transform of displayed function) :

Also called as Laplace transform of displaced function.

If, $L\,f(t) = \phi(s)$ and $g(t) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$

Then, $L\,g(t) = e^{-as}\,\phi(s)$

Example 6.24 : *Find L [g(t)]*

$$g(t) = \begin{cases} 0 & t < \frac{2\pi}{3} \\ \cos\left(t-\frac{2\pi}{3}\right) & t \geq \frac{2\pi}{3} \end{cases}$$

Solution :

Step 1 : Consider

$$g(t) = \begin{cases} 0 & t < \frac{2\pi}{3} \\ \cos\left(t-\frac{2\pi}{3}\right) & t \geq \frac{2\pi}{3} \end{cases}$$

Step 2 : We know that, If, $f(t) = \phi(s)$ and $g(t) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$

Then $L\,g(t) = e^{-as}\,\phi(s)$

Step 3 : Here $f(t-a) = \cos\left(t-\frac{2\pi}{3}\right)$ where $a = \frac{2\pi}{3}$.

$$f(t) = \cos t \text{ and } \phi(s) = \frac{s}{s^2+1}$$

Step 4 : Hence by the second shifting theorem, with $a = \frac{2\pi}{3}$,

$$L\,[g(t)] = e^{-as}\, f(s) = e^{-2\pi s/3}\left(\frac{s}{s^2+1}\right)$$

Example 6.25 : *Find L g(t)*

$$g(t) = \begin{cases} 0 & t < 3 \\ e^{2(t-3)} sin\ 3(t-3) & t \geq 3 \end{cases}$$

Solution :

Step 1 :

$$f(t) = \begin{cases} 0 & t < 3 \\ e^{2(t-3)}\sin 3\,(t-3) & t \geq 3 \end{cases}$$

Step 2 : We know that, If L $[f(t)] = \phi(s)$ and $g(t) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$

Then, L $g(t) = e^{-as}\,\phi(s)$

Step 3 : Here $f(t-a) = e^{2(t-3)}\sin 3\,(t-3)$ where $a = 3$.

$$f(t) = e^{2t}\sin 3t$$

$$\phi(s) = L\, e^{2t}\,\sin 3t$$

$$= \frac{3}{(s-2)^2+9} \qquad \text{(by 1}^{\text{st}}\text{ shifting property)}$$

Step 4 : $$L\,[g(t)] = e^{-3s}\cdot\phi(s)$$

$$= e^{-3s}\left[\frac{3}{(s-2)^2+9}\right]$$

Example 6.26 : *Find L g(t), where*

$$g(t) = \begin{cases} (t-1)^3 & t \geq 1 \\ 0 & t < 1 \end{cases}$$

Solution :

Step 1 :

$$g(t) = \begin{cases} (t-1)^3 & t \geq 1 \\ 0 & T < 1 \end{cases}$$

Step 2 :

If, L f(t) = φ(s) and $g(t) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$

Then, L g(t) = $e^{-as}\,\phi(s)$

Step 3 : Here $f(t-a) = (t-1)^3$

$f(t) = t^3 \quad \therefore \; a = 1$

$$\phi(s) = \frac{3 \times 2}{s^4} = \frac{6}{s^4}$$

Step 4 :

$$L\,g(t) = e^{-as}\,\phi(s)$$

$$= e^{-s}\left(\frac{6}{s^4}\right)$$

Example 6.27 : *Find L g(t)*

$$g(t) = \begin{cases} \sin 2(t-\pi) & t \geq \pi \\ 0 & t < \pi \end{cases}$$

Solution :

Step 1 :

$$g(t) = \begin{cases} \sin 2\,(t-\pi) & t \geq \pi \\ 0 & t < \pi \end{cases}$$

Step 2 : If, L f(t) = φ(s) and $g(t) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$

Then, L g(t) = $e^{-as}\,\phi(s)$

Step 3 : Here $f(t-a) = \sin 2\,(t-\pi) \quad \therefore \; a = \pi$

$$f(t) = \sin 2t$$

$$\phi(s) = \frac{2}{s^2+4}$$

Step 4 :

$$L\,[g(t)] = e^{-\pi s}\left[\frac{2}{s^2+4}\right]$$

Example 6.28 : *Find L g(t)*

$$g(t) = \begin{cases} \cos(t-\alpha) & t \geq \alpha \\ 0 & t < \alpha \end{cases}$$

Solution :

Step 1 :

$$g(t) = \begin{cases} \cos(t-\alpha) & t \geq \alpha \\ 0 & t < \alpha \end{cases}$$

Step 2 :

If, L f(t) = $\phi(s)$ and $g(t) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$

Then, L g(t) = $e^{-as}\,\phi(s)$

Step 3 : Here $f(t-a) = \cos(t-\alpha)$ $\therefore$ $a = \alpha$

$$f(t) = \cos t$$

$$\phi(s) = \frac{s}{s^2+1}$$

Step 4 :

$$L\,[g(t)] = e^{-\alpha s}\left[\frac{s}{s^2+1}\right]$$

Example 6.29 : $g(t) = \begin{cases} 5 \sin 3\left(t - \frac{\pi}{4}\right) & t \geq \frac{\pi}{4} \\ 0 & t < \frac{\pi}{4} \end{cases}$

Solution :

Step 1 :

$$g(t) = \begin{cases} 5 \sin 3\left(t - \frac{\pi}{4}\right) & t \geq \frac{\pi}{4} \\ 0 & t < \frac{\pi}{4} \end{cases}$$

Step 2 : If, L f(t) = $\phi(s)$ and $g(t) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$

Then, L g(t) = $e^{-as}\,\phi(s)$

Step 3 : Here $f(t-a) = 5 \sin 3\,(t-\pi/4)$ $\therefore$ $a = \pi/4$

$$f(t) = 5 \sin 3t$$

$$\phi(s) = \frac{5 \times 3}{s^2+9}$$

Step 4 :

$$L\,[g(t)] = e^{-\pi s/4}\left[\frac{15}{s^2+9}\right]$$

Exercise 6.3

1) Find Laplace transform of the following

i) $f(t) = \begin{cases} \sin(t-\alpha) & t \geq \alpha \\ 0 & t < \alpha \end{cases}$ **[Ans. : $L\,f(t) = e^{-\alpha s} \cdot \frac{1}{s^2+1}$]**

ii) $f(t) = \begin{cases} \sin 2\left(t - \frac{2\pi}{3}\right) & t > \frac{2\pi}{3} \\ 0 & t < \frac{2\pi}{3} \end{cases}$ **[Ans. : $L\,f(t) = e^{-2\pi s/3} \cdot \frac{2}{s^2+4}$]**

iii) $f(t) = \begin{cases} e^{-4(t-3)} \cdot \cos 3(t-3) & t > 3 \\ 0 & t < 3 \end{cases}$ **[Ans. : $L\,f(t) = e^{-3s} \cdot \frac{s-3}{(s-3)^2+9}$]**

iv) $f(t) = \begin{cases} e^{-4(t-1)}(t-1)^3 & t > 1 \\ 0 & t < 1 \end{cases}$ **[Ans. : $L\,f(t) = e^{-s} \cdot \frac{6}{(s+4)^4}$]**

v) $f(t) = \begin{cases} \sin 2(t-\pi) & t > \pi \\ 0 & t < \pi \end{cases}$ **[Ans. : $L\,f(t) = e^{-\pi s} \cdot \frac{2}{s^2+4}$]**

vi) $f(t) = \begin{cases} (t-2)^4 & t > 2 \\ 0 & t < 2 \end{cases}$ **[Ans. : $L\,f(t) = e^{-2s} \cdot \frac{4\,!}{s^5}$]**

6.12 Type 4 : Problems on Change of Scale Property

Change of scale property :

If, $L\,f\,(t) = \phi(s)$ then, $L\,f\,(at) = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

Example 6.30 : $L\,f(t) = \frac{s^2-s+1}{(2s+1)\,(s-1)} \cdot e^{-2s}$ *Find $L\,f(4t)$.*

Solution : Step 1 : Given that,

$$L\,f(t) = \frac{s^2-s+1}{(2s+1)\,(s-1)} \cdot e^{-2s}$$

Step 2 : We know that if, $L\,f(t) = \phi(s)$ then, $L\,f(at) = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

Step 3 :

$$\therefore \qquad L\ f(4t) = \frac{1}{4}\phi\left(\frac{s}{4}\right)$$

$$\therefore \qquad L\ [f(4t)] = \frac{1}{4}\left[\frac{\left(\frac{s}{4}\right)^2-\left(\frac{s}{4}\right)+1}{\left(2\cdot\frac{s}{4}+1\right)\left(\frac{s}{4}-1\right)}\right] e^{-2\left(\frac{s}{4}\right)}$$

Step 4 : Simplify.

$$= \frac{1}{4}\left[\frac{\left(\frac{s^2-4s+16}{16}\right)}{\frac{(s+2)\ (s-4)}{8}}\right]\cdot e^{-s/2}$$

$$= \frac{1}{8}\left[\frac{s^2-4s+16}{(s+2)\ (s-4)}\right] e^{-s/2}$$

Example 6.31 : *If* $L\left(erf\sqrt{t}\right)=\dfrac{1}{s\sqrt{s+1}}$ *then find* $L\left(erf\ 2\sqrt{t}\right)$

Solution :

Step 1 :

$$L\left(\text{erf}\sqrt{t}\right) = \frac{1}{s\sqrt{s+1}}$$

Step 2 : If, $L\ f(t) = \phi(s)$ then, $L\ f(at) = \dfrac{1}{a}\phi\left(\dfrac{s}{a}\right)$

Step 3 : Let $f(t)=\text{erf}\sqrt{t}$ thus $\text{erf}\ 2\sqrt{t} = \text{erf}\sqrt{4t} = f(4t)$

$$\therefore \qquad L\ f(4t) = \frac{1}{4}\phi\left(\frac{s}{4}\right)$$

$$\therefore \qquad L\ \text{erf}\sqrt{4t} = \frac{1}{4}\frac{1}{\left(\frac{s}{4}\right)\sqrt{\frac{s}{4}+1}}$$

Step 4 : Simplify.

$$\therefore \qquad L\ \text{erf}\sqrt{4t} = \frac{1}{s\sqrt{\frac{s+4}{4}}}$$

$$L\ \text{erf}\ 2\sqrt{t} = \frac{2}{s\sqrt{s+4}}$$

Example 6.32 : *Verify change of scale property for $f(t) = e^{2t} \cos 2t$.*

Solution :

Step 1 :

$$f(t) = e^{2t} \cos 2t$$

$$L\, f(t) = L\, e^{2t} \cos 2t$$

$$= \frac{(s-2)}{(s-2)^2 + 4} \quad \text{... (1)}$$

Step 2 : By first shifting property.

$$L\left[e^t \cos t\right] = \frac{(s-1)}{(s-1)^2 + 1}$$

Step 3 : If, $L\, f(t) = \phi(s)$ then, $L\, f(at) = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

Step 4 :

$$\therefore \quad L\, f(2t) = \frac{1}{2}\phi\left(\frac{s}{2}\right)$$

$$\therefore \quad L\, e^{2t} \cos 2t = \frac{1}{2}\left[\frac{\left(\frac{s}{2}-1\right)}{\left(\frac{s}{2}-1\right)^2 + 1}\right]$$

Step 5 : Simplify.

$$= \frac{(s-2)}{(s-2)^2 + 4} \quad \text{... (2)}$$

From (1) and (2) verified.

Example 6.33 : *If $L\left[\frac{\sin t}{t}\right] = \tan^{-1}\left(\frac{1}{s}\right)$, find $L\left[\frac{\sin at}{t}\right]$*

Solution : Step 1 : We have given that

$$L\left[\frac{\sin t}{t}\right] = \tan^{-1}\left(\frac{1}{s}\right)$$

Step 2 : If, $L\, f(t) = \phi(s)$ then, $L\, f(at) = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

Step 3 : By the Change of Scale Theorem,

$$L\left[\frac{\sin at}{at}\right] = \frac{1}{a}\tan^{-1}\frac{1}{(s/a)}$$

Step 4 : Simplify.

$$\frac{1}{a} L\left[\frac{\sin at}{t}\right] = \frac{1}{a}\tan^{-1}\left(\frac{a}{s}\right) \qquad \text{From (1)}$$

$$\therefore \qquad L\left[\frac{\sin at}{t}\right] = \tan^{-1}\left(\frac{a}{s}\right)$$

Exercise 6.4

1) *If* $L f(t) = \dfrac{8+12s-2s^2}{\left(s^2+4\right)^2}$ *find* $L f(2t)$ [**Ans. :** $L f(t) = \dfrac{4\left(16+12s-s^2\right)}{\left(s^2+16\right)^2}$]

2) *If* $L J_0(t) = \dfrac{1}{\sqrt{s^2+1}}$ *show that i)* $L J_0(at) = \dfrac{1}{\sqrt{s^2+a^2}}$ *ii)* $L e^{-at} J_0(at) = \dfrac{1}{\sqrt{s^2+2as+2a^2}}$

3) *If* $L f(t) = \dfrac{1}{s} e^{-1/s}$ *find* $L\left[e^{-t} f(3t)\right]$ [**Ans. :** $L f(t) = \dfrac{e^{-3/(s+1)}}{(s+1)}$]

4) *Verify change of scale property for i)* $e^{at}\cos at$ *ii)* $e^{2t}\sin 2t$ *iii)* $e^{3t}\cosh 3t$ *iv)* $4t^2 e^{2t}$.

6.13 Type 5 : Problems on Effect of Multiplication by t

Effect of Multiplication by 't' :

If, $L f(t) = \phi(s)$ Then, $L\, t f(t) = -\phi'(s)$

$L\, t^2 f(t) = (-1)^2 \phi''(s)$

$L\, t^3 f(t) = (-1)^3 \phi'''(s)$

$\therefore L\, t^n f(t) = (-1)^n \phi^n(s)$

Example 6.34 : *Given* $L[J_0(t)] = \dfrac{1}{\sqrt{s^2+1}}$ *show that* $L[t J_0(at)] = \dfrac{s}{\left(s^2+a^2\right)^{3/2}}$

Solution :

Step 1 : Given that $L[J_0(t)] = \dfrac{1}{\sqrt{s^2+1}}$

Step 2 : $\therefore$ By change of scale property.

$$L[J_0(at)] = \frac{1}{a}\frac{1}{\sqrt{(s/a)^2+1}} = \frac{1}{\sqrt{s^2+a^2}}$$

We have $\quad L[J_0(at)] = \dfrac{1}{\sqrt{s^2+a^2}}$

Step 3 : We know that,

If, $L f(t) = \phi(s)$ Then, $L\, t f(t) = -\phi'(s)$

Step 4 : Apply the property.

$$\therefore \quad L[tJ_0(at)] = (-1)\frac{d}{ds}\frac{1}{\sqrt{s^2+a^2}}$$

$$= \frac{s}{(s^2+a^2)^{3/2}}$$

Example 6.35 : *Show that* $L\frac{t}{2a}\sin at = \frac{s}{(s^2+a^2)^2}$

Solution :

Step 1 : We know

$$L \sin at = \frac{a}{s^2+a^2}$$

Step 2 : We know that,

If, L f(t) = $\phi(s)$ Then, L tf(t) = $-\phi'(s)$

Step 3 :

$$\therefore \quad \frac{1}{2a} L\, t\sin at = \frac{1}{2a}(-1)\frac{d}{ds}\frac{a}{s^2+a^2}$$

Step 4 : Simplify.

$$= \frac{-1}{2a}\frac{-a}{(s^2+a^2)^2}(2s)$$

$$= \frac{s}{(s^2+a^2)^2}$$

Example 6.36 : *Show that* $L\frac{1}{2a^3}(\sin at - at\cos at) = \frac{1}{(s^2+a^2)^2}$

Solution :

Step 1 :

$$L\frac{1}{2a^3}(\sin at - at\cos at) = \frac{1}{2a^3}L(\sin at) - \frac{a}{2a^3}L(t\cos at)$$

$$= \frac{1}{2a^3}\frac{a}{(s^2+a^2)} - \frac{1}{2a^2}L(t\cos at)$$

Step 2 : We know that,

If, L f(t) = $\phi(s)$ Then, L tf(t) = $-\phi'(s)$

Step 3 : Apply the property.

$$\therefore \qquad L\,[f(t)] = \frac{1}{2a^3}\,\frac{a}{\left(s^2+a^2\right)} - \frac{1}{2a^2}\left(-\frac{d}{ds}\right)\frac{s}{s^2+a^2}$$

Step 4 : Simplify.

$$= \frac{1}{2a^2}\left[\frac{1}{s^2+a^2}\right] + \frac{1}{2a^2}\left[\frac{s^2+a^2-2s^2}{\left(s^2+a^2\right)^2}\right]$$

$$= \frac{1}{2a^2}\left[\frac{1}{s^2+a^2} + \frac{s^2+a^2-2s^2}{\left(s^2+a^2\right)^2}\right]$$

$$= \frac{1}{2a^2}\left[\frac{2a^2}{\left(s^2+a^2\right)^2}\right]$$

$$= \frac{1}{\left(s^2+a^2\right)^2}$$

Example 6.37 : *Show that* $\frac{1}{2a}\left[L\left(\sin at + at\cos at\right)\right] = \frac{s^2}{\left(s^2+a^2\right)^2}$

Solution :

Step 1 :

$$= \frac{1}{2a}\left[L\sin at + a\,L\,t\cos at\right]$$

Step 2 : We know that,

If, $L\,f(t) = \phi(s)$ Then, $L\,tf(t) = -\,\phi'(s)$

Step 3 :

$$= \frac{1}{2a}\left[\frac{a}{s^2+a^2} + a\,(-1)\frac{d}{ds}\,\frac{s}{s^2+a^2}\right]$$

Step 4 : Simplify.

$$= \frac{1}{2a}\left[\frac{a}{s^2+a^2} - a\left(\frac{a^2-s^2}{\left(s^2+a^2\right)^2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{s^2+a^2+s^2-a^2}{\left(s^2+a^2\right)^2}\right]$$

$$= \frac{s^2}{\left(s^2+a^2\right)^2}$$

Example 6.38 : *Evaluate* $L\left[t^2 sin(4t)\right]$

Solution :

Step 1 : We have,

$$L \sin 4t = \frac{4}{s^2+16}$$

Step 2 : If, $L f(t) = \phi(s)$, $L\left[t^2 f(t)\right]=(-1)^2\phi''(s) = \phi''(s)$

Step 3 :

Here $$\frac{d}{ds}\frac{4}{s^2+16} = \frac{-4}{\left(s^2+16\right)^2}2s$$

$$\frac{d}{ds}\frac{4}{s^2+16} = \frac{-8s}{\left(s^2+16\right)^2}$$

$$\frac{d^2}{ds^2}\frac{4}{s^2+16} = \frac{d}{ds}\frac{-8s}{\left(s^2+16\right)^2}$$

$$= \frac{-\left(s^2+16\right)^2 8+2\left(s^2+16\right)\cdot 2s\cdot 8s}{\left(s^2+16\right)^4}$$

$$= \frac{-8\left(s^2+16\right)^2+32s^2\left(s^2+16\right)}{\left(s^2+16\right)^4}$$

$$= \frac{\left(s^2+16\right)\left[32s^2-8s^2-128\right]}{\left(s^2+16\right)^4}$$

$$= \frac{24s^2-128}{\left(s^2+16\right)^3}$$

Step 4 :

$$L\, t^2 \sin 4t = \frac{24s^2-128}{\left(s^2+16\right)^3}$$

Example 6.39 : $L\left(t\, erf\, 2\sqrt{t}\right)=\dfrac{3s+8}{s^2(s+4)^{3/2}}$

Solution : Step 1 : We know that

$$L\, erf\sqrt{t} = \frac{1}{s\sqrt{s+1}}$$

By change of scale property.

$$L\,\text{erf}\sqrt{4t} = \frac{1}{4}\frac{1}{\left(\frac{s}{4}\right)\sqrt{\left(\frac{s}{4}\right)+1}}$$

$$L\,\text{erf}\sqrt{4t} = \frac{2}{s\sqrt{s+4}}$$

Step 2 : We know that,

If, L f(t) = ϕ(s) Then, L t f(t) = – ϕ'(s)

Step 3 :

$$L\,t\,\text{erf}\,2\sqrt{t} = -\frac{d}{ds}\frac{2}{s\sqrt{s+4}}$$

$$= -2\frac{d}{ds}\frac{1}{(s^3+4s^2)^{1/2}}$$

$$= -2\cdot\frac{d}{ds}(s^3+4s^2)^{-1/2}$$

Step 4 : Simplify.

$$= -2\times\left(\frac{-1}{2}\right)\frac{1}{(s^3+4s^2)^{3/2}}(3s^2+8s)$$

$$= \frac{s\,(3s+8)}{s^3\,(s+4)^{3/2}}$$

Example 6.40 : *Evaluate* $L\left(t\sin^3 t\right)$

Solution :

Step 1 : Use $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\therefore \quad L\sin^3 t = \left[\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right]$$

$$= \frac{3}{4}\frac{1}{s^2+1} - \frac{1}{4}\frac{3}{s^2+9}$$

$$= \frac{3}{4}\left[\frac{1}{(s^2+1)} - \frac{1}{(s^2+9)}\right]$$

Step 2 : We know that,

If, L f(t) = ϕ(s) Then, L t f(t) = – ϕ'(s)

Step 3 :

$$\therefore \quad L\, t \sin^3 t = (-1)\,\frac{3}{4}\left[\frac{d}{ds}\left(\frac{1}{(s^2+1)}-\frac{1}{(s^2+9)}\right)\right]$$

Step 4 : Find the derivative.

$$= (-1)\,\frac{3}{4}\left[\frac{-1\,(2s)}{(s^2+1)^2}-\frac{(-1)\cdot 2s}{(s^2+9)^2}\right]$$

$$= \frac{3}{4}\left[\frac{2s}{(s^2+1)^2}-\frac{2s}{(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{1}{(s^2+1)^2}-\frac{1}{(s^2+9)^2}\right]$$

Example 6.41 : *Find Laplace transform of $t^2 \cos at$*

Solution :

Step 1 : We have

$$L\,[\cos at] = \frac{s}{s^2+a^2}$$

Step 2 : If, $L\, f(t) = \phi(s)$, $L\, t^2\, f(t) = (-1)^2 \phi''(s)$

Step 3 :

$$\therefore \quad L\left[t^2 \cos at\right] = (-1)^2\,\frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

Step 4 : Simplify.

$$= \frac{d}{ds}\left[\frac{(s^2+a^2)\,(1)-s\,(2s)}{(s^2+a^2)^2}\right]=\frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \left[\frac{(s^2+a^2)^2\,(-2s)-(a^2-s^2)\cdot 2(s^2+a^2)\,(2s)}{(s^2+a^2)^4}\right]$$

$$= \left[\frac{-2s^3-2a^2s-4a^2s+4s^3}{(s^2+a^2)^3}\right]=\frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

Exercise 6.5

Find the Laplace transform of the following

1) $f(t) = t^2 \sin 2t$ [Ans. : $L\, f(t) = \frac{4(3s^2-4)}{(s^2+4)^3}$]

2) $f(t) = t^2 \sinh t$ [Ans.: $L\, f(t) = \frac{6s^2+2}{(s^2-1)^3}$]

3) $f(t) = t^3 \cos t$ [Ans. : $L\, f(t) = \frac{6s^4-36s^2+6}{(s^2+1)^4}$]

4) $f(t) = t \cos(4t)$ [Ans. : $L\, f(t) = \frac{s^2-16}{(s^2+16)^2}$]

5) $f(t) = t\, e^{3t} \sin 2t$ [Ans. : $L\, f(t) = \frac{4\,(s-3)}{(s^2-6s+10)^2}$]

6) $f(t) = t^3\, e^{2t}$ [Ans. : $L\, f(t) = \frac{6}{(s-2)^4}$]

7) $f(t) = e^{-3t} \cdot t \cos 3t$ [Ans. : $L\, f(t) = \frac{s^2+6s}{(s^2+6s+18)^2}$]

8) $f(t) = t\, e^{-2t}\left(2\cosh 3t - 4\sinh 2t\right)$ [Ans. : $L\, f(t) = \frac{2s^2+8s+26}{(s^2+4s-s)^2} - \frac{16\,(s+2)}{(s^2+4s)^2}$]

9) $f(t) = t \cos(4t+3)$ [Ans. : $L\, f(t) = \frac{s^2\cos 3 - 8s \sin 3 - 16 \cos 3}{(s^2+16)^2}$]

10) $f(t) = t^2 \cos t$ [Ans. : $L\, f(t) = \frac{2s(s^3-s^2-s-1)}{(s^2+1)^3}$]

6.14 Type 6 : Problems on Division by t

Effect of Division by 't'

If, $L\, f(t) = \phi(s)$ then, $L \frac{f(t)}{t} = \int_s^{\infty} \phi(s)\, ds$

Example 6.42 : *Find the Laplace transform of* $\frac{\sin at}{t}$ *and hence show that* $\int_s^{\infty} \frac{\sin t}{t} = \frac{\pi}{2}$

Solution :

Step 1 : We have

$$L\,[\sin at] = \frac{a}{s^2+a^2}$$

Step 2 : We know that if, $L\ f(t) = \phi(s)$

Then, $$L\frac{f(t)}{t} = \int_s^\infty \phi(s)\,ds$$

Step 3 : Thus we get

$$\therefore \quad L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

Step 4 : Integrate w.r.t. s.

$$= \left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}$$

Step 5 : When a = 1, we have

$$L\left[\frac{\sin t}{t}\right] = \cot^{-1} s$$

$$\int_0^\infty e^{-st}\frac{\sin t}{t}\,dt = \cot^{-1} s \qquad \text{[(By definition)]}$$

by substituting s = 0 we get

$$\int_0^\infty \frac{\sin t}{t}\,dt = \cot^{-1}(0) = \frac{\pi}{2}$$

Example 6.43 : *Find Laplace of* $\frac{\sin^2 t}{t^2}$

Solution :

Step 1 : We have

$$L[\sin^2 t] = L\left[\frac{1-\cos 2t}{2}\right] = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

Step 2 : If, $L\ f(t) = \phi(s)$ then, $L\frac{f(t)}{t} = \int_s^\infty \phi(s)\,ds$

Step 3 : Substitute the value of $\phi(s)$.

$$\therefore \quad L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds$$

Integrate w.r.t. s.

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log s^2 - \log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log \frac{s^2}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{4}\left[\log \frac{1}{1+\left(\frac{4}{s^2}\right)}\right]_s^\infty \qquad \text{(note this step)}$$

$$= \frac{1}{4}\left[\log \frac{1}{1+0} - \log \frac{1}{1+\frac{4}{s^2}}\right]$$

$$= \frac{1}{4}\left[0 - \log \frac{s^2}{s^2+4}\right]$$

$$= \frac{1}{4}\log \frac{s^2+4}{s^2}$$

Step 4 :

$$\therefore \quad L\left[\frac{\sin^2 t}{t^2}\right] = L\left[\frac{1}{t}\left(\frac{\sin^2 t}{t}\right)\right]$$

$$= \frac{1}{4}\int_s^\infty \left(\log \frac{s^2+4}{s^2}\right)\cdot(1)\, ds$$

Integrating by parts w.r.t. s.

$$= \frac{1}{4}\left[\left(\log \frac{s^2+4}{s^2}\right)(s) - \int \left(\frac{s^2}{s^2+4}\right)\left(\frac{-8}{s^3}\right)(s)\cdot ds\right]_s^\infty$$

$$= \frac{1}{4}\left[s\log\left(1+\frac{4}{s^2}\right) + 8\int \frac{1}{s^2+4}\, ds\right]_s^\infty$$

$$= \frac{1}{4}\left[s\log\left(1+\frac{4}{s^2}\right) + \frac{8}{2}\tan^{-1}\frac{s}{2}\right]_s^\infty$$

$$= \frac{1}{4}\left\{\left[0+4\left(\frac{\pi}{2}\right)\right] - \left[s\log\left(\frac{s^2+4}{s^2}\right) + 4\tan^{-1}\left(\frac{s}{2}\right)\right]\right\}$$

Step 5 :

$$\therefore \qquad L\frac{\sin^2 t}{t^2} = \frac{1}{4}\left[4\cot^{-1}\left(\frac{s}{2}\right)+s\log\left(\frac{s^2}{s^2+4}\right)\right]$$

Example 6.44 : *Prove that* $L\left(\frac{e^{-at}-e^{-bt}}{t}\right)=log\left(\frac{s+b}{s+a}\right)$

Solution :

Step 1 :

$$L\left(e^{-at}-e^{-bt}\right) = L\,e^{-at}-L\,e^{-bt}$$

$$= \frac{1}{s+a}-\frac{1}{s+b}$$

Step 2 : We know that if, L f(t) = ϕ(s) Then, $L\frac{f(t)}{t} = \int_s^\infty \phi(s)\,ds$

Step 3 : Substitute the value of ϕ(s).

$$L\frac{\left(e^{-at}-e^{-bt}\right)}{t} = \int_s^\infty \frac{1}{s+a}-\frac{1}{(s+b)}\cdot ds$$

Step 4 : Integrate w.r.t. s.

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$= \log\left[\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right]_s^\infty$$

Step 5 : Substitute the limits of s and simplify.

$$= \log 1-\log\frac{s+a}{s+b}$$

$$= 0-\log\frac{s+a}{s+b}$$

$$= \log\frac{(s+b)}{(s+a)}$$

Example 6.45 : $L\left(\frac{e^x \sin x}{x}\right)$

Solution :

Step 1 :

$$L\,(\sin ax) = \frac{a}{s^2+a^2}$$

$$\therefore \quad L \sin x = \frac{1}{s^2+1}$$

Step 2 : By first shifting property.

$$L\, e^x \sin x = \frac{1}{(s-1)^2+1}$$

Step 3 : If, $L\, f(t) = \phi(s)$ Then, $L\frac{f(t)}{t} = \int_s^\infty \phi(s)\, ds$

Step 4 : Substitute the value of $\phi(s)$.

$$L\frac{e^x \sin x}{x} = \int_s^\infty \frac{1}{(s-1)^2+1} \cdot ds$$

Step 5 : Integrate w.r.t. s.

$$= \left[\tan^{-1}\left(\frac{s-1}{1}\right)\right]_s^\infty$$

$$= \left[\tan^{-1}(s-1)\right]_s^\infty$$

Substitute the limits of s.

$$= \frac{\pi}{2} - \tan^{-1}(s-1)$$

$$= \cot^{-1}(s-1)$$

Example 6.46 : $L\dfrac{\cos at - \cos bt}{t}$

Solution :

Step 1 :

$$L(\cos at - \cos bt) = L \cos at - L \cos bt$$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

Step 2 : If, $L\, f(t) = \phi(s)$ Then, $L\frac{f(t)}{t} = \int_s^\infty \phi(s)\, ds$

Step 3 : Substitute the value of $\phi(s)$.

$$L\left(\frac{\cos at - \cos bt}{t}\right) = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

Step 4 : Integrate w.r.t. s.

$$= \int_s^{\infty} \frac{s}{s^2+a^2}\cdot ds - \int_s^{\infty} \frac{s}{s^2+b^2}\cdot ds$$

$$= \frac{1}{2}\int_s^{\infty} \frac{2s}{s^2+a^2}\cdot ds - \frac{1}{2}\int_s^{\infty} \frac{2s}{s^2+b^2}\cdot ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2)\right]_s^{\infty} - \frac{1}{2}\left[\log(s^2+b^2)\right]_s^{\infty}$$

Step 5 : Substitute limits of s

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^{\infty}$$

$$= \frac{1}{2}\log\left[\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]_s^{\infty}$$

$$= \frac{1}{2}\left\{\log 1 - \log\left[\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]_s^{\infty}\right\}$$

$$= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

$$= \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

Example 6.47 : $L\, e^{3t}\, \frac{\sin 2t}{t}$

Solution :

Step 1 : $L\, e^{3t} \sin 2t$

$$L \sin 2t = \frac{2}{s^2+4}$$

By first shifting property.

$$L\, e^{3t} \sin 2t = \frac{2}{(s-3)^2+4}$$

Step 2 : If, $L\, f(t) = \phi(s)$ Then, $L\frac{f(t)}{t} = \int_s^{\infty} \phi(s)\, ds$

Step 3 : Substitute the value of $\phi(s)$.

$$\therefore \quad L\frac{e^{3t}\sin 2t}{t} = \int_s^\infty \frac{2}{(s-3)^2+4}\cdot ds$$

Step 4 : Integrate w.r.t. s.

$$= 2\cdot\frac{1}{2}\left[\tan^{-1}\frac{(s-3)}{2}\right]_s^\infty$$

Step 5 : Substitute the limits of s.

$$= \left[\frac{\pi}{2}-\tan^{-1}\frac{(s-3)}{2}\right]$$

$$= \cot^{-1}\frac{(s-3)}{2}$$

Example 6.48 : *Find* $L\frac{1-\cos t}{t}$

Solution :

Step 1 :

$$L(1-\cos t) = L(1)-L(\cos t)$$

$$= \frac{1}{s}-\frac{1}{s^2+1}$$

Step 2 : If, $L\ f(t) = \phi(s)$ Then, $L\frac{f(t)}{t} = \int_s^\infty \phi(s)\,ds$

Step 3 : Substitute the value of $\phi(s)$.

$$L\left(\frac{1-\cos t}{t}\right) = \int_s^\infty\left(\frac{1}{s}-\frac{s}{s^2+1}\right)\cdot ds$$

Step 4 : Integrate w.r.t s.

$$= \int_s^\infty \frac{1}{s}\cdot ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+1}\cdot ds$$

$$= [\log s]_s^\infty - \frac{1}{2}\left[\log(s^2+1)\right]_s^\infty = \left[\log\frac{s}{\sqrt{s^2+1}}\right]_s^\infty$$

Step 5 : Substitute the limits of s and simplify.

$$= \left[\log\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]_s^\infty$$

$$= \log 1 - \log \frac{1}{\sqrt{1+\frac{1}{s^2}}}$$

$$= 0 - \log \frac{s}{\sqrt{s^2+1}}$$

$$= \frac{1}{2} \log \frac{s^2+1}{s^2}$$

Example 6.49 : *Find* $L \frac{1-\cos t}{t^2}$

Solution :

Step 1 : We know that

$$L\left(\frac{1-\cos t}{t}\right) = \frac{1}{2} \log\left(\frac{s^2+1}{s^2}\right)$$

$$= [\log s]_s^{\infty} \frac{-1}{2} [\log(s^2+1)]_s^{\infty}$$

Step 2 :

$$L\left(\frac{1-\cos t}{t^2}\right) = \frac{1}{2} \int_s^{\infty} \log\left(\frac{s^2+1}{s^2}\right) \cdot ds$$

Step 3 : Use integration by parts formula.

$$= \frac{1}{2}\left\{\log\left(\frac{s^2+1}{s^2}\right) s - \int \frac{s^2}{s^2+1} \frac{d}{ds}\left(\frac{s^2+1}{s^2}\right) s \cdot ds\right\}_s^{\infty}$$

$$= \frac{1}{2}\left\{0 - s \log\left(\frac{s^2+1}{s^2}\right) - \int_s^{\infty} \frac{s^2}{s^2+1} \frac{(-2s)}{s^4} s \cdot ds\right\}$$

$$= \frac{1}{2}\left\{- s \log\left(\frac{s^2+1}{s^2}\right) - \int_s^{\infty} \frac{(-2)}{s^2+1} \cdot ds\right\}$$

$$= \frac{1}{2}\left\{s \log \frac{s^2}{s^2+1} + 2[\tan^{-1} s]_s^{\infty}\right\}$$

$$= \frac{1}{2} s \log \frac{s^2}{s^2+1} + \cot^{-1} s$$

Example 6.50 : *Obtain Laplace transform of* $\frac{1-e^{-bt}}{t}$

Solution :

Step 1 : We have

$$L[1-e^{-bt}] = \frac{1}{s} - \frac{1}{s+b}$$

Step 2 : If, $L\ f(t) = \phi(s)$ Then, $L\frac{f(t)}{t} = \int_s^\infty \phi(s)\, ds$

Step 3 : Substitute the value of $\phi(s)$.

$$\therefore \quad L\left[\frac{1-e^{-bt}}{t}\right] = \int_s^\infty \left(\frac{1}{s}-\frac{1}{s+b}\right) ds$$

Step 4 : Integrate w.r.t. s.

$$= [\log(s) - \log(s+b)]_s^\infty$$

$$= \left[\log \frac{s}{s+b}\right]_s^\infty$$

Step 5 : Substitute the limits and simplify.

$$= \left[\log \frac{1}{1+\frac{b}{s}}\right]_s^\infty$$

$$= \log 1 - \log \frac{1}{1+\frac{b}{s}} = 0 - \log \frac{s}{s+b} = \log \frac{s+b}{s}$$

Example 6.51 : *Find Laplace transform of* $\frac{\cosh at - \cosh bt}{t}$.

Solution :

Step 1 : We have

$$L\ [\cosh at - \cosh bt] = \frac{s}{s^2-a^2} - \frac{s}{s^2-b^2}$$

Step 2 : If, $L\ f(t) = \phi(s)$ Then, $L\frac{f(t)}{t} = \int_s^\infty \phi(s)\, ds$

Step 3 :

$$\therefore \ L\left[\frac{\cosh at - \cosh bt}{t}\right] = \int_s^\infty \left(\frac{s}{s^2-a^2} - \frac{s}{s^2-b^2}\right) ds$$

Step 4 : Integrate w.r.t. s.

$$= \left[\frac{1}{2}\log\left(s^2 - a^2\right) - \frac{1}{2}\log\left(s^2 - b^2\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2 - a^2}{s^2 - b^2}\right]_s^\infty$$

Step 5 : Substitute the limits and simplify.

$$= \frac{1}{2}\left[\log\frac{1 - \frac{a^2}{s^2}}{1 - \frac{b^2}{s^2}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log\frac{s^2 - a^2}{s^2 - b^2}\right]$$

$$= -\frac{1}{2}\log\frac{s^2 - a^2}{s^2 - b^2} = \frac{1}{2}\log\frac{s^2 - b^2}{s^2 - a^2}$$

Exercise 6.6

Obtain Laplace transform of the following

1) $f(t) = \frac{1 - e^{-t}}{t}$ **[Ans. : L f(t) = $\log\frac{s+1}{s}$]**

2) $f(t) = \frac{e^{-4t}\sin 3t}{t}$ **[Ans. : L f(t) = $\cot^{-1}\left(\frac{s+4}{3}\right)$]**

3) $f(t) = \frac{\cos 6t - \cos 4t}{t}$ **[Ans. : L f(t) = $\frac{1}{2}\log\frac{s^2+16}{s^2+36}$]**

4) $f(t) = \frac{\sin ht}{t}$ **[Ans. : L f(t) = $\frac{1}{2}\log\frac{s+1}{s-1}$]**

5) $f(t) = \frac{1 - \cos 3t}{t}$ **[Ans. : L f(t) = $\frac{1}{2}\log\frac{s^2+9}{s^2}$]**

6) $f(t) = \frac{e^{t} - \cos 2t}{t}$ **[Ans. : L f(t) = $\log\frac{\sqrt{s^2+4}}{(s-1)}$]**

7) $f(t) = t^{-1}e^{-t}\sin t$ **[Ans. : L f(t) = $\cot^{-1}(s+1)$]**

8) $f(t) = \frac{1 - \cos ht}{t}$ **[Ans. : L f(t) = $\log\frac{\sqrt{s^2-1}}{s}$]**

6.15 Type 7 Laplace Transform of Derivative

If	$L\ f(t)$	$=$	$\phi(s)$
Then,	$L\ f'(t)$	$=$	$-f(0)+s\,\phi(s)$
	$L\ f''(t)$	$=$	$f'(0)-s\,f(0)+s^2\,\phi(s)$
	$L\ f'''(t)$	$=$	$-f''(0)-s\,f'(0)-s^2\,f(0)+s^3\phi(s)$

Example 6.52 : *If $f(t)=e^{-5t}\ sint$, find $L[f'(t)]$*

Solution :

Step 1 : $L\ f(t) = L\ e^{-5t}\sin t$

By first shifting property,

$$= \frac{1}{(s+5)^2+1}$$

Step 2 : We know that

$$L\ f'(t) = -f(0)+s\,\phi(s)$$

Step 3 :

$$\therefore \quad L\ f'(t) = -f(0)+s\times\frac{1}{(s+5)^2+1} \quad \text{... (1)}$$

Step 4 :

$$f(0) = \lim_{t\to 0} f(t)$$

$$= 0$$

$$\therefore \quad L\ f'(t) = \frac{s}{(s+5)^2+1} = \frac{s}{s^2+10\,s+26} \quad \text{from (1)}$$

Example 6.53 : *$f(t)=sin^2 t$, find $L[f'(t)]$*

Solution :

Step 1 :

$$L\ f(t) = L\sin^2 t$$

$$= L\left(\frac{1-\cos 2t}{2}\right)$$

$$= \frac{1}{2}\ L(1)-\frac{1}{2}\ L\cos 2t$$

Step 2 : $$\phi(s) = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Also $$f(0) = \sin^2 0 = 0$$

Step 3 : We know that

$$L[f'(t)] = -f(0) + s\,\phi(s)$$

Step 4 :

$$\therefore \quad L[f'(t)] = -0 + s \cdot \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= \frac{1}{2} - \frac{s^2}{2(s^2+4)}$$

$$= \frac{2s^2+8-2s^2}{4(s^2+4)}$$

$$= \frac{8}{4(s^2+4)}$$

$$= \frac{2}{(s^2+4)}$$

Example 6.54 : *If,* $L[f''(t)] = \tan^{-1}\left(\frac{1}{s}\right)$, $f(0) = 2$, $f'(0) = -1$. *Find* $L[f(t)]$

Solution :

Step 1 : We know that

$$L[f''(t)] = -f'(0) - s\,f(0) + s^2\,\phi(s)$$

Step 2 : Here $[f''(t)] = \tan^{-1}\left(\frac{1}{s}\right)$, $f(0) = 2$, $f'(0) = -1$

Step 3 : Substituting in above equation.

$$\therefore \quad \tan^{-1}\frac{1}{s} = -(-1) - s \times 2 + s^2\,\phi(s)$$

$$\tan^{-1}\frac{1}{s} = 1 - 2s + s^2\,\phi(s)$$

Step 4 : Thus we get,

$$\therefore \quad \phi(s) = \frac{\tan^{-1}(1/s) - 1 + 2s}{s^2}$$

$$\therefore \quad L[f(t)] = \frac{\tan^{-1}(1/s) - 1 + 2s}{s^2}$$

Example 6.55 : *Find L of* $\frac{d}{dt}\left(\frac{\sin t}{t}\right)$.

Solution :

Step 1 :

$$f(t) = \frac{\sin t}{t}$$

$$\therefore \quad f(0) = \lim_{t \to 0} \frac{\sin t}{t} = 1$$

Step 2 :

$$L(\sin t) = \frac{1}{s^2+1}$$

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1}\, ds$$

$$= \left[\tan^{-1} s\right]_s^\infty$$

$$= \left[\frac{\pi}{2} - \tan^{-1} s\right]$$

$$\phi(s) = \cot^{-1} s$$

Step 3 : $L\, f'(t) = -f(0) + s\,\phi(s)$

Step 4 : Using above property simplify.

$$L[f'(t)] = -f(0) + s\,\phi(s)$$

$$= s\cot^{-1} s - 1$$

Example 6.56 : *Obtain the Laplace transform of* $y(t)$, *if* $\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 4y = t$, *given that* $y(0) = y'(0) = y''(0) = 1$.

Solution :

Step 1 : Taking Laplace transforms of both sides, we get

$$L\left[\frac{d^3y}{dt^3}\right] - L\left[\frac{d^2y}{dt^2}\right] + 4L\left[\frac{dy}{dt}\right] - 4L[y(t)] = L(t)$$

Step 2 : Now we know that

$$L\, f'(t) = -f(0) + s\,\phi(s)$$

$$L\, f''(t) = f'(0) - s\, f(0) + s^2\,\phi(s)$$

$$L\, f'''(t) = -f''(0) - s\, f'(0) - s^2\, f(0) + s^3\,\phi(s)$$

Step 3 : Using above formula we get

$$\therefore \{s^3\, Y(s) - s^2\, y(0) - s\, y'(0) - y''(0)\} - \{s^2\, Y(s) - s\, y(0) - y'(0)\}$$
$$+ 4\,\{s\, Y(s) - y(0)\} - 4\, Y(s) = \frac{1}{s^2}$$

Step 4 : Simplify.

$$\therefore (s^3 - s^2 + 4s - 4)\; Y(s) + (-s^2 - s - 1) - (-s - 1) - 4 = \frac{1}{s^2}$$

$$\therefore \qquad y(0) = y'(0) = y''(0) = 1$$

$$\therefore (s^2 + 4)(s - 1)\, Y(s) = s^2 + 4 + \frac{1}{s^2}$$

$$\therefore \qquad Y(s) = \frac{1}{(s-1)} + \frac{1}{s^2(s-1)(s^2+4)}$$

Exercise 6.7

1. *Obtain Laplace transform of* $\frac{d^2y}{dx^2} - 3\frac{dy}{dt} + 5y$ *if* $y(0) = 2,\; y'(0) = -4$

[Ans. : $(s^2 - 3s + 5)\, Y(s) - 2s + 10$**]**

2. *Given that* $4f''(t) + f(t) - 0,\; f(0) = 0,\; f'(0) = 2$ *show that* $Lf(t) = \frac{8}{4s^2+1}$.

3. *Use theorem of derivative, to derive the following Laplace transforms*

i) $L(e^{at}) = \frac{1}{s-a}$ *ii)* $L(\sin at) = \frac{a}{s^2 + a^2}$

4. *Given that* $y'' + 2y' - 8y = 0,\; y(0) = 1,\; y'(0) = 8$ *show that* $Ly(t) = \frac{2}{s-2} - \frac{1}{s+4}$.

6.16 : Type 8 Laplace Transform of Integral

If	$L\, f(t)$	$= \phi(s)$
Then	$L\int_0^t f(t)\, dt$	$= \frac{1}{s}\phi(s)$

Example 6.57 : *Find L of* $\cosh t \int_0^t e^t \cosh t\, dt$

Solution :

Step 1 :

Let $\qquad f(t) = \int_0^t e^t \cosh t\, dt$

Step 2 : We know that if,

$$L\,f(t) = \phi(s)$$

Then $$L\int_0^t f(t)\,dt = \frac{1}{s}\,\phi(s)$$

Step 3 : $$\therefore \quad \phi(s) = L\left[\int_0^t e^t \cosh t\,dt\right] = \frac{1}{s}\,L\left[e^t \cosh t\right]$$

$$= \frac{1}{s}\,\frac{s-1}{\left[(s-1)^2-1\right]} = \frac{s-1}{s\left(s^2-2\,s\right)}$$

$$= \frac{s-1}{s^2\,(s-2)}$$

Step 4 : now

$$L\left[\cosh t\int_0^t e^t \cosh t\right] = L\left[\left(\frac{e^t+e^{-t}}{2}\right)f(t)\right]$$

$$= \frac{1}{2}\left\{L\left[e^t\,f(t)\right]+L\left[e^{-t}\,f(t)\right]\right\}$$ By first shifting property

$$= \frac{1}{2}\left\{\frac{(s-1)-1}{(s-1)^2\left[(s-1)-2\right]}+\frac{(s+1)-1}{(s+1)^2\left[(s+1)-2\right]}\right\}$$

$$= \frac{1}{2}\left\{\frac{s-2}{(s-1)^2\,(s-3)}+\frac{s}{(s+1)^2\,(s-1)}\right\}$$

Example 6.58 : $L\left(\int_0^t \frac{\sin t}{t}\,dt\right)$.

Solution :

Step 1 : Consider $$f(t) = \left(\int_0^t \frac{\sin t}{t}\,dt\right)$$

Step 2 : If, $L\,f(t) = \phi(s)$ Then $L\int_0^t f(t)\,dt = \frac{1}{s}\,\phi(s)$

Step 3 :

$$\therefore \quad L\int_0^t \frac{\sin t}{t} = \frac{1}{s} L \frac{\sin t}{t} \quad \text{... (1)}$$

Step 4 : Let $\quad f(t) = \dfrac{\sin t}{t}$

$$\therefore \quad L \sin t = \frac{1}{s^2+1}$$

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1}\, ds$$

$$= \left[\tan^{-1} s\right]_s^\infty$$

$$L\left(\frac{\sin t}{t}\right) = \frac{\pi}{2} - \tan^{-1} s$$

$$L\left(\frac{\sin t}{t}\right) = \cot^{-1} s$$

Step 5 : Substituting in (1)

$$L\int_0^t \frac{\sin t}{t} = \frac{1}{s} \cot^{-1} s$$

Example 6.59 : *Find* $L\left(\int_0^t \dfrac{e^{-4t} \sin 3t}{t}\, dt\right)$

Solution :

Step 1 : Consider $\quad f(t) = \left(\int_0^t \dfrac{e^{-4t} \sin 3t}{t}\, dt\right)$

Step 2 : If, $\quad L f(t) = \phi(s) \quad$ Then $\quad L\int_0^t f(t)\, dt = \dfrac{1}{s} \phi(s)$

Step 3 : $\therefore \quad L\int_0^t e^{-4t} \dfrac{\sin 3t}{t} = \dfrac{1}{s} L\left(e^{-4t} \dfrac{\sin 3t}{t}\right) \quad \text{... (1)}$

$$L \sin 3t = \frac{3}{s^2+9}$$

$$L\frac{\sin 3t}{t} = \int_s^{\infty} \frac{3}{s^2+9}\,ds$$

$$= 3\int_s^{\infty} \frac{1}{s^2+9}\,ds$$

$$= 3\times\frac{1}{3}\left[\tan^{-1}\frac{s}{3}\right]_s^{\infty}$$

$$= \left[\frac{\pi}{2}-\tan^{-1}\frac{s}{3}\right]$$

$$= \cot^{-1}\frac{s}{3}$$

$$L\,e^{-4t}\frac{\sin 3t}{t} = \cot^{-1}\frac{(s+4)}{3}$$

Step 4 : Substitute in (1)

$$L\left(\int_0^t e^{-4t}\,\frac{\sin 3t}{t}\cdot dt\right) = \frac{1}{s}\cot^{-1}\frac{(s+4)}{3}$$

Example 6.60 : *Find* $L\left(e^{-4t}\int_0^t \frac{\sin 3t}{t}\,dt\right)$

Solution :

Step 1 : Let $f(t) = \int_0^t \frac{\sin 3t}{t}\,dt$

Step 2 : We know that if, $L\,f(t) = \phi(s)$ Then $L\int_0^t f(t)\,dt = \frac{1}{s}\phi(s)$

Step 3 :

$$\therefore \quad L[f(t)] = \frac{1}{s}\,L\frac{\sin 3t}{t}$$

$$= \frac{1}{s}\int_s^{\infty}\frac{3}{s^2+9}\cdot dt$$

$$= \frac{1}{s}\times 3\times\frac{1}{3}\left[\tan^{-1}\frac{s}{3}\right]_s^{\infty}$$

$$= \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}\frac{s}{3}\right]$$

$$= \frac{1}{s}\cot^{-1}\frac{s}{3}$$

Step 4 : $L\, e^{-4t}\int_0^t \frac{\sin 3t}{t}\, dt$

$$= \frac{1}{s+4}\cot^{-1}\frac{s+4}{3}$$

Example 6.61 : *Find* $L\left(e^{-4t}\int_0^t t \sin 3t\ dt\right)$.

Solution :

Step 1 : Let $f(t) = \int_0^t t \sin 3t \cdot dt$

Step 2 : If, $L\, f(t) = \phi(s)$ Then $L\int_0^t f(t)\, dt = \frac{1}{s}\phi(s)$

Step 3 : $\therefore\ L[f(t)] = \frac{1}{s} L\, t \sin 3t$

$$= \frac{-1}{s}\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{-3}{s}\frac{-1}{(s^2+9)^2}\, 2s$$

$$= \frac{6}{(s^2+9)^2}$$

Step 4 : Use 1^{st} shifting property

$$L\, e^{-4t}\int_0^t t \sin 3t = \frac{6}{\left[(s+4)^2+9\right]^2}$$

$$= \frac{6}{(s^2+8s+16+9)^2}$$

$$= \frac{6}{(s^2+8s+25)^2}$$

Example 6.62 : *Find* $L\left(\int_0^t t\,e^{-4t} \sin 3t\,dt\right)$.

Solution :

Step 1 : Let

$$f(t) = \left(\int_0^t t\,e^{-4t} \sin 3t\ dt\right)$$

Step 2 : If, $\quad L\,f(t) = \phi(s) \quad$ Then $\quad L\int_0^t f(t)\,dt = \frac{1}{s}\,\phi(s)$

Step 3 :

$$\therefore\ L\left(\int_0^t t\,e^{-4t} \sin 3t\,dt\right) = \frac{1}{s}\ L\,t\,e^{-4t} \sin 3t \qquad \text{... (i)}$$

$$L \sin 3t = \frac{3}{s^2+9}$$

$$L\,e^{-4t} \sin 3t = \frac{3}{(s^2+4)^2+9}$$

$$= \frac{3}{s^2+8s+16+9}$$

$$= \frac{3}{s^2+8s+25}$$

Step 4 : Using property of multiplication by t.

$$L\,t\left(e^{-4t} \sin 3t\right) = \frac{-d}{ds}\,\frac{3}{s^2+8\,s+25}$$

$$= -3\,\frac{-1}{(s^2+8\,s+25)^2}\,(2s+8)$$

$$= \frac{3(2s+8)}{(s^2+8s+25)^2}$$

Substituting in (i) we get

$$\therefore\ L\left(\int_0^t t\,e^{-4t} \sin 3t\cdot dt\right) = \frac{1}{s}\,\frac{3(2s+8)}{(s^2+8s+25)^2}$$

$$= \frac{1}{s}\,\frac{6(s+4)}{(s^2+8s+25)^2}$$

Example 6.63 : *Find* $L\left(\int_0^t \frac{1-e^{-x}}{x}\,dx\right)$

Solution :

Step 1 :

Let $$f(t) = \left(\int_0^t \frac{1-e^{-x}}{x}\,dx\right)$$

Step 2 : If, $L\,f(t) = \phi(s)$ Then $L\int_0^t f(t)\,dt = \frac{1}{s}\,\phi(s)$

Step 3 : Using above property we get,

$$L\left(\int_0^t \frac{1-e^{-x}}{x}\,dx\right) = \frac{1}{s}\,L\,\frac{1-e^{-x}}{x} \qquad \text{... (i)}$$

$$L\left(1-e^{-x}\right) = \frac{1}{s} - \frac{1}{(s+1)}$$

$$L\,\frac{\left(1-e^{-x}\right)}{x} = \int_s^{\infty} \frac{1}{s} - \frac{1}{(s+1)}\,ds$$

$$= \left[\log s - \log(s+1)\right]_s^{\infty}$$

$$= \left[\log \frac{s}{s+1}\right]_s^{\infty}$$

$$\therefore \quad L\left(\frac{1-e^{-x}}{x}\right) = \log \frac{(s+1)}{s}$$

Step 4 : Substituting in (i)

$$\therefore \; L\left(\int_0^t \frac{1-e^{-x}}{x}\,dx\right) = \frac{1}{s}\,\log \frac{s+1}{s}$$

Example 6.64 : *Find* $L\left(\int_0^t u\cosh u\,du\right)$.

Solution :

Step 1 :

Let $$f(t) = \left(\int_0^t u \cosh u\,du\right)$$

Step 2 : If, $\quad L\,f(t) = \phi(s)$ Then $L\int_0^t f(t)\,dt = \frac{1}{s}\,\phi(s)$

Step 3 : $\therefore \quad L[f(t)] = \frac{1}{s}\,L\,(u\cosh u)$ ... (i)

$$L\cosh u = \frac{s}{s^2-1}$$

Using property of multiplication by t.

$$L\,u\cosh u = -\frac{d}{ds}\frac{s}{s^2-1}$$

$$= -\frac{(s^2-1)-s(2s)}{(s^2-1)^2}$$

$$= -\frac{s^2-1-2s^2}{(s^2-1)^2}$$

$$= \frac{2s^2-s^2+1}{(s^2-1)^2}$$

$$= \frac{s^2+1}{(s^2-1)^2}$$

Step 4 : Substituting in (i)

$$\therefore L\left(\int_0^t u\cosh u\,du\right) = \frac{1}{s}\left\{\frac{s^2+1}{(s^2-1)^2}\right\}$$

Example 6.65 : *Find L erf* $\sqrt{t}$.

Solution : Step 1 : From the definition of $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$

We get, $\quad \operatorname{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du$

Step 2 : If, $\quad L\,f(t) = \phi(s)$ Then $L\int_0^t f(t)\,dt = \frac{1}{s}\,\phi(s)$

Step 3 : Put $u^2 = x$

$$u = x^{1/2}$$

$$u = \frac{1}{2} x^{-1/2} dx$$

u	0	$\sqrt{t}$
x	0	t

$$\therefore \quad \text{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x} \frac{1}{2} x^{-1/2} dx$$

$$\text{erf}\sqrt{t} = \frac{1}{\sqrt{\pi}} \int_0^t e^{-x} x^{-1/2} dx$$

Step 4 : Now we can find the Laplace of above integral.

$$L\, x^{-1/2} = \frac{\lceil 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$L\, x^{-1/2} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$\therefore$ by first shifting property

$$L\left(e^{-x} x^{-1/2}\right) = \frac{\sqrt{\pi}}{\sqrt{s+1}}$$

$$\therefore \quad L\left(\int_0^t e^{-x} x^{-1/2} dx\right) = \frac{\sqrt{\pi}}{s\sqrt{s+1}}$$

$$L\, \text{erf}\sqrt{t} = \frac{1}{s} \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{s+1}}$$

$$L\, \text{erf}\sqrt{t} = \frac{1}{s\sqrt{s+1}}$$

Example 6.66 : *Find* $L\, erf_c \sqrt{t}$

Solution : We know that $\text{erf}\sqrt{t} + \text{erf}_c\sqrt{t} = 1$

$$\therefore \quad \text{erf}_c\sqrt{t} = 1 - \text{erf}\sqrt{t}$$

Take Laplace transform on both sides.

$$L\, \text{erf}_c\sqrt{t} = L(1) - L\,\text{erf}\sqrt{t}$$

$$L\, \text{erf}_c\sqrt{t} = \frac{1}{s} - \frac{1}{s\sqrt{s+1}}$$

Example 6.67 : $\displaystyle\int_0^t \frac{e^t - \cos 2t}{t} dt$

Solution :

Step 1 :

Let $$f(t) = \left(\int_0^t \frac{e^t - \cos 2t}{t} dt\right)$$

Step 2 : If, $f(t) = \phi(s)$ Then $L\int_0^t f(t)\,dt = \frac{1}{s}\phi(s)$

Step 3 :

$$\therefore \quad L[f(t)] = \frac{1}{s} L\left(\frac{e^t}{t} - \frac{\cos 2t}{t}\right)$$

Step 4 : Use effect of division by t.

$$L\ f(t) = \frac{1}{s}\left[\int_s^\infty \frac{1}{s-1} - \frac{s}{s^2+4}\,ds\right]$$

$$= \frac{1}{s}\left[\log \frac{s-1}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \frac{1}{s}\left[\log \frac{\sqrt{s^2+4}}{s-1}\right]$$

Example 6.68 : $\int_0^t t\, e^{-3t} \sin 2t\, dt$

Solution :

Step 1 :

Let $f(t) = \int_0^t t\, e^{-3t} \sin 2t\ dt$

Step 2 : We know that if,

$$L\, f(t) = \phi(s) \text{ Then } L\int_0^t f(t)\,dt = \frac{1}{s}\phi(s)$$

Step 3 :

$$\therefore \quad L[f(t)] = \frac{1}{s} L\left(t\, e^{-3t} \sin 2t\right)$$

$$L \sin 2t = \frac{2}{s^2+4}$$

$$L\, e^{-3t} \sin 2t = \frac{2}{(s+3)^2+4}$$

$$= \frac{2}{s^2+6s+9+4}$$

$$= \frac{2}{s^2+6s+13}$$

Step 4 : Using property of multiplication by t.

$$L\, t\, e^{-3t} \sin 2t = \frac{-d}{ds} \frac{2}{s^2+6s+13}$$

$$= \frac{2(2s+6)}{(s^2+6s+13)^2} = \frac{4(s+3)}{(s^2+6s+13)^2}$$

$$\therefore L\int_0^t t\, e^{-3t} \sin 2t\, dt = \frac{1}{s} \frac{2(2s+6)}{(s^2+6s+13)^2}$$

Example 6.69 : *Obtain Laplace transform of* $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 4y + 2\int_0^t y(t)\,dt.$

Solution : $L\left[\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 4y + 2\int_0^t y(t)\,dt\right]$

$$= L\left[\frac{d^2y}{dt^2}\right] + 3L\left[\frac{dy}{dt}\right] + 4L\,[y(t)] + 2L\left[\int_0^t y(t)\,dt\right]$$

$$= \{s^2\, Y(s) - s\, y(0) - y'(0)\} + 3\{s\, Y(s) - y(0)\} + 4\, Y(s) + 2\frac{1}{s}\, Y(s)$$

$$= \left(s^2 + 3s + 4 + \frac{2}{s}\right) Y(s) - (s+3)\, y(0) - y'(0)$$

Example 6.70 : *Verify,* $L\left[\int_0^t u^2 e^{-u}\, du\right] = \frac{1}{s} L\,[t^2 e^{-t}]$

Solution :

$$\text{L.H.S.} = L\left[\int_0^t u^2\, e^{-u}\, du\right]$$

$$= L\left[\left\{u^2(-e^{-u}) - (2u)(e^{-u}) + (2)(-e^{-u})\right\}_0^t\right]$$

$$= L\left[\left\{-(u^2+2u+2)\, e^{-u}\right\}_0^t\right] = L\left[2 - (t^2+2t+2)\, e^{-t}\right]$$

$$= 2\, L(1) - L\left[(t^2+2t+2)\, e^{-t}\right]$$

Using 1^{st} shifting property we get

$$= \frac{2}{s} - 2\left\{\frac{1}{(s+1)^3} + \frac{1}{(s+1)^2} + \frac{1}{s+1}\right\} = \frac{2}{s(s+1)^3} \qquad \text{... (1)}$$

$$\text{R.H.S.} = \frac{1}{s} L\left[t^2 e^{-t}\right] = \frac{1}{s} L\left[e^{-t}(t^2)\right]$$

$$= \frac{2}{s(s+1)^3} \qquad \text{... (2)}$$

Using 1^{st} shifting property.

Exercise 6.8

Obtain the Laplace transform of the following :

1. $f(t) = e^{-2t}\int_0^t \frac{\sin^2 t}{t}\, dt$ [Ans. : $L f(t) = \frac{1}{2(s+2)} \log \frac{\sqrt{s^2+4s+8}}{s+2}$]

2. $f(t) = t\int_0^t e^{-2t}\cos t\, dt$ [Ans. : $L f(t) = \frac{2s^3+10s^2+16s+10}{s^2(s^2+4s+5)^2}$]

3. $f(t) = \int_0^t t\, e^{-3t}\sin 2t\, dt$ [Ans. : $L f(t) = \frac{4(s+3)}{s(s^2+6s+13)^2}$]

4. $f(t) = \int_0^t e^{-at}\frac{\sin bt}{t}\, dt$ [Ans. : $L f(t) = \frac{1}{s}\cot^{-1}\left(\frac{s+a}{b}\right)$]

5. $f(t) = t\int_0^t e^{-3t}\sin 2t\, dt$ [Ans. : $L f(t) = \frac{6s^2+24s+26}{s^2(s^2+6s+13)^2}$]

6. $f(t) = e^{-3t}\int_0^t t\sin 2t\, dt$ [Ans. : $L f(t) = \frac{4}{(s^2+6s+13)^2}$]

7. $f(t) = \cosh t\int_0^t t\cosh t\, dt$ [Ans. : $\frac{1}{2}\left[\frac{s^2-2s+2}{(s-1)(s^2-2s)^2} + \frac{s^2+2s+2}{(s+1)(s^2+2s)^2}\right]$]

8. $f(t) = t\int_0^t e^{-4t}\sin 3t\, dt$ [Ans. : $L f(t) = \frac{3}{s^2+8s+25}$]

9. *Verify directly* $L\int_0^t (u^2-u+e^{-u})\,du = \frac{1}{s}L\left(t^2-t+e^{-t}\right)$

10. $f(t) = \int_0^t e^x\cos x\, dx$ [Ans. : $L f(t) = \frac{s-1}{s(s^2-2s+2)}$]

11. $f(t) = \int_0^t \frac{e^x-\cos x}{x}\, dx$ [Ans. : $L f(t) = \frac{1}{s}\log\frac{\sqrt{s+1}}{s-1}$]

12. $f(t) = \int_0^t t\, e^{3t}\cos 2t\, dt$ [Ans. : $L f(t) = \frac{1}{s}\frac{s^2-6s+5}{(s^2-6s+13)^2}$]

6.17 : Type 9 Laplace Transform using Series

i)	e^x	$= 1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}\ldots$
ii)	$\sin x$	$= x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}\ldots$
iii)	$\cos x$	$= 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}\ldots$
iv)	$\dfrac{1}{(1-x)}$	$= 1+x+x^2+x^3+x^4\ldots$

Example 6.71 : $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$

Solution : **Step 1 :** Using series for cos function we have

$$\cos\sqrt{t} = 1-\frac{(\sqrt{t})^2}{2!}+\frac{(\sqrt{t})^4}{4!}-\frac{(\sqrt{t})^6}{6!}+\ldots$$

$$= 1-\frac{t}{2!}+\frac{t^2}{4!}-\frac{t^3}{6!}+\ldots$$

Step 2 : Dividing by $\sqrt{t}$ we get,

$$\therefore \quad \frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-1/2}-\frac{t^{1/2}}{2!}+\frac{t^{3/2}}{4!}-\frac{t^{5/2}}{6!}+\ldots$$

Step 3 : Taking Laplace transform,

$$\therefore \quad L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = L\left[t^{-1/2}\right]-\frac{1}{2!}L\left[t^{1/2}\right]+\frac{1}{4!}L\left[t^{3/2}\right]-\frac{1}{6!}L\left[t^{5/2}\right]+\ldots$$

Step 4 : Using formula

$$L\left[t^n\right] = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}}-\frac{1}{2!}\frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}}+\frac{1}{4!}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{5/2}}-\frac{1}{6!}\frac{\Gamma\left(\frac{7}{2}\right)}{s^{7/2}}+\ldots$$

$$= \frac{\sqrt{\pi}}{s^{1/2}}-\frac{1}{2!}\frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}}+\frac{1}{4!}\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{5/2}}-\frac{1}{6!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{7/2}}+\ldots$$

$$= \frac{\sqrt{\pi}}{s^{1/2}} \left\{1 - \frac{1}{(4s)} + \frac{1}{2!}\frac{1}{(4s)^2} - \frac{1}{3!}\frac{1}{(4s)^3} + \ldots\right\}$$

$$= \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

Example 6.72 : $L\, sin t^2$.

Solution : Step 1 : Using series for sin we have,

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} \ldots$$

Step 2 : Hence given function becomes,

$$\sin t^2 = t^2 - \frac{(t^2)^3}{3!} + \frac{(t^2)^5}{5!} - \frac{(t^2)^7}{7!}$$

$$= t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!}$$

Step 3 : Taking Laplace transform,

$$L \sin t^2 = L\, t^2 - \frac{1}{3!} L\left[t^6\right] + \frac{1}{5!} L\left[t^{10}\right] - \frac{1}{7!} L\left[t^{14}\right]$$

Step 4 : Using formula

$$L\left[t^n\right] = \frac{\overline{|n+1}}{s^{n+1}}$$

$$= \frac{\overline{|3}}{s^3} - \frac{1}{3!}\frac{\overline{|7}}{s^7} + \frac{1}{5!}\frac{\overline{|11}}{s^{11}} - \frac{1}{7!}\frac{14!}{s^{15}}$$

$$= \frac{2}{s^3} - \frac{1}{3 \times 2}\frac{6 \times 4 \times 5 \times 3 \times 2}{s^7}$$

$$+ \frac{10 \times 9 \times 8 \times 7 \times 6}{s^{11}} - \frac{8 \times 9 \times 10 \times 11 \times 12 \times 13 \times 14}{s^{15}}$$

$$= \frac{2}{s^3} - \frac{120}{s^7} + \frac{30240}{s^{11}} - \frac{17297280}{s^{15}}$$

Example 6.73 : $L\left\{\frac{t^{n-1}}{1-e^{-t}}\right\}$.

Solution : Step 1 : We have

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots$$

$$= \sum_{a=0}^{\infty} z^a$$

Step 2 : Substitute $z = e^{-t}$

$$= \sum_{a=0}^{\infty} e^{-at}$$

Step 3 : Multiply by e^{-at}

$$\frac{t^{n-1}}{1-e^{-t}} = \sum_{a=0}^{\infty} e^{-at}\, t^{n-1} \qquad \text{... (i)}$$

$$L\left(\frac{t^{n-1}}{1-e^{-t}}\right) = \sum_{a=0}^{\infty} L\, e^{-at}\, t^{n-1}$$

Step 4 : Consider $\quad L\, t^{n-1} = \frac{\Gamma n}{s^n}$ (By 1st shifting property)

$$\therefore \qquad L\, e^{-at}\, t^{n-1} = \frac{\Gamma n}{(s+a)^n}$$

Substituting in (i) we get

$$\therefore \qquad L\left(\frac{t^{n-1}}{1-e^{-t}}\right) = \sum_{a=0}^{\infty} \frac{\Gamma n}{(s+a)^n}$$

Example 6.74 : $L\left[\sin\sqrt{t}\right]$

Solution : Step 1 : We know that

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} \dots$$

Step 2 :

$$\sin\sqrt{t} = \sqrt{t} - \frac{\left(\sqrt{t}\right)^3}{3!} + \frac{\left(\sqrt{t}\right)^5}{5!} - \frac{\left(\sqrt{t}\right)^7}{7!}$$

$$= \sqrt{t} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!}$$

Step 3 : Taking Laplace transform

$$L\,\sin\sqrt{t} = L\left(t^{1/2}\right) - \frac{1}{3!}\, L\left(t^{3/2}\right) + \frac{1}{5!}\, L\left(t^{5/2}\right) - \frac{1}{7!}\left(t^{7/2}\right)$$

$$= \frac{\Gamma\, 3/2}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma\, 5/2}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma\, 7/2}{s^{7/2}} - \frac{1}{7!}\frac{\Gamma\, 9/2}{s^{9/2}} + \dots$$

Step 4 : Using formula

$$L\left[t^n\right] = \frac{\Gamma\, n+1}{s^{n+1}}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{3!}\,\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{s^{5/2}} + \frac{1}{5!}\,\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{7/2}} - \frac{1}{7!}\,\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{9/2}} + \dots$$

$$= \frac{\sqrt{\pi}}{2\,s^{3/2}}\left[1-\frac{1}{4s}+\frac{1}{2!(4s)^2}-\frac{1}{3!(4s)^3}+\ldots\right]$$

$$= \frac{\sqrt{\pi}}{2s\sqrt{s}}\,e^{-1/4s}$$

Example 6.75 : *Find Laplace transform of* $\frac{\sin\sqrt{t}}{\sqrt{t}}$

Solution : **Step 1 :** We know that

$$\sin t = t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}\ldots$$

$$\therefore \quad \sin\sqrt{t} = \sqrt{t}-\frac{(\sqrt{t})^3}{3!}+\frac{(\sqrt{t})^5}{5!}-\frac{(\sqrt{t})^7}{7!}\ \ldots.\infty$$

Step 2 : Divide by $\sqrt{t}$

$$\therefore \quad \frac{\sin\sqrt{t}}{\sqrt{t}} = 1-\frac{t}{3!}+\frac{t^2}{5!}-\frac{t^3}{7!}\ldots$$

Step 3 : Taking Laplace transform

$$\therefore \quad L\left[\frac{\sin\sqrt{t}}{\sqrt{t}}\right] = L\left[1-\frac{t}{3!}+\frac{t^2}{5!}-\frac{t^3}{7!}\ldots\right]$$

Step 4 : Using formula

$$L\left[t^n\right] = \frac{\overline{|n+1}}{s^{n+1}}$$

$$= \left[\frac{1}{s}-\frac{1}{3!\,s^2}+\frac{2!}{5!\,s^3}-\frac{3!}{7!\,s^4}+\ldots\right]$$

Exercise 6.9

1. *Find* $LJ_0(t)$ *where* $J_0(t)$ *is the Bessel function of order zero defined by*
$J_0(t) = 1-\frac{t^2}{2^2}+\frac{t^4}{2^2.4^2}-\frac{t^6}{2^2.4^2.6^2}\ldots$ **[Ans. :** $\frac{1}{\sqrt{s^2+1}}$**]**

2. *Use infinite series to obtain Laplace transform of* $\int_0^t \frac{\sin u}{u}\,du$. **[Ans. :** $\frac{1}{s}\tan^{-1}\frac{1}{s}$**]**

[Hint : $\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}\ldots$**]**

3. *Find* $L\cos\sqrt{t}$. **[Ans. :** $\frac{1}{s}\left[1-\left(\frac{1}{2s}\right)+\frac{(1/2s)^2}{3}-\frac{(1/2s)^3}{3.5}+\frac{(1/2s)^4}{3.5.7}-\frac{(1/2s)^5}{3.5.7.9}\ldots\right]$**]**

6.18 : Type 10 : Evaluate using Laplace Transform

Example 6.76 : *Evaluate* $\int_0^\infty t\, e^{-t^2}\, erf(t)\, dt.$

Solution : Step 1 : Let

$$f(t) = \int_0^\infty t\, e^{-t^2}\, \text{erf}(t)\, dt$$

Step 2 : Put $t^2 = y \quad \therefore 2t\, dt = dy$

when $t = 0, \quad y = 0$ and $t \to \infty, \quad y \to \infty$

$$\therefore \int_0^\infty t\, e^{-t^2} \text{erf}(t)\, dt = \int_0^\infty e^{-y}\, \text{erf}\left(\sqrt{y}\right) \frac{dy}{2}$$

As the variable is not important.

$$= \frac{1}{2} \int_0^\infty e^{-t}\, \text{erf}\left(\sqrt{t}\right) dt$$

Step 3 : By definition of Laplace transform, we have

$$\int_0^\infty e^{-st}\, f(t)\, dt = L[f(t)]$$

Step 4 : Now we know that

$$L\left[\text{erf}\left(\sqrt{t}\right)\right] = \frac{1}{s\sqrt{s+1}}$$

$$\therefore \int_0^\infty e^{-st}\, \text{erf}\left(\sqrt{t}\right) dt = \frac{1}{s\sqrt{s+1}}$$

Step 5 : To get the required answer put s = 1, we get

$$\int_0^\infty e^{-t}\, \text{erf}\left(\sqrt{t}\right) dt = \frac{1}{\sqrt{2}}$$

$$\therefore \int_0^\infty t\, e^{t^2}\, \text{erf}(t)\, dt = \frac{1}{2\sqrt{2}}$$

Example 6.77 : *Evaluate the integral* $\int_0^\infty \frac{\sin t}{t}\, dt.$

Solution : Step 1 : Let

$$f(t) = \frac{\sin t}{t}$$

Step 2 : We have,

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \cot^{-1} s$$

Step 3 : By definition of Laplace transform, we have

$$\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$$

Step 4 :

$$\therefore \quad \int_0^\infty e^{-st} \frac{\sin t}{t}\,dt = \cot^{-1} s$$

Step 5 : To get the required answer put s = 0

$$\int_0^\infty \frac{\sin t}{t}\,dt = \cot^{-1} 0$$

$$= \frac{\pi}{2}$$

Example 6.78 : *Evaluate* $\int_0^\infty e^{-2t}\, t\, erf\left(2\sqrt{t}\right) dt$

Solution : Step 1 : Let $f(t) = t\,\text{erf}\left(2\sqrt{t}\right)$

Step 2 : $L\,\text{erf}\sqrt{t} = \frac{1}{s\sqrt{s+1}}$

$$L\,\text{erf}\sqrt{4t} = \frac{1}{4}\,\frac{1}{\frac{s}{4}\sqrt{\frac{s}{4}+1}} \qquad \text{By change of scale property.}$$

$$= \frac{2}{s\sqrt{s+4}}$$

$\therefore$ Using effect of multiplication by t.

$$L\,t\,\text{erf}\sqrt{4t} = (-1)\,\frac{d}{ds}\,\frac{2}{s\sqrt{s+4}}$$

$$= (-1)\,\frac{d}{ds}\,\frac{2}{\sqrt{s^3+4\,s^2}}$$

$$= -2\,\frac{d}{ds}\,\frac{1}{\sqrt{s^3+4\,s^2}}$$

$$= -2\,\frac{-1}{2}\left(s^3+4s^2\right)^{-3/2}\left(3s^2+8s\right)$$

$$= \frac{\left(3s^2+8s\right)}{\left(s^3+4s^2\right)^{3/2}}$$

Step 3 : By definition of Laplace transform, we have

$$\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$$

Step 4 :

$$\therefore \int_0^\infty e^{-st}\, t\,\text{erf}\,2\sqrt{t}\,dt = \frac{3s^2+8s}{\left(s^3+4s^2\right)^{3/2}}$$

Step 5 : To get the required answer put s = + 2

$$\therefore \int_0^\infty e^{-2t}\, t\,\text{erf}\,2\sqrt{t}\,dt = \frac{12+16}{(8+16)^{3/2}}$$

$$= \frac{28}{(24)^{3/2}}$$

$$= \frac{28}{(4)^{3/2}(6)^{3/2}}$$

$$= \frac{28}{2\times 2\times 2\times (6)^{3/2}}$$

$$= \frac{7}{2(6)^{3/2}}$$

Example 6.79 : *Evaluate* $\int_0^\infty e^{-3t}\,\frac{\sin ht}{t}\,dt$

Solution :

Step 1 : Let $f(t) = \frac{\sin ht}{t}$

Step 2 : We know that

$$L[\sinh t] = \frac{1}{s^2-1}$$

$$L\left[\frac{\sin ht}{t}\right] = \int_s^\infty \frac{1}{s^2-1}\,ds \quad \text{effect of division by t.}$$

$$= \frac{1}{2}\left[\log\left(\frac{s-1}{s+1}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[-\log\frac{s-1}{s+1}\right]$$

$$\phi(s) = \frac{1}{2}\log\frac{s+1}{s-1}$$

Step 3 : By definition of Laplace transform, we have

$$\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$$

Step 4 :

$$\therefore \int_0^\infty e^{-st}\frac{\sinh t}{t}\,dt = \frac{1}{2}\log\frac{s+1}{s-1}$$

Step 5 : To get the required answer put s = 3

$$\int_0^\infty e^{-3t}\frac{\sinh t}{t}\,dt = \frac{1}{2}\log\frac{4}{2}$$

$$= \frac{1}{2}\log 2$$

Example 6.80 : *Evaluate* $\int_0^\infty \frac{\cos 3t - \cos 2t}{t}\,dt$

Solution : Step 1 : Let

$$f(t) = \frac{\cos 3t - \cos 2t}{t}$$

Step 2 : We find out Laplace transform first

$$L\cos 3t - \cos 2t = \left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

$$\therefore L\left(\frac{\cos 3t - \cos 2t}{t}\right) = \int_s^\infty \left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)ds$$

$$= \frac{1}{2}\left[\log(s^2+9) - \log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2+9}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{2}\left[0-\log\left(\frac{s^2+9}{s^2+4}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)$$

$$\therefore\ L\left(\frac{\cos 3t-\cos 2t}{t}\right) = \frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)$$

Step 3 : By definition of Laplace transform, we have

$$\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$$

Step 4 :

$$\therefore \int_0^\infty e^{-st}\left(\frac{\cos 3t-\cos 2t}{t}\right)dt = \frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)$$

Step 5 : To get the required answer put s = 0

$$\therefore \quad \int_0^\infty \left(\frac{\cos 3t-\cos 2t}{t}\right)dt = \frac{1}{2}\log\frac{4}{9}$$

$$= \log\frac{2}{3}$$

Example 6.81 : *Evaluate* $\int_0^\infty e^{-2t}\sin^3 t\,dt$

Solution : **Step 1 :** Let $f(t) = \sin^3 t$

Step 2 : $L(\sin^3 t) = L\left(\frac{3}{4}\sin t-\frac{1}{4}\sin 3t\right)$

$$= \frac{-1}{4}L[\sin 3t-\sin t]$$

$$= \left(\frac{3}{4}\cdot\frac{1}{s^2+1}-\frac{1}{4}\cdot\frac{3}{s^2+9}\right)$$

$$= \frac{3}{4}\left[\frac{1}{s^2+1}-\frac{1}{s^2+9}\right]$$

Step 3 : By definition of Laplace transform, we have

$$\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$$

Step 4 :

$$\int_0^\infty e^{-st} \sin^3 t\, dt = \frac{3}{4}\left[\frac{1}{s^2+1}-\frac{1}{s^2+9}\right]$$

Step 5 : To get the required answer put s = 2

$$\int_0^\infty e^{-2t} \sin^3 t\, dt = \frac{3}{4}\left[\frac{1}{5}-\frac{1}{13}\right]$$

$$= \frac{6}{65}$$

Example 6.82 : *Evaluate* $\int_0^\infty e^{2t}\left(\frac{\cosh 6t-\cosh 4t}{t}\right)dt$

Solution : Step 1 : Let

$$f(t) = \left(\frac{\cosh 6t}{t}-\frac{\cosh 4t}{t}\right)$$

Step 2 : $L\left[\frac{\cosh 6t}{t}\right]-L\left[\frac{\cosh 4t}{t}\right]$

$$= \int_s^\infty \frac{s}{s^2-36}\, ds-\int_s^\infty \frac{s}{s^2-16}\, ds$$

$$= \frac{1}{2}\int_s^\infty \frac{2s}{s^2-36}\, ds-\frac{1}{2}\int_s^\infty \frac{2s}{s^2-16}\, ds$$

$$= \frac{1}{2}\left[\log(s^2-36)-\log(s^2-16)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2-36}{s^2-16}\right)_s^\infty\right]$$

$$= \frac{1}{2}\left[0-\log\frac{s^2-36}{s^2-16}\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2-16}{s^2-36}\right)$$

Step 3 : By definition of Laplace transform, we have

$$\int_0^\infty e^{-st} f(t)\, dt = L[f(t)]$$

Step 4 :

$$\therefore \int_0^\infty e^{-st}\left(\frac{\cosh 6t - \cosh 4t}{t}\right) dt = \frac{1}{2} \log\left(\frac{s^2-16}{s^2-36}\right)$$

Step 5 : To get the required answer put s = – 2

$$\therefore \int_0^\infty e^{2t}\left(\frac{\cosh 6t - \text{cish } 2t}{t}\right) dt = \frac{1}{2} \log\left(\frac{4-16}{4-36}\right)$$

$$= \frac{1}{2} \log\left(\frac{-12}{-32}\right)$$

$$= \frac{1}{2} \log\left(\frac{3}{8}\right)$$

Example 6.83 : *Evaluate* $\int_0^\infty e^{2t}\left(\frac{e^{-at}-e^{-bt}}{t}\right) dt$

Solution : Step 1 : let

$$f(t) = \left(\frac{e^{-at}-e^{-bt}}{t}\right)$$

Step 2 : $L\left(\frac{e^{-at}-e^{-bt}}{t}\right) = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$

$$= \left[\log \frac{s+a}{s+b}\right]_s^\infty$$

$$= \left[0 - \log\frac{s+a}{s+b}\right]$$

$$= -\log\frac{s+b}{s+a}$$

Step 3 : By definition of Laplace transform, we have

$$\int_0^\infty e^{-st} f(t)\, dt = L[f(t)]$$

Step 4 :

$$\therefore \int_0^\infty e^{-st}\left(\frac{e^{-at}-e^{-bt}}{t}\right) dt = \log\frac{s+b}{s+a}$$

Step 5 : To get the required answer put s = – 2

$$\int_0^\infty e^{2t}\left(\frac{e^{-at}-e^{-bt}}{t}\right)dt = \log\frac{b-2}{a-2}$$

Example 6.84 : *Evaluate* $\int_0^\infty e^{-t}\frac{\sin^2 t}{t}\,dt$

Solution : Step 1 : Let

$$f(t) = \frac{\sin^2 t}{t}$$

Step 2 :

$$L(\sin^2 t) = L\sin t\cdot\sin t$$

$$= L\frac{1}{2}(\cos 0-\cos 2t)$$

$$= L\left[\frac{1}{2}-\frac{1}{2}\cos 2t\right]$$

$$= \frac{1}{2s}-\frac{s}{2(s^2+4)}$$

$$L\left[\frac{\sin^2 t}{t}\right] = \int_s^\infty \frac{1}{2s}-\frac{s}{2(s^2+4)}\,ds$$

$$= \left[\frac{1}{2}\log s-\frac{1}{4}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[2\log s-\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\frac{s^2}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{4}\left[0-\log\frac{s^2}{s^2+4}\right]$$

$$= \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

Step 3 : By definition of Laplace transform, we have

$$\int_0^\infty e^{-st}f(t)\,dt = L[f(t)]$$

Step 4 :

$$\int_0^\infty e^{-st}\frac{\sin^2 t}{t}\,dt = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

Step 5 : To get the required answer put s = 1

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t}\,dt = \frac{1}{4}\log\left(\frac{5}{1}\right)$$

Example 6.85 : *Evaluate* $\int_0^\infty e^t(\sin t + t\cos t)\,dt$

Solution : Step 1 : Let

$$f(t) = (\sin t + t\cos t)$$

Step 2 : $L(\sin t + t\cos t) = L\sin t + L\,t\cos t$

$$= \frac{1}{s^2+1} + \left(-\frac{d}{ds}\frac{s}{s^2+1}\right)$$

$$= \frac{1}{s^2+1} - \frac{(s^2+1)\,1 - s\times 2s}{(s^2+1)^2}$$

$$= \frac{1}{(s^2+1)} - \frac{s^2+1-2s^2}{(s^2+1)^2}$$

$$= \frac{s^2+1-(s^2+1-2s^2)}{(s^2+1)^2}$$

$$= \frac{2s^2}{(s^2+1)^2}$$

Step 3 : By definition of Laplace transform, we have

$$\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$$

Step 4 :

$$\int_0^\infty e^{-st}(\sin t + t\cos t)\,dt = \frac{2s^2}{(s^2+1)^2}$$

Step 5 : To get the required answer put s = – 1

$$\therefore \quad \int_0^\infty e^t (\sin t + t \cos t)\, dt = \frac{2}{(1+1)^2}$$

$$= \frac{1}{2}$$

Example 6.86 : *Evaluate* $\int_0^\infty e^t \cos t \cos 2t\, dt$

Solution : Step 1 : Let

$$f(t) = \cos t \cos 2t$$

Step 2 :

$$\therefore \quad L(\cos t \cos 2t) = \frac{1}{2} L(\cos t + \cos 3t)$$

$$= \frac{1}{2}\left\{\frac{s}{s^2+1} + \frac{s}{s^2+9}\right\}$$

Step 3 : By definition of Laplace transfom, we have

$$\int_0^\infty e^{-st} f(t)\, dt = L[f(t)]$$

Step 4 :

$$\therefore \int_0^\infty e^{-st} \cos t \cos 2t\, dt = \frac{1}{2}\left\{\frac{s}{s^2+1} + \frac{s}{s^2+9}\right\}$$

Step 5 : To get the required answer put s = – 1

$$\therefore \int_0^\infty e^t \cos t \cos 2t\, dt = \frac{1}{2}\left\{\frac{-1}{1+1} - \frac{1}{1+9}\right\}$$

$$= \frac{1}{2}\left\{\frac{-1}{2} - \frac{1}{10}\right\} = \frac{-3}{10}$$

Example 6.87 : *Evaluate* $\int_0^\infty e^{-2t}\, t \cos t\, dt = \frac{3}{25}$

Solution : Step 1 :

Let $\quad f(t) = t \cos t$

Step 2 :

$$L\, t \cos t = \frac{-d}{ds}\frac{s}{s^2+1}$$

$$= \frac{\left[(s^2+1)\,1 - s(2s)\right]}{(s^2+1)^2}$$

$$= \frac{-[s^2+1-2s^2]}{(s^2+1)^2}$$

$$= \frac{2s^2-s^2-1}{(s^2+1)^2}$$

$$= \frac{s^2-1}{(s^2+1)^2}$$

Step 3 : By definition of Laplace transform, we have

$$\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$$

Step 4 :

$$\therefore \quad \int_0^\infty e^{-st}\, t\cos t\,dt = \frac{s^2-1}{(s^2+1)^2}$$

Step 5 : To get the required answer put s = 2

$$\therefore \quad \int_0^\infty e^{-2t}\, t\cos dt = \frac{4-1}{(4+1)^2}$$

$$= \frac{3}{25}$$

Example 6.88 : *Given that* $L J_0(t) = \frac{1}{\sqrt{1+s^2}}$, *show that*

$$\int_0^\infty t e^{-3t} J_0(4t)\,dt = \frac{3}{125}$$

Solution : By change of scale property if L f(t) = $\phi(s)$

Then $\quad L f(at) = \frac{1}{a}\,\phi\left(\frac{s}{a}\right)$

$$\therefore \quad L J_0(4t) = \frac{1}{4}\,\phi\left(\frac{s}{4}\right) = \frac{1}{4}\,\frac{1}{\sqrt{1+\left(\frac{s}{4}\right)^2}} = \frac{1}{\sqrt{s^2+16}}$$

Now $\quad L\, t J_0\, d(4t) = -\frac{d}{ds}\,\frac{1}{\sqrt{s^2+16}} = \frac{s}{(s^2+16)^{3/2}}$

Step 1 : By definition of Laplace transform, we have

$$\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$$

By definition of Laplace transform,

$$L\, t J_0(4t) = \int_0^\infty e^{-st}\, t J_0(4t)\, dt = \frac{s}{(s^2+16)^{3/2}}$$

put s = 3

then $\int_0^\infty t e^{-3t} J_0(4t)\, dt = \frac{3}{(9+16)^{3/2}} = \frac{3}{125}$

Exercise 6.10

Evaluate the following integrals using Laplace transforms.

1. $\int_0^\infty t e^{-3t} \sin t\, dt = \frac{3}{50}$

2. $\int_0^\infty t^2 e^{-t} \sin t\, dt = \frac{1}{2}$

3. $\int_0^\infty t^3 e^{-t} \sin t\, dt = 0$

4. $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt = \log \frac{b}{a}$

5. $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\, dt = \log \frac{2}{3}$

6. $\int_0^\infty e^{-2t} \frac{\sin ht}{t}\, dt = \frac{1}{2} \log 3$

7. $\int_0^\infty e^{-t} \frac{\sin t}{t}\, dt = 7/4$

8. $\int_0^\infty t^2 e^{-3t} \sinh 2t\, dt = \frac{124}{125}$

9. $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\, dt = \log 3$

10. $\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t}\, dt = \log 2$

11. $\int_0^\infty e^{3t} \frac{\sin ht}{t}\, dt = -\log \sqrt{2}$

12. $\int_0^\infty \frac{1 - \cos t}{t^2}\, dt = 7/2$

13. $\int_0^{\infty} \frac{\sin^2 t}{t^2} dt = 7/2$

14. $\int_0^{\infty} e^{-2t} \frac{\sinh t \sin t}{t} dt = \frac{1}{2} \tan^{-1}\left(\frac{1}{2}\right)$

15. $\int_0^{\infty} e^{3t} \cos^3 t \, dt = \frac{4}{15}$

16. $\int_0^{\infty} \frac{\sin^3 t}{t} dt = 7/4$

17. $\int_0^{\infty} e^{-t} \frac{1-\cos t}{t} dt = \frac{1}{2} \log 2$

University Questions

Dec. - 98

1. *Evaluate any two :*

   i) $L\left\{\frac{1-\cos t}{t}\right\}$

   ii) $L\left\{t^2 U(t-4)\right\}$

   iii) $L\left\{\int_0^t t\, e^{-t} \sin 2t \, dt\right\}$

[6 Marks]

May - 99

1. *Find Laplace transform of the following functions (any two) :*

   i) $t \sin^3 t$

   ii) $e^{-4t} \int_0^t \frac{\sin 3t}{t} dt$

   iii) $\sin t \, U(t-4)$.

[6 Marks]

Dec. - 99

1. *Find Laplace transform of the following :*

   i) $t^3 e^{-3t}$

   ii) $\frac{\cos at - \cos bt}{t}$

   iii) $\int_0^t t\, e^{-3t} \sin 2t \, dt$.

[9 Marks]

May - 2000

1. *Find Laplace transform of any two of the following :*

   i) $\frac{d}{dt}\left(\frac{\sin t}{t}\right)$

ii) $\frac{e^{-at} - e^{-at}}{t}$

iii) $\frac{\cos\left(\sqrt{t}\right)}{\sqrt{t}}$

[6 Marks]

Dec. - 2000

1. *Find Laplace transform of :*

i) $\frac{d}{dt}\left(\frac{\sin t}{t}\right)$

ii) $\int_0^t \frac{e^{-4t}\sin 3t}{t}\,dt$

iii) $t\sin^3 t$

[9 Marks]

May - 2001

1. *Find Laplace transform of :*

i) $t \cdot e^{3t} \cdot \sin 2t$

ii) $\int_0^t \frac{\sin t}{t}\,dt$

iii) $\frac{\cos at - \cos bt}{t}$

[9 Marks]

Dec. - 2001

1. *Find the Laplace transform of :*

i) $e^{3t}\cosh^3 3t$ ii) $(t-1)^3\,U(t-1) + \sin 3t\,U(t-\pi)$

iii) $\int_0^t \frac{\cos at - \cos bt}{t}\,dt.$

[9 Marks]

May - 2002

1. *Find the Laplace transforms of any three of the following :*

i) $\frac{1-\cos t}{t}$

ii) $\frac{\cos\sqrt{t}}{\sqrt{t}}$

iii) $\left(t^2 - 4\right)\sin 2t$

iv) $t\int_0^t e^{-3t}\cos 4t\,dt$

[12 Marks]

Dec. - 2002

1. *Find the Laplace transform of any three of the following :*

i) $t\,e^{3t}\cos 2t$

ii) $\int_0^t \frac{1-e^{-4}}{4}\,du$

iii) $e^{-2t}\int_0^t t\sin 3t\,dt$

iv) $\sin\sqrt{t}$ **[12 Marks]**

May - 2003

1. *Find Laplace transform (any three) :*

i) $t\sin^3 t$

ii) $\int_0^t e^{2t}\frac{\sin t}{t}\,dt$

iii) $f(t)=\begin{cases}\frac{t}{T}, & 0<t<T\\ 1, & t>T \qquad (T \text{ is constant})\end{cases}$

iv) $\frac{\sin^2 t}{t^2}$ **[12 Marks]**

Dec. - 2003

1. *Find Laplace transform of (any three) :*

i) $4^t+\left(e^{-2t}+e^{3t}\right)^2$

ii) $e^{-3t}\sin^2 t$

iii) $\frac{\cos at-\cos bt}{t}$

iv) $\int_0^t \frac{1-e^{-x}}{x}\,dx$ **[12 Marks]**

May - 2004

1. *Find Laplace transforms of any two of the following :*

i) $t^2\sin^3 t$

ii) $t\int_0^t e^{-2t}\left\{\frac{\sin^2 t}{t}\right\}dt$

iii) $e^{2t}\,\mathrm{erf}\,\sqrt{t}$

iv) $F(t)=\sin^2 t\;U(t-\pi)+t^3\,\delta(t-2)$ **[10 Marks]**

Dec. - 2004

1. *Find Laplace Transform (any three) :*

i) $\frac{1-\cos 3t}{t}$

ii) $t\,e^{5t}\sin 3t$

iii) $\int_0^t u\sinh u\,du$

iv) $f(t) = \begin{matrix} t & 0 < t < 4 \\ 5 & t > 4 \end{matrix}$ **[12 Marks]**

May - 2005

1. *Find the Laplace transform of the following (any two) :*

i) $\int_0^t \frac{\sin t}{t} dt$

ii) $\frac{\cos 2t - \cos t}{t}$

iii) $\frac{\cos\sqrt{t}}{\sqrt{t}}$ **[6 Marks]**

Dec. - 2005

1. *Find Laplace transform of* $f(t) = t\sin^3 t$ *and evaluate*

$\int_0^\infty e^{-2t} t\sin^3 t\,dt$ **[5 Marks]**

May - 2006

1. *Find the Laplace transform :*

i) $e^{-2t}\frac{\sin t}{t}$

ii) $f(t) = \begin{cases} t^2, & 0 < t < 2\pi \\ \sin t, & t > 2\pi \end{cases}$ **[6 Marks]**

❑❑❑

7

Inverse Laplace Transforms

7.1 Introduction

Using Laplace transforms we transform a problem from time domain (t) into a simple algebraic problem in frequency domain (s). Hence to recover the solution in terms of time we have to use inverse laplace transforms and is denoted by L^{-1}.

i.e. If $L\, f(t) = \phi(s)$ then $L^{-1}\, \phi(s) = f(t)$

7.2 Properties and Theorems on Inverse Laplace Transforms

a) Linearity property :

$L^{-1}\left[C_1\phi_1(s) + C_2\phi_2(s)\right] = C_1\, L^{-1}\,\phi_1(s) + C_2\, L^{-1}\,\phi_2(s)$

Note : The property of Laplace transform is linear, therefore inverse Laplace is also linear i.e. inverse Laplace transforms are distributive over addition. Also this result can be extended for more functions also.

b) First shifting theorem :

If $L^{-1}\,\phi(s) = f(t)$ then $L^{-1}\,\phi(s-a) = e^{at}\, f(t)$

Proof : We know first shifting theorem of Laplace transforms

i.e. $\quad L\, e^{at}\, f(t) = \phi(s-a)$

Thus
$$L^{-1}\,\phi(s-a) = e^{at}\, f(t)$$
$$= e^{at}\, L^{-1}\,\phi(s)$$

Note : The replacement of s by s – a in $\phi(s)$ corresponds to multiplication of original function f (t) by e^{at}.

c) Second shifting theorem :

If $L^{-1}\,\phi(s) = f(t)$ then $L^{-1}\, e^{-as}\,\phi(s) = \begin{cases} f(t-a) & t \geq a \\ 0 & t < a \end{cases}$

d) Change of scale property :

If $L^{-1}\,\phi(s) = f(t)$ then $L^{-1}\,\phi(Ks) = \dfrac{1}{K}\, f\left(\dfrac{t}{K}\right)$

i.e. $L^{-1}\,\phi\left(\dfrac{s}{a}\right) = a\, f(at)$

e) Inverse transform of derivatives :

If $L^{-1}\,\phi(s) = f(t)$ then $L^{-1}\phi'(s) = -\,t\,f(t)$

i.e. $$L^{-1}\,\phi'(s) = -\,t\,L^{-1}\,\phi(s) \quad \text{... (1)}$$

i.e. $$\boxed{L^{-1}\,\phi(s) = -\frac{1}{t}\,L^{-1}\,\phi'(s)}$$

From (1) $$L^{-1}\,\phi''(s) = (-\,t)^2\,L^{-1}\,\phi(s)$$

and so on

$$L^{-1}\,\phi^n(s) = (-\,t)^n\,L^{-1}\,\phi(s)$$

f) Inverse Laplace transforms of integrals :

If $L^{-1}\,\phi(s) = f(t)$ then $L^{-1}\left[\int_s^{\infty}\phi(s)\,ds\right] = \frac{f(t)}{t}$ and so on.

g) Effect of multiplication by s :

If $L^{-1}\,\phi(s) = f(t)$ then $L^{-1}\,s\,\phi(s) = f'(t)$ if $f(0) = 0$

Proof : We know that $L\,f'(t) = -\,f(0) + s\,\phi(s)$

If $f(0) = 0$ then $L\,f'(t) = s\,\phi(s)$

i.e. $L^{-1}\,s\,\phi(s) = f'(t)$

h) Effect of division by s :

If $L^{-1}\,\phi(s) = f(t)$ then $L^{-1}\,\frac{\phi(s)}{s} = \int_0^t f(t)$

Proof : We know $L\int_0^t f(t)\,dt = \frac{1}{s}\,\phi(s)$ hence the proof.

i) **Convolution theorem :** If the function $\phi(s)$ can be expressed as a product of two functions $\phi_1(s)$ and $\phi_2(s)$ whose inverses $f_1(t)$ and $f_2(t)$ respectively are known then the inverse of the product $\phi(s) = \phi_1(s)\cdot\phi_2(s)$ can be calculated by using convolution theorem.

$$L^{-1}\,\phi_1(s)\cdot\phi_2(s) = \int_0^t f_1(u)\cdot f_2(t-u)\,du$$

$$= \int_0^t f_1(t-u)\cdot f_2(u)\,du$$

As convolution of $f_1(t)$ and $f_2(t)$ is commutative

$\therefore f_1(t)$ and $f_2(t)$ are interchangable.

Table of Inverse Laplace Transforms

	$\phi(s)$	$L^{-1}\phi(s) = f(t)$
1.	$\frac{1}{s-a}$	e^{at}
2.	$\frac{1}{s^{n+1}}$	$\frac{t^n}{\lfloor n+1} = \frac{t^n}{n!}$ (n = integer)
3.	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
4.	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
5.	$\frac{s}{s^2+a^2}$	$\cos at$
6.	$\frac{s}{s^2-a^2}$	$\cosh at$
7.	$\frac{1}{s}$	1
8.	1	$\delta(t)$
9.	$\phi(s-a)$	$e^{at} f(t) = e^{at}\ L^{-1}\phi(s)$
10.	$\phi'(s)$	$-t\,f(t)$
11.	$\int_s^\infty \phi(s)\,ds$	$\frac{f(t)}{t}$
12.	$\frac{1}{s}\phi(s)$	$\int_0^t f(t)\,dt$
13.	$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a}\sin at$
14.	$\frac{s^2}{(s^2+a^2)^2}$	$\frac{1}{2a}(\sin at + at\cos at)$
15.	$\frac{1}{(s^2+a^2)^2}$	$\frac{1}{2a^3}(\sin at - at\cos at)$
16.	$e^{-as}\phi(s)$	$f(t-a)\,U(t-a)$
17.	$f(a)\,e^{-as}$	$f(t)\,\delta(t-a)$

7.3 Procedure : Inverse Laplace Transform using Partial Fractions

1. The degree of numerator must be less than that of the denominator.

2. If factorisation of denominator is possible then factorise it. Use partial fractions and then L^{-1}.

3. If factorisation of denominator is not possible then adjust the perfect square in the denominator and adjust the same term in numerator. Then the function will be of the form $\phi(s-a)$. Use formula 9 and proceed.

7.4 Type I

Find Laplace inverse of following

Example 7.1 : $L^{-1}\left[\dfrac{3s-8}{s^2+4}-\dfrac{4s-24}{s^2-16}\right]$

Solution : Step 1 : We have to find

$$L^{-1}\left[3\left(\frac{s}{s^2+4}\right)-8\left(\frac{1}{s^2+4}\right)-4\left(\frac{s}{s^2-16}\right)+24\left(\frac{1}{s^2-16}\right)\right]$$

Step 2 : Simplify.

$$= 3\,L^{-1}\left(\frac{s}{s^2+4}\right)-8L^{-1}\left(\frac{1}{s^2+4}\right)-4L^{-1}\left(\frac{s}{s^2-16}\right)+24L^{-1}\left(\frac{1}{s^2-16}\right)$$

Step 3 : Use the formula $L^{-1}\dfrac{1}{s^2+a^2}=\dfrac{1}{a}\sin at$ $L^{-1}\dfrac{s}{s^2+a^2}=\cos at$, $L^{-1}\dfrac{1}{s^2-a^2}=\dfrac{1}{a}\sinh at$

Step 4 : Simplify

$$f(t) = 3\cos 2t-\frac{8}{2}\sin 2t-4\cosh 4t+\frac{24}{4}\sinh 4t$$

$$= 3\cos 2t-4\sin 2t-4\cosh 4t+6\sinh 4t$$

Example 7.2 : $L^{-1}\dfrac{2s+5}{s^2+4s+13}$

Solution :

Step 1 : Let $f(t) = L^{-1}\dfrac{2s+5}{s^2+4s+13}$

Step 2 : Adjusting the perfect square we get,

$$= L^{-1}\left[\frac{2(s+2)+1}{(s+2)^2+9}\right]$$

Step 3 : $L^{-1}\phi(s+2) = e^{-2t}L^{-1}\phi(s)$

Now from first shifting property we get

$$\therefore\ f(t) = e^{-2t}L^{-1}\frac{2s+1}{s^2+9} = e^{-2t}L^{-1}\left[\frac{2s}{s^2+9}+\frac{1}{s^2+9}\right]$$

Step 4 : Use the formula given by, $L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at,\ L^{-1}\frac{s}{s^2+a^2} = \cos at$

$$\therefore\qquad f(t) = e^{-2t}\left\{2\cos 3t+\frac{1}{3}\sin 3t\right\}$$

Example 7.3 : $L^{-1}\frac{s}{(s+7)^4}$

Solution :

Step 1 : Let $f(t) = L^{-1}\frac{s}{(s+7)^4}$

Step 2 : Adding and subtracting 7 from numerator to get

$$f(t) = L^{-1}\frac{s+7-7}{(s+7)^4}$$

Step 3 : $L^{-1}\phi(s+7) = e^{-7t}L^{-1}\phi(s)$

Using first shifting property we get

$$f(t) = e^{-7t}L^{-1}\frac{s-7}{s^4}$$

$$\therefore\qquad f(t) = e^{-7t}L^{-1}\left[\frac{1}{s^3}-\frac{7}{s^4}\right]$$

Step 4 : Simplify using the formula $L^{-1}\frac{1}{s^{n+1}} = \frac{t^n}{\Gamma(n+1)} = \frac{t^n}{n!}$ (n = integer)

$$f(t) = e^{-7t}\left(\frac{t^2}{2!}-\frac{7t^3}{3!}\right)$$

Example 7.4 : $L^{-1}\frac{3s+1}{(s+1)^4}$

Solution :

Step 1 : Let $f(t) = L^{-1}\frac{3s+1}{(s+1)^4}$

Step 2 : Adjusting the numerator to get s + 1 we get,

$$f(t) = L^{-1} \frac{3(s+1)-2}{(s+1)^4}$$

Step 3 : $L^{-1}\phi(s+1) = e^{-t}L^{-1}\phi(s)$

By first shifting property.

$$f(t) = e^{-t}L^{-1}\left(\frac{3}{s^3}-\frac{2}{s^4}\right)$$

Step 4 : Use the formula $L^{-1}\frac{1}{s^{n+1}} = \frac{t^n}{\lceil n+1} = \frac{t^n}{n!}$ (n = integer)

Step 5 :

$$\therefore \quad f(t) = e^{-t}\left(\frac{3}{2}t^2-\frac{1}{3}t^3\right)$$

Example 7.5 : $L^{-1}\left\{\frac{2s+3}{s^2+2s+17}\right\}$

Solution :

Step 1 : Let $f(t) = L^{-1}\left\{\frac{2s+3}{s^2+2s+17}\right\}$

Step 2 : Adjusting the perfect square

$$f(t) = L^{-1}\frac{2(s+1)+1}{(s+1)^2+16}$$

Step 3 : $L^{-1}\phi(s+1) = e^{-t}L^{-1}\phi(s)$

By first shifting property,

$$f(t) = e^{-t}L^{-1}\frac{2s+1}{s^2+16}$$

Step 4 : Use the formula given by, $L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$, $L^{-1}\frac{s}{s^2+a^2} = \cos at$

$$\therefore \quad f(t) = e^{-t}\left\{2\cos 4t+\frac{1}{4}\sin 4t\right\}$$

Example 7.6 : $\dfrac{1}{\sqrt{7s+6}}$

Solution :

Step 1 : Let $f(t) = \dfrac{1}{\sqrt{7s+6}}$

Step 2 : Adjusting the coefficient of s in denominator we get

$$f(t) = \frac{1}{\sqrt{7}} L^{-1}\left[\frac{1}{\sqrt{s+6/7}}\right] = \frac{1}{\sqrt{7}} L^{-1}\left[\frac{1}{(s+6/7)^{1/2}}\right]$$

Step 3 : $L^{-1}\phi(s+6/7) = e^{-\frac{6}{7}t} L^{-1}\phi(s)$

By first shifting property.

$$f(t) = \frac{1}{\sqrt{7}} e^{-\frac{6}{7}t} L^{-1}\left[\frac{1}{s^{1/2}}\right]$$

Step 4 : Simplify using the formula.

$$L^{-1}\frac{1}{s^{n+1}} = \frac{t^n}{\lceil n+1} = \frac{t^n}{n!} \text{ (n = integer)}$$

Step 5 :

$$\therefore \quad f(t) = \frac{1}{\sqrt{7}} e^{-\frac{6}{7}t} \frac{t^{1/2-1}}{\lceil 1/2} = \frac{e^{-\frac{6}{7}t}}{\sqrt{\pi}\sqrt{t}}$$

Example 7.7 : $\dfrac{s+1}{(s^2-2s+2)^2}$

Solution :

Step 1 : Let $f(t) = L^{-1}\dfrac{s+1}{(s^2-2s+2)^2}$

Step 2 : Adjust the perfect square in numerator as well as in denominator. Hence we get

$$f(t) = L^{-1}\frac{(s-1)+2}{[(s-1)^2+1]^2}$$

Step 3 : Use the formula $L^{-1}\phi(s-1) = e^t L^{-1}\phi(s)$

Step 4 : Simplify.

$$= e^t\left[L^{-1}\frac{s+2}{(s^2+1)^2}\right]$$

$$= e^t\left[L^{-1}\frac{s}{(s^2+1)^2} + 2L^{-1}\frac{1}{(s^2+1)^2}\right]$$

Step 5 : Use the formula given by

$$L^{-1} \frac{s}{(s^2+a^2)^2} = \frac{t}{2a} \sin at$$

$$L^{-1} \frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3} (\sin at - at \cos at)$$

$$\therefore \quad L^{-1} \frac{s+1}{(s^2-2s+2)^2} = e^t \left[\frac{t}{2} \sin t + 2\frac{1}{2} (\sin t - t \cos t)\right]$$

$$= e^t \left[\frac{t}{2} \sin t + \sin t - t \cos t\right]$$

Example 7.8 : $L^{-1} \dfrac{s^2-6s+7}{(s^2-4s+5)^2}$

Solution :

Step 1 : Let $f(t) = L^{-1} \dfrac{s^2-6s+7}{(s^2-4s+5)^2}$

Step 2 : Adjust the perfect square in numerator and denominator.

Thus we get

$$f(t) = L^{-1} \frac{(s-2)^2 - 2(s-2) - 1}{[(s-2)^2+1]^2} \quad \text{... (1)}$$

Step 3 : Use the formula $L^{-1} \phi(s-1) = e^t L^{-1} \phi(s)$

Step 4 : Simplify

$$= e^{2t} L^{-1} \frac{s^2-2s-1}{(s^2+1)}$$

$$= e^{2t} \left[L^{-1} \frac{s^2}{(s^2+1)^2} - 2L^{-1} \frac{s}{(s^2+1)^2} - L^{-1} \frac{1}{(s^2+1)^2}\right]$$

Step 5 : Use the formula

$$= e^{2t} \left[\frac{1}{2}(\sin t + t \cos t) - 2\frac{t}{2} \sin t - \frac{1}{2}[\sin t - t \cos t]\right]$$

$$= e^{2t} \left[\frac{\sin t}{2} + \frac{t \cos t}{2} - \frac{\sin t}{2} + \frac{t \cos t}{2} - t \sin t\right]$$

$$= e^{2t} [t \cos t - t \sin t]$$

$$= e^{2t} [t (\cos t - \sin t)]$$

$$= t\ e^{2t} (\cos t - \sin t)$$

Example 7.9 : $L^{-1}\dfrac{s+1}{\left(s^2+2s+2\right)^2}$

Solution :

Step 1 : Let $f(t) = L^{-1}\dfrac{s+1}{\left(s^2+2s+2\right)^2}$

Step 2 : Adjust the perfect square in numerator and denominator.

$$f(t) = L^{-1}\frac{s+1}{\left[(s+1)^2+1\right]^2}$$

Step 3 : Use the formula $L^{-1}\phi(s+1) = e^{-t}L^{-1}\phi(s)$

$$L^{-1}\phi(s+1) = e^{-t}\left[L^{-1}\frac{s}{\left(s^2+1\right)^2}\right]$$

Step 5 : Use the formula $L^{-1}\dfrac{s}{\left(s^2+a^2\right)^2} = \dfrac{t}{2a}\sin at$

$$\therefore \quad f(t) = e^{-t}\frac{t}{2}\sin t$$

$$= \frac{1}{2}t\,e^{-t}\sin t$$

Exercise 7.1

Find inverse Laplace tranforms of the following

Problems on Type 1

1. $\dfrac{2s+1}{s(s+1)}, \dfrac{1}{\sqrt{7s+6}}$ **[Ans. : $1+e^{-t}$, $e^{-6t/7}\dfrac{1}{\sqrt{7\pi t}}$]**

2. $\dfrac{s+1}{s^{4/3}}$ **[Ans. : $\dfrac{1}{\left|\overline{1/3}\right.}\left(t^{-2/3}+3\,t^{1/3}\right)$]**

3. $\dfrac{3\left(s^2-1\right)^2}{2s^5}$ **[Ans. : $\dfrac{3}{2}\left(1-t^2+\dfrac{t^4}{24}\right)$]**

4. $\dfrac{\left(1-\sqrt{s}\right)^2}{s^2}$ **[Ans. : $\dfrac{t^3}{6}+\dfrac{t^2}{2}-\dfrac{16}{15\sqrt{\pi}}t^{5/2}$]**

5. $\dfrac{1}{(s+4)^6}$ **[Ans. : $e^{-4t}\dfrac{t^5}{5!}$]**

6. $\dfrac{s}{s^2+6s+25}$ **[Ans. : $e^{-3t}\left(\cos 4t-\dfrac{3}{4}\sin 4t\right)$]**

7. $\dfrac{1}{s^2+2s+2}$ [Ans. : e^{-t} (cos t + 6 sin t)]

8. $\dfrac{6s-4}{s^2-4s+20}$ [Ans. : $2\,e^{2t}$ (3 cos 4t + sin 4t)]

9. $\dfrac{2s+5}{s^2-2s-3}$ [Ans. : $e^{t}\left(2\cosh 2t+\dfrac{7}{2}\sinh 2t\right)$]

10. $\dfrac{s-1}{s^2-6s+25}$ [Ans. : $e^{3t}\left(\cos 4t+\dfrac{1}{2}\sin 4t\right)$]

11. $\dfrac{s+7}{s^2+2s+2}$ [Ans. : e^{-t} (cos t + 6 sin t)]

12. $\dfrac{1}{\left(s^2+2s+5\right)^2}$ [Ans. : $\dfrac{e^{-t}}{16}$ (sin 2t − 2t cos 2t)]

13. $\dfrac{s^3+3s^2-s-3}{\left(s^2+2s+5\right)^2}$ [Ans. : e^{-t} (cos 2t − 2t sin t)]

14. *If* $L^{-1}\dfrac{e^{-\frac{1}{s}}}{s^{\frac{1}{2}}}=\dfrac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$ *find* $L^{-1}\left[\dfrac{e^{-a/s}}{s^{1/2}}\right]$, $a>0$.

Hint : Use change of scale $L^{-1}\phi\left(\dfrac{s}{a}\right)=a\cdot f(at)$ [Ans. : $\dfrac{1}{\sqrt{\pi t}}\cos\left(2\sqrt{at}\right)$]

7.4.1 Type II : Problems on Partial Fractions

Example 7.10 : $\dfrac{1}{(s+1)^2\left(s^2+1\right)}$

Solution : Step 1 :

Let
$$f(t) = L^{-1}\frac{1}{(s+1)^2\left(s^2+1\right)} \qquad \text{... (1)}$$

Step 2 : As we have two factors. Using the partial fraction,

$$\frac{1}{(s+1)^2\left(s^2+1\right)} = \frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{CS+D}{\left(s^2+1\right)} \qquad \text{... (2)}$$

$$\therefore \quad 1 = A\,(s+1)\left(s^2+1\right)+B\left(s^2+1\right)+(CS+D)\,(s+1)^2$$

here we find the value of A B C and D.

Putting s = − 1, we have, 1 = 2B $\Rightarrow$ B = $\dfrac{1}{2}$

Equating coefficients of s^3, s^2 and constant terms we have,

$A + C = 0; \quad A + B + 2C + D = 0; \quad A + B + D = 1$

Putting $B = \frac{1}{2}$, we have

$A + 2C + D = -\frac{1}{2}$ and $A + D = \frac{1}{2}$

These gives $C = -\frac{1}{2}; \quad A = \frac{1}{2}; \quad D = 0.$

Substituting the values of A, B, C, D in (2) we have,

$$\frac{1}{(s+1)^2(s^2+1)} = \frac{1/2}{s+1} + \frac{1/2}{(s+1)^2} - \frac{1}{2}\frac{s}{s^2+1}$$

$$\therefore L^{-1}\frac{1}{(s+1)^2(s^2+1)} = \frac{1}{2}L^{-1}\frac{1}{s+1} + \frac{1}{2}L^{-1}\frac{1}{(s+1)^2} - \frac{1}{2}L^{-1}\frac{s}{s^2+1}$$

Step 3 : Using first shifting property and the formula.

$$L^{-1}\frac{s}{s^2+a^2} = \cos at$$

Step 4 :

$$\therefore \quad f(t) = \frac{1}{2}\left[e^{-t} + e^{-t}L^{-1}\frac{1}{s^2} - \cos t\right]$$

$$f(t) = \frac{1}{2}\left[e^{-t} + e^{-t}(t) - \cos t\right]$$

Example 7.11 : $L^{-1}\frac{1}{s^2-3s+2}$

Solution :

Step 1 : Let $f(t) = L^{-1}\frac{1}{s^2-3s+2}$

Step 2 : Factorizing the denominator.

$$f(t) = L^{-1}\frac{1}{(s-1)(s-2)} = L^{-1}\left[\frac{1}{s-2} - \frac{1}{s-1}\right]$$

Step 3 : Using the formula $L^{-1}\frac{1}{s-a} = e^{at}$

Step 4 :

$$\therefore \quad f(t) = e^{2t} - e^{t}$$

Example 7.12 : $L^{-1}\dfrac{2s^3-2s^2-3}{(s-1)^3(s+2)}$

Solution :

Step 1 : Let $$f(t) = L^{-1}\frac{2s^3-2s^2-3}{(s-1)^3(s+2)}$$

Step 2 : By partial fractions,

$$\frac{2s^3-2s^2-3}{(s-1)^3(s+2)} = \frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{(s-1)^3}+\frac{D}{(s+2)} \quad \text{... (1)}$$

$$2s^3-2s^2-3 = A(s-1)^2(s+2)+B(s-1)(s+2)+C(s+2)+D(s-1)^3$$

We find the value of A B C and D.

Putting $s = 1, -2$, we have,

$2-2-3 = 3C \Rightarrow C = -1, \quad -16-8-3 = -27D \Rightarrow D = 1$

Equating coefficients of s^3 and s^2, we have,

$2 = A + D \Rightarrow A = 2-1 = 1.$

and $-2 = B - 3D \Rightarrow B = -2 + 3(1) = 1$

Step 3 : Substituting these in (1), we have,

$$\frac{2s^3-2s^2-3}{(s-1)^3(s+2)} = \frac{1}{s-1}+\frac{1}{(s-1)^2}-\frac{1}{(s-1)^3}+\frac{1}{(s+2)}$$

$$\therefore \quad L^{-1}\frac{2s^3-2s^2-3}{(s-1)^3(s+2)} = L^{-1}\frac{1}{s-1}+L^{-1}\frac{1}{(s-1)^2}-L^{-1}\frac{1}{(s-1)^3}+L^{-1}\frac{1}{(s+2)}$$

Step 4 : By first shifting property and using the formula.

$$L^{-1}\frac{1}{s^{n+1}} = \frac{t^n}{\lceil n+1} = \frac{t^n}{n!} \quad (n = \text{integer})$$

Step 5 :

$$\therefore \quad f(t) = e^t + e^t L^{-1}\frac{1}{s^2} - e^t L^{-1}\frac{1}{s^3} + e^{-2t}$$

$$= e^t + e^t(t) - e^t\left(\frac{t^2}{2}\right) + e^{-2t}$$

$$= e^t\left(1+t-\frac{t^2}{2}\right) + e^{-2t}$$

Example 7.13 : $L^{-1}\dfrac{2s^3+2s^2+4s+1}{(s^2+1)(s^2+s+1)}$

Solution :

Step 1 : Let $f(t) = L^{-1}\dfrac{2s^3+2s^2+4s+1}{(s^2+1)(s^2+s+1)}$

Step 2 : By partial fraction,

$$\frac{2s^3+2s^2+4s+1}{(s^2+1)(s^2+s+1)} = \frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+s+1} \qquad \dots (1)$$

$$(2s^3+2s^2+4s+1) = (As+B)(s^2+s+1)+(Cs+D)(s^2+1)$$

Here we find out the value of A B C and D.

Equating, coefficients of s^3, s^2, s and constant terms,

$2 = A + C,\qquad 2 = A + B + D$

$4 = A + B + C,\qquad 1 = B + D$

Solving above equation we get A = 1, B = 2, C = 1, D = – 1.

Substituting in (1), and then taking Laplace inverse.

$$L^{-1}\frac{2s^3+2s^2+4s+1}{(s^2+1)(s^2+s+1)} = L^{-1}\frac{s+2}{s^2+1}+L^{-1}\frac{s-1}{s^2+s+1}$$

Step 3 : Adjusting the perfect square in second term

$$= L^{-1}\frac{s}{s^2+1}+2L^{-1}\frac{1}{1+s^2}+L^{-1}\frac{\left(s+\frac{1}{2}\right)-\frac{3}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

Step 4 : Use the formulae given by, $L^{-1}\dfrac{1}{s^2+a^2} = \dfrac{1}{a}\sin at$

$L^{-1}\dfrac{s}{s^2+a^2} = \cos at$ and $L^{-1}\,\phi\left(s+\dfrac{1}{2}\right) = e^{-t/2}\,L^{-1}\,\phi(s)$

Step 5 : $\therefore\quad f(t) = \cos t+2\sin t+e^{-\frac{1}{2}t}L^{-1}\dfrac{s-\frac{3}{2}}{s^2+\frac{3}{4}}$

$$= \cos t+2\sin t+e^{-\frac{1}{2}t}\left[L^{-1}\frac{s}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}-\frac{3}{2}L^{-1}\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= \cos t + 2\sin t + e^{-\frac{1}{2}t}\left[\cos\frac{\sqrt{3}}{2}t - \frac{3}{2}\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]$$

$$= \cos t + 2\sin t + e^{-\frac{1}{2}t}\left[\cos\frac{\sqrt{3}}{2}t - \sqrt{3}\sin\frac{\sqrt{3}}{2}t\right]$$

Example 7.14 : $\dfrac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}$

Solution :

Step 1 : Let $f(t) = L^{-1}\dfrac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}$

Step 2 : Adjusting the perfect square in numerator and denominator we get

$$\therefore \quad f(t) = L^{-1}\frac{(s+1)^2-5}{[(s+1)^2+4]\,[(s+1)^2+1]}$$

$$= e^{-t}L^{-1}\left[\frac{s^2-5}{(s^2+4)(s^2+1)}\right] \quad \text{... (1)}$$

Step 3 : Now consider, put $s^2=\lambda$ for convenience.

$$\frac{s^2-5}{(s^2+4)(s^2+1)} = \frac{\lambda-5}{(\lambda+4)(\lambda+1)} \quad \text{... (2)}$$

$$= \frac{3}{\lambda+4}-\frac{2}{\lambda+1} \quad \text{(Using partial fractions)}$$

$$= \frac{3}{s^2+4}-\frac{2}{s^2+1}$$

Step 4 : Substituting in (1) we get

$$\therefore \quad f(t) = e^{-t}\left[L^{-1}\frac{3}{s^2+2^2}-L^{-1}\frac{2}{s^2+1}\right]$$

Step 5 : Use the formula given by, $L^{-1}\dfrac{1}{s^2+a^2}=\dfrac{1}{a}\sin at$; $L^{-1}\dfrac{s}{s^2+a^2}=\cos at$ and simplify.

$$= e^{-t}\left[3\frac{\sin 2t}{2}-2\sin t\right]$$

Example 7.15 : $L^{-1}\left(\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right)$

Solution : Step 1 : We can write

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1}+\frac{Bs+C}{s^2+2s+5}$$

Step 2 : Here substituting s = 1 we get A = 1

$$\therefore \quad \text{Let} \quad \phi(s) = \frac{1}{s-1}+\frac{Bs+C}{s^2+2s+5}$$

We find the value of A B and C.

$$5s+3 = (s^2+2s+5)+(s-1)(Bs+C) \qquad \text{... (1)}$$

Now put s = 0 in (1)

$3 = 5 + (-1)(C)$

$\Rightarrow \quad C = 2$

Now put s = – 1 in (1)

$-5+3 = (1-2+5)+(-2)(-B+2)$

$-2 = 4+(-2)(-B+2)$

$-6 = 2B-4$

$\Rightarrow \; B = -1$

$$\therefore \quad \phi(s) = \frac{1}{s-1}+\frac{-s+2}{s^2+2s+5}$$

Step 3 : Now adjust the perfect square in numerator and denominator.

$$= \frac{1}{s-1}-\frac{s-2}{(s+1)^2+4}$$

$$= \frac{1}{s-1}-\frac{(s+1)-3}{(s+1)^2+4}$$

Step 4 : Use the formula $L^{-1}\,\phi(s+1) = e^{-t}L^{-1}\phi(s)$

$$\therefore \quad L^{-1}\,\phi(s) = e^{t}-e^{-t}L^{-1}\left(\frac{s-3}{s^2+4}\right)$$

Step 5 : Use the formula given by

$$L^{-1}\frac{s}{s^2+a^2} = \cos at \; L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$$

$$\therefore \quad L\,[f(t)] = e^{t}-e^{-t}\left[\cos 2t-\frac{3}{2}\sin 2t\right]$$

Example 7.16 : $L^{-1} \dfrac{s^2-3}{(s+2)(s-3)(s^2-2s+5)}$

Solution : Step 1 : We can write

$$\frac{s^2-3}{(s+2)(s-3)(s^2-2s+5)} = \frac{A}{s+2}+\frac{B}{s-3}+\frac{Cs+D}{s^2-2s+5}$$

Step 2 :

Let $$\phi(s) = \frac{A}{s+2}+\frac{B}{s-3}+\frac{Cs+D}{s^2-2s+5}$$

We find the values of A, B, C and D.

To find A put $s=-2$ $\therefore$ $A = \dfrac{-1}{65}$, To find B put $s=3$ $\therefore$ $B = \dfrac{3}{20}$

$$\therefore\ s^2-3 = \frac{-1}{65}(s-3)(s^2-2s+5)+\frac{3}{20}(s+2)(s^2-2s+5)+(Cs+D)(s+2)(s-3) \qquad \text{... (1)}$$

Put $s=0$ in (1)

$$-3 = -\frac{1}{65}\times(-3)\times(5)+\frac{3}{20}\times 2\times 5+D(2)(-3)$$

$$-3 = \frac{3}{13}+\frac{3}{2}-6D = \frac{6+39}{26}-6D$$

$$6D = \frac{45}{26}+\frac{73}{26} = \frac{123}{26} \qquad \therefore\ D = \frac{123}{26\times 6} = \frac{41}{52}$$

$$D = \frac{41}{52}$$

Put $s=1$ in (1)

$$-2 = \frac{-1}{65}(-2)(4)+\frac{3}{20}(3)(4)+(C+D)(3)(-2)$$

$$C = \frac{-7}{52}$$

Substituting A, B, C, D we get,

$$\phi(s) = \frac{\frac{-1}{65}}{s+2}+\frac{\frac{3}{20}}{s-3}+\frac{\frac{-7}{52}(s)+\frac{41}{52}}{s^2-2s+5}$$

$$= \frac{-1/65}{s+2}+\frac{3/20}{s-3}-\frac{7}{52}\,\frac{s}{s^2-2s+5}\ \frac{+41}{52}\,\frac{1}{s^2-2s+5}$$

Step 3 : Adjust the perfect square.

Step 4 : Thus

$$L^{-1}\phi(s) = \frac{3}{20}e^{3t}-\frac{1}{65}e^{-2t}-\frac{7}{52}L^{-1}\frac{s-1+1}{(s-1)^2+4}+\frac{41}{52}L^{-1}\frac{1}{(s-1)^2+4}$$

Step 5 : Use the formula $L^{-1}\,\phi(s-1) = e^t\, L^{-1}\,\phi(s)$

$$L^{-1}\,\phi(s) = \frac{3}{20}e^{3t} - \frac{1}{65}e^{-2t} - \frac{7}{52}e^t\, L^{-1}\frac{s+1}{(s^2+4)} + \frac{41}{52}e^t\, L^{-1}\frac{1}{(s^2+4)}$$

$$= \frac{3}{20}e^{3t} - \frac{1}{65}e^{-2t} - \frac{7}{52}e^t\, L^{-1}\frac{s}{s^2+4} + \frac{34}{52}e^t\, L^{-1}\frac{1}{s^2+4}$$

$$= \frac{3}{20}e^{3t} - \frac{1}{65}e^{-2t} - \frac{7}{52}e^t\cos 2t + \frac{17}{52}e^t\sin 2t$$

Example 7.17 : $\dfrac{s^2-2s+3}{(s-1)^2(s+1)}$

Solution : Step 1 : We can write

$$\frac{s^2-2s+3}{(s-1)^2(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{(s-1)^2}$$

Step 2 : Using partial fractions we get

Let $$\phi(s) = \frac{A}{s+1} + \frac{Bs+C}{(s-1)^2}$$

We find the value of A, B and C.

$s^2-2s+3 = A\,(s-1)^2 + (s+1)\,(Bs+C)$

Solving above for A, B, C we get

$A = \frac{3}{2},\ B = \frac{-1}{2}\ ;\ C = \frac{3}{2}$

$$\therefore \quad \phi(s) = \frac{3/2}{s+1} + \frac{-s/2+3/2}{(s-1)^2}$$

$$\phi(s) = \frac{3/2}{s+1} - \frac{1}{2}\frac{(s-3)}{(s-1)^2}$$

$$= \frac{3}{2\,(s+1)} - \frac{1}{2}\left[\frac{s-1-2}{(s-1)^2}\right]$$

Step 3 : Use the formula $L^{-1}\phi\,(s-1) = e^t L^{-1}\phi\,(s)$

$$\therefore \quad L^{-1}\,\phi(s) = \frac{-3}{2}e^{-t} - \frac{1}{2}e^t L^{-1}\left[\frac{s-2}{s^2}\right]$$

$$f(t) = \frac{-3}{2}e^{-t} - \frac{1}{2}e^t L^{-1}\left[\frac{1}{s} - \frac{2}{s^2}\right]$$

Step 4 : Use the formula given by

$$L^{-1}\frac{1}{s^{n+1}} = \frac{t^n}{\lceil n+1} = \frac{t^n}{n!} \quad (n = \text{integer})$$

Step 5 :

$$\therefore \quad f(t) = \frac{-3}{2}e^{-t} - \frac{1}{2}e^{t}[1-2t]$$

Note : In problem nos. 18 to 23 we use short cut for finding partial fractions.

Example 7.18 : $L^{-1}\dfrac{s}{s^4+s^2+1}$

Solution :

Step 1 : Let $\quad f(t) = L^{-1}\dfrac{s}{s^4+s^2+1}$

Step 2 : We have,

$$s^4+s^2+1 = s^4+2s^2+1-s^2$$
$$= (s^2+1)^2 - s^2$$
$$= (s^2+1-s)(s^2+1+s)$$
$$\therefore \quad f(t) = L^{-1}\frac{s}{(s^2-s+1)(s^2+s+1)}$$
$$= L^{-1}\frac{1}{2}\left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right] \quad \leftarrow \text{note this step.}$$

Step 3 : Adjusting the perfect square,

$$f(t) = \frac{1}{2}L^{-1}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

Step 4 : Use the formula $L^{-1}\phi(s-a) = e^{at}L^{-1}\phi(s)$

$$f(t) = \frac{1}{2}\left\{e^{t/2}\left(L^{-1}\frac{1}{s^2+\frac{3}{4}}\right) - e^{-t/2}L^{-1}\left(\frac{1}{s^2+\frac{3}{4}}\right)\right\}$$

Step 5 : Again use the formula given by $L^{-1}\dfrac{1}{s^2+a^2} = \dfrac{1}{a}\sin at$

$$\therefore \quad f(t) = \left(\frac{e^{t/2}-e^{t/2}}{2}\right)\frac{1}{\sqrt{\frac{3}{4}}}\sin\frac{\sqrt{3}t}{2}$$
$$= \frac{2}{\sqrt{3}}\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t$$

Example 7.19 : $L^{-1} \dfrac{s}{s^4+4a^4}$

Solution :

Step 1 : Let $\quad f(t) = L^{-1} \dfrac{s}{s^4+4a^4}$

Step 2 : We have,

$$s^4+4a^4 = s^4+4a^2s^2+4a^4-4a^2s^2$$

$$= \left(s^2+2a^2\right)^2-(2as)^2$$

$$= \left(s^2+2a^2-2as\right)\left(s^2+2a^2+2as\right)$$

$$\therefore \quad f(t) = L^{-1} \frac{s}{\left(s^2-2as+2a^2\right)\left(s^2+2as+2a^2\right)}$$

$$= L^{-1} \frac{1}{4a}\left[\frac{1}{s^2-2as+2a^2}-\frac{1}{s^2+2as+2a^2}\right] \leftarrow \text{note this step.}$$

Step 3 : Adjusting the perfect square,

$$= \frac{1}{4a} L^{-1}\left\{\frac{1}{(s-a)^2+a^2}-\frac{1}{(s+a)^2+a^2}\right\}$$

Step 4 : Use the formula $L^{-1}\phi(s-1) = e^t L^{-1}\phi(s)$

$$= \frac{1}{4a}\left\{e^{at}\left(L^{-1}\frac{1}{s^2+a^2}\right)-e^{-at}L^{-1}\left(\frac{1}{s^2+a^2}\right)\right\}$$

Step 5 : Again use the formula given by

$$L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at,$$

Step 6 :

$$\therefore \quad f(t) = \left(\frac{e^{at}-e^{-at}}{4a}\right)\cdot\frac{1}{a}\sin at$$

$$= \frac{1}{2a^2}\sin at \sinh at$$

Example 7.20 : $L^{-1} \dfrac{s^3}{s^4-a^4}$

Solution :

Step 1 : Let $\quad f(t) = L^{-1} \dfrac{s^3}{s^4-a^4}$

$$= L^{-1} \frac{s \cdot s^2}{(s^2 - a^2)(s^2 + a^2)}$$

Step 2 : Adjusting the terms we get

$$f(t) = L^{-1} \frac{s}{2}\left[\frac{1}{s^2 - a^2} + \frac{1}{s^2 + a^2}\right]$$

$$= L^{-1} \frac{1}{2}\left[\frac{s}{s^2 - a^2} + \frac{s}{s^2 + a^2}\right]$$

Step 4 : Use the formula $L^{-1} \frac{s}{s^2 - a^2} = \cosh at$ $L^{-1} \frac{s}{s^2 + a^2} = \cos at$

$$\therefore \quad f(t) = \frac{1}{2}(\cosh at + \cos at)$$

Example 7.21 : $L^{-1} \frac{s}{(s^2+1)(s^2+2)}$

Solution :

Step 1 : Let $f(t) = L^{-1} \frac{s}{(s^2+1)(s^2+2)}$

Step 2 : Adjusting the terms we get

$$= L^{-1} s\left[\frac{1}{s^2+1} - \frac{1}{s^2+2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+1} - \frac{s}{s^2+2}\right]$$

Step 3 : Using the formula $L^{-1} \frac{s}{s^2 + a^2} = \cos at$

$$\therefore \quad f(t) = \cos t - \cos\sqrt{2}\, t$$

Example 7.22 : $L^{-1} \frac{s}{(s^2+1)(s^2+2)(s^2+3)}$

Solution :

Step 1 : Let $f(t) = L^{-1} \frac{s}{(s^2+1)(s^2+2)(s^2+3)}$

Step 2 :

Let consider $\left[\frac{1}{(s^2+1)(s^2+2)(s^2+3)}\right]$

Put $s^2 = u$ for finding partial fractions.

$$= \frac{A}{u+1} + \frac{B}{u+2} + \frac{C}{u+3}$$

Step 3 : We find the values of A, B and C using partial fractions. We get A = 1/2 , B = 1, C = 1/2

$$\therefore \quad \phi(s) = s\left[\frac{\frac{1}{2}}{s^2+1} + \frac{-1}{s^2+2} + \frac{\frac{1}{2}}{s^2+3}\right]$$

$$= L^{-1}\left\{\frac{1}{2}\frac{s}{s^2+1} - \frac{s}{s^2+2} + \frac{1}{2}\frac{s}{s^2+3}\right\}$$

Step 4 : Using the formula $L^{-1}\dfrac{s}{s^2+a^2} = \cos at$

$$\therefore \quad f(t) = \frac{1}{2}\cos t - \cos\sqrt{2}\,t + \frac{1}{2}\cos\sqrt{3}\,t$$

Example 7.23 : $L^{-1}\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$

Solution :

Step 1 : Let $\quad f(t) = L^{-1}\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$

Step 2 : Put $s^2+2s = u$

and Let $\quad \phi(s) = \dfrac{u+3}{(u+2)(u+5)}$

$$= \frac{A}{u+2} + \frac{B}{u+5}$$

We find the value of A and B, using partial fractions we get A = 1/3, B = 2/3.

$$= \frac{\frac{1}{3}}{u+2} + \frac{\frac{2}{3}}{u+5}$$

$$= \frac{1}{3}\left[\frac{1}{s^2+2s+2} + \frac{2}{s^2+2s+5}\right]$$

Step 3 : Adjusting the perfect square we get,

$$= \frac{1}{3}\left[\frac{1}{(s+1)^2+1} + \frac{2}{(s+1)^2+4}\right]$$

Step 4 : Using the formula $L^{-1}\phi(s+1) = e^{-t}L^{-1}\phi(s)$

Step 5 :

$$\therefore \qquad \phi(s) = \frac{e^{-t}}{3}\left[L^{-1}\frac{1}{s^2+1}+L^{-1}\frac{2}{s^2+4}\right]$$

Step 6 : Again use the formula given by $L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$

$$\therefore \qquad f(t) = \frac{e^{-t}}{3}(\sin t + \sin 2t)$$

Example 7.24 : $L^{-1}\frac{1}{s^4-a^4}$

Solution : Step 1 : use $(a^2-b^2) = (a-b)(a+b)$

$$= L^{-1}\frac{1}{(s^2-a^2)(s^2+a^2)}$$

Step 2 : Adjustment

$$= L^{-1}\frac{1}{2a^2}\left[\frac{1}{s^2-a^2}-\frac{1}{s^2+a^2}\right]$$

Step 3 : Use inverse Laplace formula.

$$= \frac{1}{2a^2}\left[\frac{1}{a}\sinh at - \frac{1}{a}\sin at\right]$$

$$= \frac{1}{2a^3}[\sinh at - \sin at]$$

Exercise 7.2

Find Laplace inverse of the following using partial fractions.

1. $\frac{11s^2-2s+5}{(s-2)(2s-1)(s+1)}$ **[Ans. : $5e^{2t} - \frac{3}{2}e^{t/2} + 2e^{-t}$]**

2. $\frac{3s+1}{(s-1)(s^2+1)}$ **[Ans. : $2e^{t} - 2\cos t + \sin t$]**

3. $\frac{2s^2-1}{(s^2+1)(s^2+4)}$ **[Ans. : $-\sin t + \frac{3}{2}\sin 2t$]**

4. $\frac{1}{s^3+a^3}$

Hint : $\frac{1}{s^3+a^3} = \frac{1}{(s+a)(s^2-as+a^2)} = \frac{A}{s+a}+\frac{Bs+C}{s^2-as+a^2}$

$A = \frac{1}{3a^2} \quad B = \frac{-1}{3a^2} \quad C = \frac{2}{3a}$ **[Ans. : $\frac{e^{-at}}{3a^2}-\frac{e^{at/2}}{3a^2}\left(\cos\frac{\sqrt{3}\,a}{2}t - \sin\frac{\sqrt{3}\,a}{2}t\right)$]**

5. $\dfrac{1}{(s+2)(s^2+2s+2)}$

Hint : $A=\dfrac{1}{2}$ $B=\dfrac{-1}{2}$ $C=0$ **[Ans. : $\dfrac{1}{2}e^{-2t}-\dfrac{e^{-t}}{2}(\cos t-\sin t)$]**

6. $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$

Hint : $A=\dfrac{1}{2}$ $B=-1,\ C=\dfrac{5}{2}$ **[Ans. : $\dfrac{1}{2}e^{t}-e^{2t}+\dfrac{5}{2}e^{3t}$]**

7. $\dfrac{3s^3+s^2+12s+2}{(s-3)(s+1)^3}$

Hint : $\dfrac{A}{s-3}+\dfrac{B}{s+1}+\dfrac{C}{(s+1)^2}+\dfrac{D}{(s+1)^3}$

$A=2,\ B=1,\ C=-4,\ D=3$ **[Ans. : $2e^{3t}+e^{-t}-4te^{-t}+\dfrac{3}{2}t^2e^{-t}$]**

8. $\dfrac{s^3+16s-24}{s^4+20s^2+64}$ **[Ans. : $\dfrac{1}{2}\sin 4t+\cos 2t-\sin 2t$]**

9. $\dfrac{1}{s^3-a^3}$ **[Ans. : $\dfrac{e^{at}}{3a^2}-\dfrac{e^{-at/2}}{3a^2}\left[\cos\dfrac{\sqrt{3}\,at}{2}+\sqrt{3}\,\dfrac{\sin\sqrt{3}\,at}{2}\right]$]**

10. $\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}$ **[Ans. : $\dfrac{e^{-t}}{6}-\dfrac{4}{3}e^{2t}+\dfrac{7}{2}e^{3t}$]**

11. $\dfrac{2s+1}{(s+2)^2(s-1)^2}$ **[Ans. : $\dfrac{1}{3}t\left(e^{t}-e^{-2t}\right)$]**

12. $\dfrac{1}{(s-2)(s+2)^2}$ **[Ans. : $\dfrac{1}{16}\left(e^{2t}-e^{-2t}-4t\,e^{-2t}\right)$]**

13. $\dfrac{s-2}{s(s+1)^3}$ **[Ans. : $e^{-t}\left(2+2t+\dfrac{3}{2}t^2\right)-2$]**

14. $\dfrac{s^2-2s+3}{(s-1)^2(s+1)}$ **[Ans. : $\left(t-\dfrac{1}{2}\right)e^{t}+\dfrac{3}{2}e^{-t}$]**

7.5 Type III

Laplace inverse using $L^{-1}\dfrac{1}{s}\phi(s)=\int_0^t f(t)\,dt$

Example 7.25 : $L^{-1}\dfrac{1}{s^3(s^2+1)(s^2+4)}$

Solution : **Step 1 :** Let $f(t)=L^{-1}\dfrac{1}{s^3(s^2+1)(s^2+4)}$

$$=L^{-1}\frac{1}{s}\left(\frac{1}{s^2(s^2+1)(s^2+4)}\right)$$

Step 2 :

Let $$\phi(s) = \frac{1}{s^2(s^2+1)(s^2+4)}$$

$$= \frac{A}{u} + \frac{B}{u+1} + \frac{C}{u+4}$$ where $s^2 = u$ for finding partial fractions.

Using partial fractions we get

$A = \frac{1}{4}$, $B = \frac{-1}{3}$, $C = \frac{1}{12}$

Step 3 : $$L^{-1}\phi(s) = \frac{1/4}{s^2} + \frac{-1/3}{s^2+1} + \frac{1/12}{s^2+4}$$

Step 4 : Using the formula given by $L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$

$$L^{-1}\phi(s) = \frac{1}{4}t - \frac{1}{3}\sin t + \frac{1}{24}\sin 2t$$

Step 5 : Now using $L^{-1}\frac{1}{s}\phi(s) = \int_0^t f(t)\,dt$ we get

Step 6 : $$\therefore L^{-1}\frac{1}{s}\phi(s) = \int_0^t \left(\frac{t}{4} - \frac{1}{3}\sin t + \frac{1}{24}\sin 2t\right)dt$$

$$\therefore \quad f(t) = \left[\frac{t^2}{8}\right]_0^t + \left[\frac{\cos t}{3}\right]_0^t - \left[\frac{\cos 2t}{48}\right]_0^t$$

$$\therefore \quad f(t) = \frac{t^2}{8} + \frac{1}{3}\cos t - \frac{1}{48}\cos 2t - \frac{1}{3} + \frac{1}{48}$$

Example 7.26 : $L^{-1}\dfrac{1}{s^3(s^2+1)}$

Solution :

Step 1 : Let $$f(t) = L^{-1}\frac{1}{s^3(s^2+1)}$$

Step 2 : Let $$\phi(s) = \frac{1}{s^2(s^2+1)}$$

$$= \left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$$

Step 3 : Use the formula given by $L^{-1} \frac{1}{s^2+a^2} = \frac{1}{a} \sin at$

$$L^{-1} \frac{1}{s^{n+1}} = \frac{t^n}{\lceil n+1} = \frac{t^n}{n!} \quad (n = \text{integer})$$

$$\therefore \quad L^{-1} \phi(s) = t - \sin t$$

Step 4 : Now using $\frac{1}{s} \phi(s) = \int_0^t f(t)\, dt$ we get

Step 5 :

$$L^{-1} \frac{1}{s} \phi(s) = \int_0^t (t - \sin t)\, dt$$

$$\therefore \quad f(t) = \left[\frac{t^2}{2}\right]_0^t + [\cos t]_0^t$$

$$= \frac{t^2}{2} + \cos t - 1$$

Example 7.27 : $L^{-1} \frac{1}{s(s+1)^3}$

Solution :

Step 1 : Let $f(t) = L^{-1} \frac{1}{s(s+1)^3}$

Step 2 : Let $\phi(s) = \frac{1}{(s+1)^3}$

Step 3 : $L^{-1} \phi(s-a) = e^{at} L^{-1} \phi(s)$

$$L^{-1} \phi(s) = e^{-t} L^{-1} \frac{1}{s^3}$$

$$L^{-1} \phi(s) = e^{-t} \frac{t^2}{2!}$$

Step 4 : Now using $\frac{1}{s} \phi(s) = \int_0^t f(t)\, dt$ we get

Step 5 : $\therefore$ $f(t) = L^{-1}\frac{1}{s}\phi(s) = \frac{1}{2}\int_0^t t^2 e^{-t}\,dt$

$$f(t) = \left[\frac{-t^2}{2!}e^{-t} - \frac{2\,te^{-t}}{2!} - e^{-t} + e^{-0}\right]_0^t$$

$$= \left[\frac{-t^2e^{-t}}{2!} - \frac{2\,t\,e^{-t}}{2!} - e^{-t} + e^{-0}\right]$$

$$= \left[\frac{-t^2e^{-t}}{2!} - t\,e^{-t} - e^{-t} + 1\right]$$

$\therefore$ $$f(t) = 1 - e^{t}\left[1 + t + \frac{t^2}{2}\right]$$

Example 7.28 : $\frac{1}{s(s+2)}$

Solution :

Step 1 : Let $f(t) = \frac{1}{s(s+2)}$

Step 2 : Let $\phi(s) = \frac{1}{s+2}$

$$L^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}$$

Step 3 : Now using $\frac{1}{s}\phi(s) = \int_0^t f(t)\,dt$ we get

Step 4 : $\therefore$ $L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s+2}\right] = f(t)$

$$= \int_0^t e^{-2t}\,dt$$

$$= \left[\frac{e^{-2t}}{-2}\right]_0^t$$

$\therefore$ $$f(t) = \frac{1-e^{-2t}}{2}$$

Example 7.29 : $\dfrac{1}{s(s^2+4)}$

Solution :

Step 1 : Let $f(t) = \dfrac{1}{s(s^2+4)}$

Step 2 : Using $L^{-1} \dfrac{1}{s^2+a^2} = \dfrac{1}{a} \sin at$

Step 3 : We have $L^{-1}\left[\dfrac{1}{s^2+4}\right] = \dfrac{\sin 2t}{2}$

Step 4 : Now using $\dfrac{1}{s}\phi(s) = \int_0^t f(t)\,dt$ we get

Step 5 : $\therefore \quad f(t) = L^{-1}\left[\dfrac{1}{s}\cdot\dfrac{1}{s^2+4}\right] = \int_0^t \dfrac{\sin 2t}{2}\,dt$

$$= \frac{1}{2}\left[\frac{-\cos 2t}{2}\right]_0^t = \frac{1-\cos 2t}{4}$$

$$= \frac{1}{2}\sin^2 t$$

Example 7.30 : $\dfrac{1}{s^2(s+1)}$

Solution :

Step 1 : Let $f(t) = \dfrac{1}{s^2(s+1)}$

Step 2 : Using $L^{-1} \dfrac{1}{s+a} = e^{-at}$

Step 3 : We have $L^{-1}\left[\dfrac{1}{s+1}\right] = e^{-t}$

Step 4 : Now using $L^{-1} \dfrac{1}{s}\phi(s) = \int_0^t f(t)\,dt$ we get $L^{-1} \dfrac{1}{s^2}\phi(s) = \int_0^t \int_0^t f(t)\,dt$

Step 5 : $\therefore$ $L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s+1}\right]=\int_0^t e^{-t}\,dt=\left[-e^{-t}\right]_0^t=1-e^{-t}$

$$\therefore \quad L^{-1}\frac{1}{s^2(s+1)} = \int_0^t \left(1-e^{-t}\right)dt$$

$$= \left[t+e^{-t}\right]_0^t = t+e^{-t}-1$$

$$\text{Hence } L^{-1}\left[\frac{1}{s^2(s+1)}\right] = e^{-t}+t-1$$

Example 7.31 : $\frac{s^2+2}{s\left(s^2+4\right)}$

Solution :

Step 1 : Let $f(t)=\frac{s^2+2}{s\left(s^2+4\right)}$

Step 2 : Adjusting the terms in the numerator and denominator we get

$$L^{-1}\left[\frac{s^2+2}{s\left(s^2+4\right)}\right] = L^{-1}\left[\frac{s^2+4-2}{s\left(s^2+4\right)}\right]=L^{-1}\left[\frac{1}{s}-\frac{2}{s\left(s^2+4\right)}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{2}{s\left(s^2+4\right)}\right]$$

Step 3 : Now using $L^{-1}\frac{1}{s}\phi(s)=\int_0^t f(t)\,dt$ we get

$$= 1-\int_0^t L^{-1}\left[\frac{2}{s^2+4}\right]dt$$

Step 4 : Using the formula $L^{-1}\frac{1}{s^2+a^2}=\frac{1}{a}\sin at$

Step 5 : $\therefore$ $f(t) = 1-\int_0^t \sin 2t\,dt=1-\left[-\frac{\cos 2t}{2}\right]_0^t$

$$= 1+\frac{\cos 2t}{2}-\frac{1}{2}=\frac{1+\cos 2t}{2}=\cos^2 t$$

$$\therefore \quad L^{-1}\left[\frac{s^2+2}{s\left(s^2+4\right)}\right] = \cos^2 t$$

Example 7.32 : $\dfrac{1}{(s-2)^4\,(s+3)}$

Solution :

Step 1 : $L^{-1}\dfrac{1}{(s+3-5)^4\,(s+3)}$

Step 2 : Use $L^{-1}\,\phi(s+3) = e^{-3t}\,L^{-1}\,\phi(s)$

$$= e^{-3t}\,L^{-1}\frac{1}{s\,(s-5)^4}$$

Step 3 : Use $L^{-1}\dfrac{1}{s}\,\phi(s) = \int_0^t f(t)\,dt$

$$= e^{-3t}\int_0^t L^{-1}\frac{1}{(s-5)^4}\,ds$$

$$= e^{-3t}\int_0^t e^{5t}\,\frac{t^3}{3!}\,dt$$

$$= \frac{e^{-3t}}{6}\left[t^3\,\frac{e^{5t}}{5} - 3t^2\,\frac{e^{5t}}{25} + 6t\,\frac{e^{5t}}{125} - 6\,\frac{e^{5t}}{625}\right]_0^t$$

$$= \frac{e^{-3t}}{6}\left\{e^{5t}\left(\frac{t^3}{5} - \frac{3t^2}{25} + \frac{6t}{125} - \frac{6}{625}\right) + \frac{6}{625}\right\}$$

Exercise 7.3

1. *Given that* $L^{-1}\dfrac{s}{\left(s^2+1\right)^2} = \dfrac{t}{2}\sin t$ *Find* $L^{-1}\dfrac{1}{\left(s^2+1\right)^2}$ **[Ans. : $\frac{1}{2}$ (sin t – t cos t)]**

2. *Find inverse Laplace transform of the following.*

   i) $\dfrac{1}{s^2+s}$ **[Ans. : $1 - e^{-t}$]**

   ii) $\dfrac{s-a}{s(s+a)}$ **[Ans. : $2\,e^{-at} - 1$]**

   iii) $\dfrac{1}{s^4-2s^3}$ **[Ans. : $\frac{1}{8}\left(e^{2t} - 1 - 2t - 2t^2\right)$]**

   iv) $\dfrac{1}{s^3(s+1)}$ **[Ans. : $\frac{t^2}{2} + 1 - t - e^{-t}$]**

   v) $\dfrac{1}{s^2(s-a)}$ **[Ans. : $\frac{1}{a^2}\left(e^{at} - at - 1\right)$]**

7.6 Type IV

> Laplace Inverse using $L^{-1} \int_s^{\infty} \phi(s)\, ds = \frac{f(t)}{t}$
>
> and $L^{-1}\, s\phi(s) = \frac{d}{dt} f(t)$ if $f(0) = 0$

Example 7.33 : *Find* $L^{-1} \dfrac{s}{(s^2+a^2)^2}$

Solution : Step 1 : We know that

$$L^{-1} \int_s^{\infty} \phi(s)\, ds = \frac{f(t)}{t}$$

Step 2 : Hence integrating w.r.t. s we have

$$L^{-1} \frac{1}{2}\int_s^{\infty} \frac{2s}{(s^2+a^2)^2}\, ds = \frac{f(t)}{t}$$

Step 3 : Using formula $\int f(x)^n f'(x)\, dx = \frac{f(x)^{n+1}}{n+1}$.

Step 4 : $\therefore$

$$L^{-1} \frac{1}{2}\left[\frac{-1}{s^2+a^2}\right]_s^{\infty} = \frac{f(t)}{t}$$

$$L^{-1} \frac{1}{2}\left[0+\frac{1}{s^2+a^2}\right] = \frac{f(t)}{t} \qquad \text{... (1)}$$

Step 5 : Now using formula $L^{-1} \frac{1}{s^2+a^2} = \frac{1}{a} \sin at$ equation becomes

$$\frac{1}{2}\frac{1}{a} \sin at = \frac{f(t)}{t}$$

$$\therefore \quad f(t) = \frac{t}{2a} \sin at$$

Example 7.34 : $L^{-1} \dfrac{2s+1}{(s^2+s+1)^2}$

Solution : Step 1 : We know that

$$L^{-1} \int_s^{\infty} \phi(s)\, ds = \frac{f(t)}{t}$$

Step 2 : Hence integrating w.r.t. s we get

$$L^{-1}\int_s^{\infty} \frac{2s+1}{\left(s^2+s+1\right)^2}\, ds = \frac{f(t)}{t}$$

Step 3 : Using formula $\int f(x)^n f'(x)\, dx \frac{f(x)^{n+1}}{n+1}$

$$L^{-1}\left[\frac{-1}{s^2+s+1}\right]_s^{\infty} = \frac{f(t)}{t}$$

$$L^{-1}\left[0+\frac{1}{s^2+s+1}\right] = \frac{f(t)}{t}$$

$$L^{-1}\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{f(t)}{t} \qquad \text{... (1)}$$

Step 4 : Now using first shifting property and formula
$L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}$ sin at equation (1) becomes

Step 5 : $\frac{2}{\sqrt{3}} e^{-t/2} \sin \frac{\sqrt{3}}{2} t = \frac{f(t)}{t}$

$$\therefore \qquad f(t) = e^{-t/2}\frac{2t}{\sqrt{3}} \sin\frac{\sqrt{3}\, t}{2}$$

Example 7.35 : $\dfrac{s+1}{\left(s^2+2s+2\right)^2}$

Solution : Step 1 : We know that

$$L^{-1}\int_s^{\infty} \phi(s)\, ds = \frac{f(t)}{t}$$

Step 2 : Hence integrating w.r.t. s we have

$$L^{-1}\frac{1}{2}\int_s^{\infty} \frac{2(s+1)}{\left(s^2+2s+2\right)^2}\, ds = \frac{f(t)}{t}$$

Step 3 : Using formula $\int f(x)^n f'(x)\, dx = \frac{f(x)^{n+1}}{n+1}$

Step 4 : $L^{-1}\frac{1}{2}\left[\frac{-1}{s^2+2s+2}\right]_s^{\infty} = \frac{f(t)}{t}$

$$L^{-1}\frac{1}{2}\left[\frac{1}{(s+1)^2+1}\right] = \frac{f(t)}{t}$$

$$e^{-t} L^{-1} \frac{1}{2}\left[\frac{1}{s^2+1}\right] = \frac{f(t)}{t}$$

Step 5 : Using formula $L^{-1} \dfrac{1}{s^2 + a^2} = \dfrac{1}{a}$ sin at equation (1) becomes

$$e^{-t} \frac{1}{2} \sin t = \frac{f(t)}{t}$$

$$\therefore \quad \frac{1}{2} t e^{-t} \sin t = f(t)$$

Note : This type is useful if the denominator involves $(\ )^2$ and numerator involves the derivative of the term present inside ( ).

Example 7.36 : $\dfrac{1}{\left(s^2 + a^2\right)^2}$

Solution : Step 1 : We know

$$L^{-1} \frac{1}{s^2 + a^2} = \frac{1}{a} \sin at$$

Step 2 : Use $L^{-1} \dfrac{d}{ds} \phi(s) = -t f(t)$

$$\therefore \quad L^{-1} \frac{d}{ds} \frac{1}{s^2 + a^2} = -t \cdot \frac{1}{a} \sin at$$

$$\therefore \quad L^{-1} \frac{-2s}{\left(s^2 + a^2\right)^2} = \frac{-t}{a} \sin at$$

Step 3 : Use $L^{-1} \dfrac{1}{s} \phi(s) = \int_0^t f(t)\, dt$

$$\text{i.e.} L^{-1} \frac{1}{s} \cdot \frac{2s}{\left(s^2 + a^2\right)^2} = \int_0^t \frac{t}{a} \sin at\, dt$$

$$L^{-1} \frac{2}{\left(s^2 + a^2\right)^2} = \frac{1}{a}\left[t\left(\frac{-\cos at}{a}\right) - (1)\left(\frac{-\sin at}{a^2}\right)\right]_0^t$$

$$= \frac{1}{a}\left[-\frac{t}{a} \cos at + \frac{\sin at}{a^2} - 0\right]$$

$$L^{-1} \frac{2}{\left(s^2 + a^2\right)^2} = \frac{1}{a^3} [\sin at - at \cos at]$$

$$\text{Thus } L^{-1} \frac{1}{\left(s^2 + a^2\right)^2} = \frac{1}{2a^3} [\sin at - at \cos at]$$

Example 7.37 : $\dfrac{s^2}{(s^2+a^2)^2}$

Solution : Step 1 : We know that

$$L^{-1}\frac{1}{s^2+a^2} = \frac{\sin at}{a}$$

Step 2 : Using $L^{-1}\dfrac{d}{ds}\phi(s) = -t\,f(t)$

Thus $$L^{-1}\frac{d}{ds}\frac{1}{s^2+a^2} = -t\cdot\frac{\sin at}{a}$$

$$L^{-1}\frac{-2s}{(s^2+a^2)^2} = \frac{-t}{a}\sin at$$

$$\therefore \quad L^{-1}\frac{s}{(s^2+a^2)^2} = \frac{+t}{2a}\sin at$$

Step 3 : Using $L^{-1}\, s\phi(s) = \dfrac{d}{dt} f(t)$ if $f(0) = 0$ as $\sin 0 = 0$

$$\therefore \quad L^{-1}\, s\cdot\frac{s}{(s^2+a^2)^2} = \frac{d}{dt}\frac{+t\sin at}{2a}$$

$$\therefore \quad L^{-1}\frac{s^2}{(s^2+a^2)^2} = \frac{+1}{2a}[\sin at + at\cos at]$$

Example 7.38 : $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$

Solution :

Step 1 : $$L^{-1}\frac{(s+2)^2}{(s^2+4s+8)^2} = L^{-1}\frac{(s+2)^2}{[(s+2)^2+4]^2}$$

Step 2 : Use $L^{-1}\phi(s+2) = e^{-2t}\, L^{-1}\phi(s)$

$$\therefore \quad L^{-1}\frac{(s+2)^2}{[(s+2)^2+4]^2} = e^{-2t}\, L^{-1}\frac{s^2}{(s^2+4)^2}$$

Step 3 : Using above problem ; a = 2 we get

$$L^{-1}\frac{(s+2)^2}{(s^2+4s+8)^2} = e^{-2t}\left[+\frac{1}{4}(\sin 2t + 2t\cos 2t)\right]$$

Example 7.39 : $\dfrac{s^2}{(s+a)^3}$

Solution : Step 1 : Since $L^{-1}\,\phi(s-a) = e^{at}\,L^{-1}\,\phi(s)$

Step 2 : We have

$$L^{-1}\left[\frac{1}{(s+a)^3}\right] = e^{-at}\,L^{-1}\left[\frac{1}{s^3}\right]$$

$$= e^{-at}\,\frac{t^2}{2}$$

Step 3 : Using $L^{-1}\,[s\phi(s)] = \dfrac{d}{dt}\,f(t)$ if $f(0) = 0$

Step 4 : $\therefore\; L^{-1}\left[s\,\dfrac{1}{(s+a)^3}\right] = \dfrac{d}{dt}\left(\dfrac{1}{2}\,e^{-at}t^2\right) = \dfrac{1}{2}\,e^{-at}\left(2t - at^2\right) = f(t)$ say,

where $\phi(s) = \dfrac{s}{(s+a)^3}$ and $\phi(0) = 0$

$$\text{Again}\quad L^{-1}\,\frac{s^2}{(s+a)^3} = \frac{d}{dt}\left\{\frac{1}{2}\,e^{-at}\left(2t - at^2\right)\right\}$$

$$= \frac{1}{2}\,e^{-at}\left(2 - 4at + a^2t^2\right)$$

$$\text{Hence}\quad L^{-1}\left[\frac{s^2}{(s+a)^3}\right] = e^{-at}\left(1 - 2at + \frac{1}{2}\,a^2t^2\right)$$

Exercise 7.4

Find inverse Laplace transforms of the following

1. $\dfrac{s+2}{\left(s^2+4s+5\right)^2}$ **[Ans. : $\dfrac{t\,e^{-2t}}{2}\sin t$]**

2. $\dfrac{s}{(s+a)^2}$ **[Ans. : $e^{-at}\,(1 - at)$]**

3. $\dfrac{s^2}{\left(s^2+4\right)^2}$ **[Ans. : $\dfrac{1}{4}\,(\sin 2t + 2t\cos 2t)$]**

4. $\dfrac{s^2}{\left(s^2-a^2\right)^2}$ **[Ans. : $\dfrac{1}{2a}\,(\sinh at + at\cosh at)$]**

5. *Using* $L^{-1}\dfrac{1}{s^2+a^2} = \dfrac{1}{a}\sin at$ *show that* $L^{-1}\dfrac{s}{s^2+a^2} = \cos at$

6. *Using* $L^{-1}\dfrac{1}{s^2-a^2} = \dfrac{1}{a}\sinh at$ *show that* $L^{-1}\dfrac{s}{s^2-a^2} = \cosh at$

7. *Find* $L^{-1} \dfrac{s^2 - a^2}{(s^2 + a^2)^2}$ **[Ans. : t cos at]**

Hint : $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)^2} - a^2\left(\dfrac{1}{(s^2+a^2)^2}\right)\right]$ *find both* L^{-1} *separately:*

8. *Given* $L^{-1}\dfrac{1}{s-a} = e^{at}$ *find* $L^{-1}\dfrac{1}{(s-a)^3}$ **[Ans. : $e^{at}\dfrac{t^2}{2}$]**

9. *Given* $L^{-1}\dfrac{1}{s^2-a^2} = \dfrac{1}{a}\sinh at$ *find* $L^{-1}\dfrac{s}{(s^2-a^2)^2}$

7.7 Type V

Laplace inverse using $L^{-1}\,\phi(s) = \dfrac{-1}{t}\,L^{-1}\,\phi'(s)$

Example 7.40 : *Show that* $L^{-1}\,log\dfrac{s}{s^2+1}$

Solution :

Step 1 : Let $f(t) = L^{-1}\log\dfrac{s}{s^2+1}$

Step 2 : Let $\phi(s) = \log s - \log(s^2+1)$

$$\phi'(s) = \frac{1}{s} - \frac{1}{s^2+1}\cdot 2s$$

Step 3 : Find L^{-1} of $\phi'(s)$

$$L^{-1}\,\phi'(s) = L^{-1}\left[\frac{1}{s} - 2\frac{s}{s^2+1}\right] = 1 - 2\cos t$$

Step 4 : Using the formula $L^{-1}\,\phi(s) = \dfrac{-1}{t}\,L^{-1}\,\phi'(s)$

$$\therefore \quad L^{-1}\,\phi(s) = \frac{1-2\cos t}{-t}$$

i.e.
$$f(t) = L^{-1}\,\phi(s) = \frac{2\cos t - 1}{t}$$

Example 7.41 : $L^{-1} \log\sqrt{\alpha - \frac{a^2}{s^2}}$

Solution :

Step 1 : Let $f(t) = L^{-1} \log\sqrt{\alpha - \frac{a^2}{s^2}}$

Step 2 :

$$\phi(s) = \log\sqrt{\alpha - \frac{a^2}{s^2}}$$

$$= \frac{1}{2}\log\left(\frac{\alpha s^2 - a^2}{s^2}\right)$$

$$= \frac{1}{2}\left[\log \alpha\left(s^2 - \frac{a^2}{\alpha}\right) - \log s^2\right]$$

$$\phi(s) = \frac{1}{2}\left[\log \alpha + \log\left(s^2 - \frac{a^2}{\alpha}\right) - 2\log s\right]$$

$$\phi'(s) = \frac{1}{2}\left[0 + \frac{2s}{s^2 - \frac{a^2}{\alpha}} - \frac{2}{s}\right]$$

Step 3 : Find L^{-1} of $\phi'(s)$

$$L^{-1}\phi'(s) = \frac{1}{2}\left[2\cosh\frac{at}{\sqrt{\alpha}} - 2\right]$$

Step 4 : Using the formula $L^{-1}\phi(s) = \frac{-1}{t} L^{-1}\phi'(s)$

$$\therefore \quad f(t) = \frac{\left(\cosh\frac{at}{\sqrt{\alpha}} - 1\right)}{-t}$$

$$= \frac{1}{t}\left[\left(1 - \cosh\frac{at}{\sqrt{\alpha}}\right)\right]$$

Example 7.42 : $L^{-1} \log\left(1 - \frac{a}{s}\right)$

Solution :

Step 1 : Let $f(t) = L^{-1} \log\left(1 - \frac{a}{s}\right)$

Step 2 : $$\phi(s) = \log\left(\frac{s-a}{s}\right)$$

$$\phi(s) = \log(s-a) - \log(s)$$

$$\therefore \quad \phi'(s) = \frac{1}{s-a} - \frac{1}{s}$$

Step 3 : Find L^{-1} of $\phi'(s)$

$$L^{-1}\,\phi'(s) = e^{at} - 1$$

Step 4 : Using the formula $L^{-1}\,\phi(s) = \frac{-1}{t} L^{-1}\,\phi'(s)$

$$f(t) = \frac{1-e^{at}}{t}$$

Example 7.43 : $L^{-1} \, log\sqrt{\frac{s^2-a^2}{s^2}}$

Solution :

Step 1 : Let $$f(t) = L^{-1}\log\sqrt{\frac{s^2-a^2}{s^2}}$$

$$= L^{-1}\log\sqrt{\frac{s^2-a^2}{s^2}}$$

Step 2 : $$\phi(s) = \frac{1}{2}\log\frac{s^2-a^2}{s^2}$$

$$= \frac{1}{2}\left[\log(s^2-a^2) - \log(s^2)\right]$$

$$\phi'(s) = \frac{1}{2}\left[\frac{2s}{s^2-a^2} - \frac{2}{s}\right]$$

Step 3 : Find L^{-1} of $\phi'(s)$

$$\therefore \quad L^{-1}\,\phi'(s) = \frac{1}{2}[2\cosh at - 2]$$

$$\therefore \quad L^{-1}\,\phi'(s) = [\cosh at - 1]$$

Step 4 : Using the formula $L^{-1}\,\phi(s) = \frac{-1}{t} L^{-1}\,\phi'(s)$

$$\therefore \quad L^{-1}\,\phi(s) = \frac{1-\cosh at}{t}$$

Example 7.44 : $L^{-1}\ tan^{-1}\dfrac{2}{s^2}$

Solution :

Step 1 : Let $$f(t) = L^{-1}\tan^{-1}\frac{2}{s^2}$$

Step 2 : $$\phi(s) = \tan^{-1}\frac{2}{s^2}$$

$$\left\{\text{As } \frac{d}{ds}\frac{2}{s^2} = 2\cdot\frac{d}{ds}\frac{1}{s^2} = 2\left(\frac{-2}{s^3}\right) = \frac{-4}{s^3}\right\}$$

$$\therefore \quad \phi'(s) = \frac{1}{1+\left(\dfrac{2}{s^2}\right)^2}\times\left(\frac{-4}{s^3}\right)$$

$$= \frac{-4s}{s^4+4}$$

Step 3 : To find L^{-1} of $\phi'(s)$

and adjusting the terms we get,

$$\phi'(s) = \frac{-4s}{s^4+4s^2+4-4s^2}$$

$$\phi'(s) = \frac{-4s}{(s^2+2+2s)(s^2+2-2s)}$$

$$L^{-1}\phi'(s) = L^{-1}\left[\frac{1}{s^2+2+2s}-\frac{1}{s^2+2-2s}\right]$$

$$= L^{-1}\left[\frac{1}{(s+1)^2+1}-\frac{1}{(s-1)^2+1}\right]$$

$$= \left[e^{-t}\sin t - e^{t}\sin t\right]$$

$$= -\left(e^{t}-e^{-t}\right)\sin t$$

Thus $$L^{-1}\phi'(s) = -2\sinh t\sin t$$

Step 4 : Using $L^{-1}\phi(s) = \dfrac{-1}{t}L^{-1}\phi'(s)$

$$\therefore \quad L^{-1}\phi(s) = \left(\frac{-1}{t}\right)(-2\sinh t\sin t)$$

Thus $$f(t) = \frac{2}{t}\sin t\sinh t$$

Example 7.45 : $L^{-1} \log \dfrac{s^2 + a^2}{s^2 + b^2}$

Solution : **Step 1 :** Let $f(t) = L^{-1} \log \dfrac{s^2 + a^2}{s^2 + b^2}$

Step 2 :

$$\phi(s) = \log(s^2 + a^2) - \log(s^2 + b^2)$$

$$\phi'(s) = \frac{2s}{(s^2 + a^2)} - \frac{2s}{(s^2 + b^2)}$$

Step 3 : Find L^{-1} of $\phi'(s)$

$$L^{-1} \phi'(s) = 2(\cos at - \cos bt)$$

Step 4 : Using the formula $L^{-1} \phi(s) = \dfrac{-1}{t} L^{-1} \phi'(s)$

$$\therefore \quad L^{-1} \phi(s) = \frac{2(\cos bt - \cos at)}{t}$$

Example 7.46 : $L^{-1}\left[\dfrac{1}{s} \log \dfrac{s^2 + 1}{s^2}\right]$

Solution : **Step 1 :** Consider $\phi(s) = \log \dfrac{s^2 + 1}{s^2}$

Let
$$f(t) = L^{-1}\left[\log \frac{s^2 + 1}{s^2}\right]$$

Step 2 :

$$\phi(s) = \left[\log \frac{s^2 + 1}{s^2}\right]$$

$$\phi(s) = \log(s^2 + 1) - \log(s^2)$$

$$\phi'(s) = \frac{2s}{s^2 + 1} - \frac{2}{s}$$

$$L^{-1} \phi'(s) = 2 \cos t - 2$$

Step 3 : Using the formula $L^{-1} \phi(s) = \dfrac{-1}{t} L^{-1} \phi'(s)$

$$f(t) = \frac{2 \cos t - 2}{-t}$$

Step 4 : We know that

$$\therefore \quad L^{-1} \frac{1}{s} \phi(s) = \int_0^t f(t)\, dt$$

$$= \int_0^t \frac{2(1 - \cos t)}{t}\, dt$$

Example 7.47 : $L^{-1} \frac{1}{s} \log\left(\frac{s^2+a^2}{s^2+b^2}\right)$

Solution : Step 1 : Consider $\phi(s) = \log \frac{(s^2+a^2)}{(s^2+b^2)}$

Let $f(t) = L^{-1} \log\left(\frac{s^2+a^2}{s^2+b^2}\right)$

Step 2 :

$$\phi(s) = \log(s^2+a^2) - \log(s^2+b^2)$$

$$\phi'(s) = \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}$$

Step 3 : Using the formula $L^{-1}\phi(s) = \frac{-1}{t} L^{-1}\phi'(s)$

As $L^{-1}\phi'(s) = (2\cos at - 2\cos bt)$

$$\therefore \quad L^{-1}\phi(s) = \frac{2(\cos bt - \cos at)}{t}$$

Step 4 : Also $L^{-1}\frac{1}{s}\phi(s) = \int_0^t f(t)\,dt$

$$\therefore \quad f(t) = 2\int_0^t \frac{\cos bt - \cos at}{t}\,dt$$

Example 7.48 : $L^{-1}\left[s \log \frac{s^2}{s^2+1}\right]$.

Solution : Step 1 : Consider $\phi(s) = \log\frac{s^2}{s^2+1}$

Let $f(t) = L^{-1}\left[\log\frac{s^2}{s^2+1}\right]$

Step 2 :

$$\phi(s) = \log(s^2) - \log(s^2+1)$$

$$\phi'(s) = \frac{2s}{s^2} - \frac{2s}{s^2+1}$$

$$= 2\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

Step 3 : Using the formula $L^{-1}\phi(s) = \frac{-1}{t} L^{-1}\phi'(s)$

Step 4 : As $L^{-1}\phi'(s) = 2(1-\cos t)$

$$L^{-1}\phi(s) = 2\,\frac{(\cos t - 1)}{t}$$

Step 5 : Use $L^{-1}\, s\,\phi(s) = \frac{d}{dt} f(t)$

$$\therefore \quad L^{-1}\, s \log \frac{s^2}{s^2+1} = \frac{d}{dt}\, \frac{2(\cos t - 1)}{t}$$

Example 7.49 : $\cot^{-1}\left(\frac{s-2}{3}\right)$

Solution : Step 1 : Let $f(t) = L^{-1} \cot^{-1}\left(\frac{s-2}{3}\right) = L^{-1}\phi(s) = f(t)$ ... (1)

Step 2 :
$$\phi(s) = \cot^{-1}\left(\frac{s-2}{3}\right)$$

$$\therefore \quad \phi'(s) = \frac{-1}{1+\left(\frac{s-2}{3}\right)^2}\left(\frac{1}{3}\right) = \frac{-3}{(s-2)^2+3^2}$$

Step 3 : Using the formula $L^{-1}\phi(s) = \frac{-1}{t} L^{-1}\phi'(s)$

$$\therefore \quad L^{-1}\phi'(s) = -3L^{-1}\frac{1}{(s-2)^2+3^2} = -3e^{2t}\, L^{-1}\frac{1}{s^2+3^2}$$

$$= -3e^{2t}\,\frac{\sin 3t}{3} = -e^{2t}\sin 3t$$

Step 4 : $\therefore \quad f(t) = -\frac{1}{t} L^{-1}\phi'(s) = +\frac{1}{t} e^{2t}\sin 3t$

$$\therefore \quad L^{-1}\cot^{-1}\left(\frac{s-2}{3}\right) = \frac{1}{t} e^{2t}\sin 3t$$

Exercise 7.5

Find inverse Laplace transforms of the following.

1. $\tan^{-1}\frac{1}{s}$ **[Ans. : $\frac{\sin t}{t}$]**

 Hint : $\tan^{-1}\frac{1}{s} = \cot^{-1}s$ *and* $\frac{d}{ds}\cot^{-1}s = \frac{-1}{s^2+1}$

2. $\log\frac{s+1}{s-1}$ **[Ans. : $\frac{2\sinh t}{t}$]**

3. $\tan^{-1}(s+2)$ **[Ans. : $\frac{e^{-2t}}{-t}\sin t$]**

4. $\log\left(1+\frac{1}{s^2}\right)$ **[Ans. : $\frac{2(1-\cos t)}{t}$]**

5. $s\log\frac{s}{\sqrt{s^2+1}} + \cot^{-1}s$ **[Ans. : $\frac{1-\cos t}{t^2}$]**

6. $\log\left(1+\frac{1}{s}\right)$ [Ans. : $\frac{1-e^{t}}{t}$]

7. $\tan^{-1}\frac{a}{s}$ [Ans. : $\frac{\sin at}{a}$]

8. $\frac{1}{2}\log\frac{s^2+b^2}{(s-a)^2}$ [Ans. : $\frac{e^{-at}-\cos bt}{t}$]

9. $\frac{1}{2}\log\frac{s^2-a^2}{s^2-b^2}$ [Ans. : $\frac{\cosh bt-\cosh at}{t}$]

10. $\frac{1}{2}\log\frac{s^2-a^2}{s^2}$ [Ans. : $\frac{1-\cosh at}{t}$]

7.8 Type VI

Convolution Theorem

$$L^{-1}\,\phi_1(s)\phi_2(s) = \int_0^t f_1(u)\,f_2(t-u)\,du$$

$$= \int_0^t f_1(t-u)\,f_2(u)\,du$$

Example 7.50 : $L^{-1}\dfrac{s}{\left(s^2+a^2\right)^2}$

Solution : Step 1 : Let $f(t) = L^{-1}\dfrac{s}{\left(s^2+a^2\right)^2}$

$$= L^{-1}\left[\frac{1}{s^2+a^2}\cdot\frac{s}{s^2+a^2}\right]$$

$$= L^{-1}\,\phi_1(s)\cdot\phi_2(s)$$

$\therefore$ $$f_1(t) = \frac{1}{a}\sin at$$

$$f_2(t) = \cos at$$

Step 2 : By convolution theorem we know that,

$$L^{-1}\,\phi_1(s)\phi_2(s) = \int_0^t f_1(u)\,f_2(t-u)\,du$$

Step 3 : Thus $$f(t) = \int_0^t f_1(u)\,f_2(t-u)\,du$$

$$= \int_0^t \frac{1}{a}\sin au\cos(at-au)\,du$$

Step 4 : Integrating w.r.t. u.

$$= \frac{1}{2a}\int_0^t [\sin(au + at - au) + \sin(au - at + au)]\, du$$

$$= \frac{1}{2a}\int_0^t \sin at + \sin(2au - at)\, du$$

$$= \frac{1}{2a}\left\{\sin at\,[u]_0^t + \left[-\frac{\cos(2au - at)}{2a}\right]_0^t\right\}$$

Step 5 : Substitute the limits of u.

$$= \frac{1}{2a}\left[t \sin at - \frac{\cos at}{2a} + \frac{\cos at}{2a}\right]$$

$$= \frac{t \sin at}{2a}$$

Example 7.51 : $L^{-1}\dfrac{s^2}{\left(s^2 + a^2\right)^2}$

Solution : Step 1 : Let $f(t) = L^{-1}\dfrac{s^2}{\left(s^2 + a^2\right)^2}$

$$= L^{-1}\frac{s}{\left(s^2 + a^2\right)}\cdot\frac{s}{\left(s^2 + a^2\right)}$$

$$= L^{-1}\,\phi_1(s)\cdot\phi_2(s)$$

$\therefore \quad f_1(t) = \cos at$

$f_2(t) = \cos at$

Step 2 : By convolution theorem we know that,

$$L^{-1}\,\phi_1(s)\phi_2(s) = \int_0^t f_1(u) f_2(t-u)\, du$$

Step 3 :

$$L^{-1}\,\phi_1(s)\phi_2(s) = \int_0^t f_1(u) f_2(t-u)\cdot du$$

$$\therefore \quad f(t) = \int_0^t \cos au \cdot \cos(at - au)\cdot du$$

Step 4 : Integrating w.r.t. u we get,

$$= \frac{1}{2}\int_0^t [\cos(au - at + au) + \cos(au + at - au)]\cdot du$$

$$= \frac{1}{2}\int_0^t \cos(2au - at) + \cos at \cdot du$$

$$= \frac{1}{2}\left\{\left[\frac{\sin(2au - at)}{2a}\right]_0^t + \cos at\,[u]_0^t\right\}$$

Step 5 . Substitute the limits of u.

$$= \frac{1}{2}\left\{\frac{\sin(2at - at)}{2a} - \frac{\sin(-at)}{2a} + t\cos at\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin at}{2a} + \frac{\sin at}{2a} + t\cos at\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin at}{a} + t\cos at\right\}$$

$$= \frac{1}{2a}(at\cos at + \sin at)$$

Example 7.52 : $L^{-1}\dfrac{1}{s^4 - a^4}$

Solution : **Step 1** : Let $f(t) = L^{-1}\dfrac{1}{s^4 - a^4}$

$$= L^{-1}\frac{1}{s^2 + a^2}\cdot\frac{1}{s^2 - a^2}$$

$$= L^{-1}\,\phi_1(s)\cdot\phi_2(s)$$

$$\therefore \quad f_1(t) = \frac{1}{a}\sin at$$

$$f_2(t) = \frac{1}{a}\sinh at$$

Step 2 : By convolution theorem we know that,

$$L^{-1}\,\phi_1(s)\,\phi_2(s) = \int_0^t f_1(u)\,f_2(t-u)\,du$$

Step 3 :

$$L^{-1}\,\phi_1(s)\,\phi_2(s) = \int_0^t f_1(u) f_2(t-u)\cdot du$$

$$f(t) = \frac{1}{a^2}\int_0^t \sin au \sinh(at - au)\cdot du$$

Step 4 : Integrating w.r.t. u

$$= \frac{1}{a^2} \int_0^t \left(\frac{e^{at-au} - e^{-(at-au)}}{2} \right) \sin au \cdot du$$

$$= \frac{1}{2a^2} \int_0^t e^{at} e^{-au} \sin au - e^{-at+au} \sin au \, du$$

$$= \frac{1}{2a^2} \left\{ \int_0^t e^{at} e^{-au} \sin au \, du + \int_0^t -e^{-at} e^{au} \sin au \, du \right\}$$

$$= \frac{1}{2a^2} \{I_1 + I_2\} \qquad \text{... (i)}$$

Step 5 : We evaluate I_1 and I_2 separately,

Let $$I_1 = e^{at} \int_0^t e^{-au} \sin au$$

$$= e^{at} \left\{ \frac{e^{-au}}{a^2 + a^2} [-a \sin au - a \cos au] \right\}_0^t$$

$$= e^{at} \left\{ \frac{e^{-at}}{2a^2} [-a \sin at - a \cos at] - \left[\frac{1}{2a^2} [-a] \right] \right\}$$

$$= \frac{e^{at}}{2a^2} \left[ae^{-at} [-\sin at - \cos at] + a \right]$$

$$= \frac{e^{at} \cdot a}{2a^2} \left[e^{-at} [-\sin at - \cos at] + 1 \right]$$

$$= \frac{1}{2a} \left\{ [-\sin at - \cos at] + e^{at} \right\}$$

$$I_1 = \frac{1}{2a} \left\{ -\sin at - \cos at + e^{at} \right\}$$

To get I_2 replace a by $-$ a

$$I_2 = \frac{-1}{2a} \left[\sin at - \cos at + e^{-at} \right]$$

Step 6 : Substituting I_1 and I_2 in (i) we get

$$I = \frac{1}{2a^2} \left\{ \frac{1}{2a} (-\sin at - \cos at + e^{at}) + \frac{-1}{2a} \left[\sin at - \cos at + e^{-at} \right] \right\}$$

$$= \frac{1}{2a^2} \left\{ \frac{-\sin at}{2a} - \frac{\cos at}{2a} + \frac{e^{at}}{2a} - \frac{\sin at}{2a} + \frac{\cos at}{2a} - \frac{e^{-at}}{2a} \right\}$$

$$= \frac{1}{2a^2} \left\{ \frac{e^{at} - e^{-at}}{2a} - \frac{2 \sin at}{2a} \right\}$$

$$= \frac{1}{2a^2} \left\{ \frac{\sinh at}{a} - \frac{\sin at}{a} \right\} = \frac{1}{2a^3} \{\sinh at - \sin at\}$$

Example 7.53 : $L^{-1} \dfrac{s+29}{(s+4)\left(s^2+9\right)}$

Solution : Step 1 : Let $f(t) = L^{-1} \dfrac{s+29}{(s+4)\left(s^2+9\right)}$

$$= L^{-1} \frac{1}{(s+4)} \frac{s+29}{s^2+9}$$

$$= L^{-1} \phi_1(s) \cdot \phi_2(s)$$

$$f_1(t) = e^{-4t} \quad \text{and}$$

$$f_2(t) = L^{-1} \frac{s}{s^2+9} + L^{-1} \frac{29}{s^2+9}$$

$$= \cos 3t + \frac{29}{3} \sin 3t$$

Step 2 : By convolution theorem we know that,

$$L^{-1} \phi_1(s)\phi_2(s) = \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$f_1(t-u) = e^{-4(t-u)}$$

$$f_2(u) = \cos 3u + \frac{29}{3} \sin 3u$$

Step 3 :

$$L^{-1} \phi_1(s)\phi_2(s) = \int_0^t e^{-4(t-u)} \left[\cos 3u + \frac{29}{3} \sin 3u\right] du$$

$$= \int_0^t \left[e^{-4t} e^{4u} \cos 3u + e^{-4t} e^{4u} \frac{29}{3} \sin 3u\right] du$$

$$f(t) = e^{-4t} \int_0^t e^{4u} \cos 3u\, du + \frac{29\, e^{-4t}}{3} \int_0^t e^{4u} \sin 3u \cdot du$$

$$= I_1 + I_2 \qquad \text{... (i)}$$

Step 4 : Integrating w.r.t. u and evaluating I_1 and I_2 separately.

$$I_1 = e^{-4t} \int_0^t e^{4u} \cos 3u \cdot du$$

$$= e^{-4t} \left\{\frac{e^{4u}}{16+9} (4 \cos 3u + 3 \sin 3u)\right\}_0^t$$

$$= e^{-4t}\left\{\frac{e^{4t}}{16+9}(4\cos 3t + 3\sin 3t) - \left[\frac{1}{16+9}(4)\right]\right\}$$

$$= \frac{1}{16+9}(4\cos 3t + 3\sin 3t) - \frac{e^{-4t}}{16+9}(4)$$

$$I_2 = \frac{29}{3}e^{-4t}\int_0^t e^{4u}\sin 3u \cdot du$$

$$= \frac{29}{3}e^{-4t}\left\{\frac{e^{4u}}{16+9}(4\sin 3u - 3\cos 3u)\right\}_0^t$$

$$= \frac{29}{3}e^{-4t}\left\{\frac{e^{4t}}{16+9}(4\sin 3t - 3\cos 3t) - \frac{1}{(16+9)}(0-3)\right\}$$

$$= \frac{29}{3}\left\{\frac{1}{16+9}(4\sin 3t - 3\cos 3t) + \frac{3e^{-4t}}{16+9}\right\}$$

Step 5 : Substituting I_1 and I_2 in (i)

$$f(t) = I_1 + I_2$$

$$= \frac{1}{16+9}\left[4\cos 3t + 3\sin 3t - 4e^{-4t} + \frac{116}{3}\sin 3t - 29\cos 3t + 29e^{-4t}\right]$$

$$= \frac{1}{25}\left\{-25\cos 3t + \frac{125}{3}\sin 3t + 25e^{-4t}\right\}$$

$$= \frac{5}{3}\sin 3t - \cos 3t + e^{-4t}$$

Example 7.54 : $L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right]$

Solution : Step 1 : Let $f(t) = L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right]$

$$= L^{-1}\frac{1}{s+1}\cdot\frac{1}{s^2+1}$$

$$= L^{-1}\phi_1(s)\cdot\phi_2(s)$$

$$\therefore \quad f_1(t) = e^{-t}$$

$$f_2(t) = \sin t$$

Step 2 : By convolution theorem we know that,

$$L^{-1}\phi_1(s)\phi_2(s) = \int_0^t f_1(u)f_2(t-u)\,du$$

Step 3 :

$$L^{-1}\,\phi_1(s)\,\phi_2(s) = \int_0^t e^{-(t-u)}\sin u \cdot du$$

$$= e^{-t}\int_0^t e^{u}\sin u \cdot du$$

Step 4 : Integrating w.r.t. u.

$$= e^{-t}\left\{\frac{e^{u}}{1+1}(\sin u - \cos u)\right\}_0^t$$

Step 5 : Substituting the limits of t.

$$= e^{-t}\left\{\frac{e^{t}}{2}(\sin t - \cos t) - \frac{1}{2}(-1)\right\}$$

$$= e^{-t}\left\{\frac{e^{t}}{2}(\sin t - \cos t) + \frac{1}{2}\right\}$$

$$= \frac{1}{2}(\sin t - \cos t) + \frac{e^{-t}}{2}$$

$$= \frac{1}{2}\left[\sin t - \cos t + e^{-t}\right]$$

Example 7.55 : $L^{-1}\dfrac{1}{s\sqrt{s+4}}$

Solution : Step 1 : Let $f(t) = L^{-1}\dfrac{1}{s\sqrt{s+4}}$

$$= L^{-1}\frac{1}{s}\cdot\frac{1}{(s+4)^{1/2}}$$

$$= L^{-1}\,\phi_1(s)\cdot\phi_2(s)$$

$\therefore$ $f_1(t) = 1$

$$f_2(t) = \frac{1}{(s+4)^{1/2}}$$

Step 2 : By convolution theorem we know that,

$$L^{-1}\,\phi_1(s)\,\phi_2(s) = \int_0^t f_1(u)\,f_2(t-u)\,du$$

$$f_2(t) = e^{-4t}\,L^{-1}\frac{1}{s^{1/2}}$$

$$= e^{-4t} \frac{t^{-1/2}}{\lceil 1/2}$$

$$= e^{-4t} \frac{1}{\sqrt{\pi t}}$$

$$f_2 (u) = e^{-4u} \frac{1}{\sqrt{\pi u}}$$

Step 3 : $L^{-1} \phi_1 (s) \phi_2 (s) = \int_0^t e^{-4u} \frac{1}{\sqrt{\pi u}} \cdot du$

$$f(t) = \int_0^t \frac{e^{-4u}}{\sqrt{u\pi}} \cdot du$$

Step 4 : Integrating w.r.t. u. Put $4u = v^2$

i.e. $u = \frac{v^2}{4}$

$$du = \frac{vdv}{2}$$

Substituting in integral we get

u	0	t
v	0	$2\sqrt{t}$

$\therefore$ $f(t) = \int_0^{2\sqrt{t}} \frac{e^{-v^2}}{\sqrt{\pi\left(\frac{v^2}{4}\right)}} \cdot \frac{v\,dv}{2}$

$$f(t) = \int_0^{2\sqrt{t}} \frac{e^{-v^2}}{\sqrt{\pi}} \cdot \frac{dv}{1}$$

$$= \frac{1}{2}\left\{\frac{2}{\sqrt{\pi}} \int_0^{2\sqrt{t}} e^{-v^2} dv\right\}$$

$\therefore$ by the definition of erf (x).

$$= \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2}\, dv$$

$$f(t) = \frac{1}{2} \operatorname{erf} 2\sqrt{t}$$

Example 7.56 : $L^{-1} \dfrac{1}{s^2(s^2+a^2)}$

Solution : Step 1 : Let $f(t) = L^{-1} \dfrac{1}{s^2(s^2+a^2)}$

$$= L^{-1} \frac{1}{s^2} \cdot \frac{1}{s^2+a^2}$$

$$= L^{-1} \phi_1(s) \cdot \phi_2(s)$$

$$\therefore \quad f_1(t) = \frac{t}{1}$$

$$f_1(t-u) = (t-u)$$

$$f_2(t) = \frac{1}{a} \sin at$$

$$f_2(u) = \frac{\sin au}{a}$$

Step 2 : By convolution theorem we know that,

$$L^{-1} \phi_1(s)\phi_2(s) = \int_0^t f_1(t-u) f_2(u) \cdot du$$

$$f(t) = \int_0^t (t-u) \cdot \frac{\sin au}{a} \cdot du$$

Step 3 : Integrating w.r.t. u.

$$= \frac{1}{a}\left\{\int_0^t t \sin au - \int_0^t u \sin au \cdot du\right\}$$

$$= \frac{1}{a}\left\{t\left(\frac{-\cos au}{a}\right) - \left[u\left(\frac{-\cos au}{a}\right) - (1)\left(\frac{-\sin au}{a^2}\right)\right]\right\}_0^t$$

$$= \frac{1}{a}\left\{\left(\frac{-t\cos at}{a} + \frac{t\cos at}{a} - \frac{\sin at}{a^2}\right) - \left(\frac{-t}{a} - 0\right)\right\}$$

$$f(t) = \frac{t}{a^2} - \frac{\sin at}{a^3}$$

Example 7.57 : $L^{-1} \dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$.

Solution : Step 1 : Let $f(t) = L^{-1} \dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

$$= L^{-1} \frac{s}{s^2+a^2} \cdot \frac{s}{s^2+b^2}$$

$$= L^{-1}\, \phi_1(s) \cdot \phi_2(s)$$

$$\therefore \quad f_1(t) = \cos at$$

$$f_2(t) = \cos bt$$

Step 2 : By convolution theorem we know that,

$$L^{-1}\, \phi_1(s)\phi_2(s) = \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$f_1(u) = \cos au$$

$$f_2(t-u) = \cos(bt - bu)$$

Step 3 :

$$L^{-1}\, \phi_1(s)\phi_2(s) = \int_0^t \cos au \cos(bt - bu) \cdot du$$

Step 4 : Integrating w.r.t. u.

$$\therefore \quad f(t) = \frac{1}{2}\int_0^t \cos(au + bu - bt) + \cos(au + bt - bu)\, du$$

$$= \frac{1}{2}\left\{\int_0^t \cos[(a+b)u - bt]\, du + \int_0^t \cos[(a-b)u + bt] \cdot du\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\sin[(a+b)u - bt]}{(a+b)}\right]_0^t + \left[\frac{\sin[(a-b)u + bt]}{(a-b)}\right]_0^t\right\}$$

Step 5 : Substituting the limits of u.

$$= \frac{1}{2}\left\{\frac{\sin[(a+b)t - bt]}{a+b} - \frac{\sin(-bt)}{(a+b)} + \frac{\sin[(a-b)t + bt]}{a-b} - \frac{\sin bt}{(a-b)}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin at}{(a+b)} + \frac{\sin at}{(a-b)} + \frac{\sin bt}{(a+b)} - \frac{\sin bt}{(a-b)}\right\}$$

$$= \frac{1}{2}\left\{\frac{2a \sin at}{a^2 - b^2} - \frac{2b \sin bt}{a^2 - b^2}\right\}$$

$$f(t) = \frac{a \sin at - b \sin bt}{(a^2 - b^2)}$$

Example 7.58 : $L^{-1} \dfrac{1}{(s^2 + 1)^3}$.

Solution : Step 1 : Let $f(t) = L^{-1} \dfrac{1}{(s^2+1)^3}$

$$= L^{-1} \frac{1}{(s^2+1)} \cdot \frac{1}{(s^2+1)^2}$$

$$= L^{-1} \phi_1(s) \cdot \phi_2(s)$$

$$\therefore \quad f_1(t) = \sin t$$

$$f_2(t) = \frac{1}{2}(\sin t - t\cos t)$$

Step 2 : By convolution theorem we know that,

$$L^{-1} \phi_1(s)\phi_2(s) = \int_0^t f_2(u)\, f_1(t-u)\, du$$

Step 3 : Now $f_1(t-u) = \sin(t-u)$

$$f_2(u) = \frac{1}{2}(\sin u - u\cos u)$$

$$\therefore \quad f_1(t-u)\, f_2(u) = \frac{1}{2}\{\sin u \sin(t-u) - u[\sin(t-u)\cos u]\}$$

$$\therefore \quad f(t) = \frac{1}{2}\int_0^t \{\sin u \sin(t-u)\}\, du - \int_0^t \{u[\sin(t-u)\cos u]\, du\}$$

$$= \frac{1}{2}(I_1 - I_2) \qquad \text{... (i)}$$

Step 4 : Integrating w.r.t. u and evaluating I_1 and I_2 separately.

$$I_1 = \int_0^t \sin u \sin(t-u) \cdot du$$

$$= \frac{1}{2}\int_0^t [\cos(u-t+u) - \cos(u+t-u) \cdot du]$$

$$= \frac{1}{2}\int_0^t [\cos(2u-t) - \cos t]\, du$$

$$= \frac{1}{2}\left\{\int_0^t \cos(2u-t) \cdot du - \int_0^t \cos t \cdot du\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin(2u-t)}{2} - u\cos t\right\}_0^t$$

Step 5 : Substitute the limits of u.

$$= \frac{1}{2}\left\{\frac{\sin t}{2} - \frac{\sin(-t)}{2} - t\cos t\right\}$$

$$= \frac{1}{2}\{\sin t - t\cos t\} \quad \dots (1)$$

Consider $\quad I_2 = \int_0^t u \sin(t-u)\cos u \cdot du$

$$= \frac{1}{2}\int_0^t u\,[\sin(t-u+u) + \sin(t-u-u)]\cdot du$$

$$= \frac{1}{2}\int_0^t u\,[\sin t + \sin(t-2u)]\cdot du$$

$$= \frac{1}{2}\left\{\int_0^t u\sin t\cdot du + \int_0^t u\sin(t-2u)\cdot du\right\}$$

$$= \frac{1}{2}\left\{\sin t\left[\frac{u^2}{2}\right]_0^t + \left[\frac{-u\cos(t-2u)}{(-2)}\right]_0^t - \frac{1}{2}\left[\frac{\sin(t-2u)}{(-2)}\right]_0^t\right\}$$

$$= \frac{1}{2}\left\{\frac{t^2\sin t}{2} + \frac{t\cos(-t)}{2} + \frac{1}{4}[\sin(-t) - \sin t]\right\}$$

$$= \frac{1}{2}\left\{\frac{t^2\sin t}{2} + \frac{t\cos t}{2} - \frac{2\sin t}{4}\right\}$$

Step 6 : Substituting I_1 and I_2 in (i) we get

$$\therefore \quad f(t) = \frac{1}{2}\left[\frac{1}{2}(\sin t - t\cos t) - \frac{1}{2}\left(\frac{t^2\sin t}{2} + \frac{t\cos t}{2} - \frac{\sin t}{2}\right)\right]$$

$$= \frac{1}{4}\left[\sin t - t\cos t - \frac{t^2\sin t}{2} - \frac{t\cos t}{2} + \frac{\sin t}{2}\right]$$

$$= \frac{1}{4}\left[\frac{3\sin t}{2} - \frac{3t\cos t}{2} - \frac{t^2\sin t}{2}\right]$$

$$= \frac{1}{8}\left[\left(3 - t^2\right)\sin t - 3t\cos t\right]$$

Example 7.59 : $\dfrac{1}{(s-2)^4\,(s+3)}$

Solution : **Step 1 :** Let $f(t) = L^{-1}\left[\dfrac{1}{(s-2)^4\,(s+3)}\right]$

$$= L^{-1}\left[\frac{1}{(s-2)^4\,(s-2+5)}\right]$$

$$= e^{2t}\,L^{-1}\left[\frac{1}{s^4\,(s+5)}\right] \quad \dots (1)$$

Step 2 : Now to find $L^{-1}\left[\dfrac{1}{s^4 (s+5)}\right]$, we use convolution theorem.

i.e. $$L^{-1}\, \phi_1 (s) \phi_2 (s) = \int_0^t f_1 (u)\, f_2 (t-u)\, du$$

Step 3 : Let $\phi_1 (s) = \dfrac{1}{s^4}$ and $\phi_2 (s) = \dfrac{1}{(s+5)}$

so that $f_1 (t) = \dfrac{t^3}{6}$ and $f_2 (t) = e^{-5t}$

$$\therefore \quad L^{-1}\left[\frac{1}{s^4 (s+5)}\right] = \int_0^t f_1 (u) \cdot f_2 (t-u)\, du$$

$$= \int_0^t \frac{u^3}{6} e^{-5(t-u)}\, du = \frac{e^{-5t}}{6} \int_0^t u^3 e^{5u} du$$

Step 4 : Integrating w.r.t. u.

$$= \frac{e^{-5t}}{6}\left[u^3\left(\frac{e^{5u}}{5}\right) - 3u^2\left(\frac{e^{5u}}{25}\right) + 6u\left(\frac{e^{5u}}{125}\right) - 6\left(\frac{e^{5u}}{625}\right)\right]_0^t$$

Step 5 : Substitute the limits of u.

$$= \frac{e^{-5t}}{6}\left[\left(t^3 \frac{e^{5t}}{5} - 3t^2 \frac{e^{5t}}{25} + 6t \frac{e^{5t}}{125} - \frac{6e^{5t}}{625}\right) + \frac{6}{625}\right]$$

$$= \frac{1}{30}\left(t^3 - \frac{3t^2}{5} + \frac{6t}{25} - \frac{6}{125}\right) + \frac{e^{-5t}}{625} \qquad \text{... (2)}$$

Hence from (1) and (2), we get

$$L^{-1}\left[\frac{1}{(s-2)^4 (s+3)}\right] = e^{2t} L^{-1}\left[\frac{1}{s^4 (s+5)}\right]$$

$$= \frac{e^{2t}}{30}\left(t^3 - \frac{3}{5} t^2 + \frac{6}{25} t - \frac{6}{125}\right) + \frac{e^{-3t}}{625}$$

Example 7.60 : $\dfrac{s+2}{s^2 (s-1)^2}$.

Solution : Step 1 : Let $$f (t) = L^{-1}\left[\frac{s+2}{s^2 (s-1)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s (s-1)^2}\right] + 2 L^{-1}\left[\frac{1}{s^2 (s-1)^2}\right] \qquad \text{... (1)}$$

Step 2 : We use convolution theorem.

i.e. $$L^{-1} \phi_1(s)\phi_2(s) = \int_0^t f_1(u)\, f_2(t-u)\, du$$

Step 3 : Now $$L^{-1}\left[\frac{1}{s(s-1)^2}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s-1)^2}\right]$$

Let $\phi_1(s) = \frac{1}{s}$, $\phi_2(s) = \frac{1}{(s-1)^2}$

$$\therefore f_1(t) = 1 \quad \text{and} \quad f_2(t) = e^t \cdot L^{-1}\frac{1}{s^2}$$

$$= e^t \cdot t$$

$$\therefore f_1(t-u) = 1 \text{ and } f_2(u) = u\, e^u$$

$$\therefore \quad L^{-1}\frac{1}{s(s-1)^2} = \int_0^t 1\cdot u e^u du = \left[ue^u - e^u\right]_0^t$$

$$= 1 + e^t(t-1) \qquad \text{... (2)}$$

and $$L^{-1}\left[\frac{1}{s^2(s-1)^2}\right] = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{s(s-1)^2}\right\}\right] = \int_0^t \left[1 + e^t(t-1)\right]dt$$

Step 4 : Integrating w.r.t. t.

$$= \left[t + \{(t-1)e^t - e^t\}\right]_0^t = \left[\left(t + (t-1)e^t - e^t\right) - (0-1-1)\right]$$

$$= 2 + t + e^t(t-2) \qquad \text{... (3)}$$

Step 5 : Using (1), (2) and (3), we get

$$L^{-1}\left[\frac{s+2}{s^2(s-1)^2}\right] = 1 + e^t(t-1) + 2\left[2 + t + e^t(t-2)\right]$$

$$= 5 + 2t + e^t(3t-5)$$

Exercise 7.6

Find the inverse Laplace transforms of the following using convolution theorem.

1. $\dfrac{1}{\left(s^2+a^2\right)^2}$ **[Ans. : $\dfrac{1}{2a^3}$ (sin at − at cos at)]**

2. $\dfrac{1}{s^2(s+1)^2}$ **[Ans. : $t\,e^{-t} + 2e^{-t} + t - 2$]**

3. $\dfrac{1}{(s-2)(s-3)}$ **[Ans. : $e^{3t} - e^{2t}$]**

4. $\dfrac{1}{s^2(s-a)}$ **[Ans. : $\dfrac{1}{a^2}\left(e^{at} - at - 1\right)$]**

5. $\dfrac{1}{(s+2)^2(s-2)}$ **[Ans. : $\dfrac{1}{16}\left(e^{2t} - e^{-2t} + 4\,t\,e^{-2t}\right)$]**

6. $\dfrac{s^2}{\left(s^2+4\right)^2}$ **[Ans. : $\dfrac{1}{4}$ (sin 2t + 2t cos 2t)]**

7. $\dfrac{s}{\left(s^2+4\right)^3}$ **[Ans. : $\dfrac{t}{64}$ (sin 2t − 2t cos 2t)]**

8. $\dfrac{1}{\left(s^2+9\right)^2}$ **[Ans. : $\dfrac{1}{18}\left(\dfrac{1}{3}\sin 3t - t\cos 3t\right)$]**

9. *If* $L\,J_0(t) = \dfrac{1}{\sqrt{s^2+1}}$ *then show that* $\int_0^t J_0(t)\cdot J_0(t-u)\,du = \sin t$

Hint : *To show* $L\int_0^t J_0(t)\cdot J_0(t-u)\,du = L\cdot \sin t$

RHS $= \dfrac{1}{s^2+1}$, *LHS is convolution theorem.*

$\therefore$ *LHS* $= L J_0(t)\cdot L J_0(t)$

$$= \frac{1}{\sqrt{s^2+1}}\cdot\frac{1}{\sqrt{s^2+1}} = \frac{1}{s^2+1}$$

Thus LHS = RHS

7.9 Type VII : Laplace Inverse using Standard Series

Example 7.61 : $L^{-1}\,\dfrac{1}{s}\cos\dfrac{1}{s}$

Solution : Step 1 : Using infinite series (expansion), we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots\ldots$$

$$\therefore \quad \cos\frac{1}{s} = 1 - \frac{1}{2!}\frac{1}{s^2} + \frac{1}{4!}\frac{1}{s^4} - \frac{1}{6!}\frac{1}{s^6}\ldots$$

Step 2 :

$$\therefore \quad \frac{1}{s}\cos\frac{1}{s} = \frac{1}{s} - \frac{1}{2!}\frac{1}{s^3} + \frac{1}{4!}\frac{1}{s^5} - \frac{1}{6!}\frac{1}{s^7}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r)!}\frac{1}{s^{2r+1}}$$

Step 3 : $\therefore \quad L^{-1}\left[\frac{1}{s}\cos\frac{1}{s}\right] = L^{-1}\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r)!}\frac{1}{s^{2r+1}}\right]$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{(2r)!}L^{-1}\left[\frac{1}{s^{2r+1}}\right]$$

Step 4 : Using the formula $L^{-1}\frac{1}{s^{n+1}} = \frac{t^n}{n!}$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{(2r)!}\frac{t^{2r}}{(2r)!}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r\, t^{2r}}{[(2r)!]^2}$$

Example 7.62 : *If s is sufficiently large, show using series expansion of* $\tan^{-1}\left(\frac{a}{s}\right)$ *that* $L^{-1}\left[\tan^{-1}\left(\frac{a}{s}\right)\right] = \frac{\sin at}{t}$.

Solution : Step 1 : Using infinite series (expansion), we have,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Step 2 : $\therefore \tan^{-1}\left(\frac{a}{s}\right) = \left(\frac{a}{s}\right) - \frac{1}{3}\left(\frac{a^3}{s^3}\right) + \frac{1}{5}\left(\frac{a^5}{s^5}\right) - \frac{1}{7}\left(\frac{a^7}{s^7}\right)\ldots.$

$$\therefore \quad L^{-1}\left[\tan^{-1}\left(\frac{a}{s}\right)\right] = aL^{-1}\left[\frac{1}{s}\right] - \frac{a^3}{3}L^{-1}\left[\frac{1}{s^3}\right] + \frac{a^5}{5}L^{-1}\left[\frac{1}{s^5}\right] - \frac{a^7}{7}L^{-1}\left[\frac{1}{s^7}\right]\ldots$$

Step 3 : Using the formula $L^{-1}\frac{1}{s^{n+1}} = \frac{t^n}{n!}$

Step 4 : $\therefore \quad f(t) = a\cdot 1 - \frac{a^3}{3}\cdot\frac{t^2}{2!} + \frac{a^5}{5}\cdot\frac{t^4}{4!} - \frac{a^7}{7}\cdot\frac{t^6}{6!}$

$$= \frac{1}{t}\left[(at) - \frac{(at)^3}{3!} + \frac{(at)^5}{5!} - \frac{(at)^7}{7!}\ldots\right]$$

$$= \frac{\sin at}{t} \qquad \text{... By infinite series expansion}$$

Example 7.63 : $L^{-1}\left[\dfrac{1}{\sqrt{1+s^2}}\right]$

Solution : Step 1 : We expand $\dfrac{1}{\sqrt{1+s^2}}$ in powers of $\dfrac{1}{s}$

Using $$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots$$

$$\frac{1}{\sqrt{1+s^2}} = \frac{1}{s\sqrt{1+\frac{1}{s^2}}} = \frac{1}{s}\left(1+\frac{1}{s^2}\right)^{1/2}$$

$$= \frac{1}{s}\left\{1+\left(\frac{-1}{2}\right)\left(\frac{1}{s^2}\right)+\frac{(-1/2)(-1/2-1)}{2!}\left(\frac{1}{s^2}\right)^2+\ldots\right\}$$

$$= \frac{1}{s}-\frac{1}{2}\frac{1}{s^3}+\frac{3}{8}\frac{1}{s^5}\ldots$$

Step 2 : $$\therefore L^{-1}\left[\frac{1}{\sqrt{1+s^2}}\right] = L^{-1}\left[\frac{1}{s}-\frac{1}{2}\cdot\frac{1}{s^3}+\frac{3}{8}\frac{1}{s^5}\ldots\right]$$

$$= L^{-1}\left[\frac{1}{s}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{s^3}\right]+\frac{3}{8}L^{-1}\left[\frac{1}{s^5}\right]\ldots.$$

Step 3 : Using the formula $L^{-1}\dfrac{1}{s^{n+1}} = \dfrac{t^n}{n!}$

$$= 1-\frac{1}{2}\frac{t^2}{2!}+\frac{3}{8}\frac{t^4}{4!}\ldots.$$

Example 7.64 : $L^{-1}\left[\log\left(1+\dfrac{1}{s}\right)\right]$

Solution : Step 1 : Using infinite series, we have

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots.$$

$$\therefore \quad \log\left(1+\frac{1}{s}\right) = \frac{1}{s}-\frac{1}{2}\cdot\frac{1}{s^2}+\frac{1}{3}\cdot\frac{1}{s^3}-\frac{1}{4}\cdot\frac{1}{s^4}\ldots.$$

Step 2 : $$\therefore L^{-1}\left[\mathrm{Log}\left(1+\frac{1}{s}\right)\right] = L^{-1}\left[\frac{1}{s}-\frac{1}{2}\frac{1}{s^2}+\frac{1}{3}\frac{1}{s^3}-\frac{1}{4}\frac{1}{s^4}\ldots.\right]$$

$$= L^{-1}\left[\frac{1}{s}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{s^2}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{s^3}\right]-\frac{1}{4}L^{-1}\left[\frac{1}{s^4}\right]\ldots$$

Step 3 : Using the formula $L^{-1}\dfrac{1}{s^{n+1}} = \dfrac{t^n}{n!}$

Step 4 :

$$f(t) = 1 - \frac{1}{2}t + \frac{1}{3}\frac{t^2}{2!} - \frac{1}{4}\frac{t^3}{3!} \ldots$$

$$= \frac{1}{t}\left(t - \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} \ldots\right) = \frac{1 - e^{-t}}{t}$$

Example 7.65 : $\frac{1}{s} \sin \frac{1}{s}$.

Solution : Step 1 : Let $\phi(s) = \frac{1}{s} \sin \frac{1}{s}$

Using expansion of sin x, we have, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\ldots$

$$\therefore \quad \sin \frac{1}{s} = \frac{1}{s} - \frac{1}{3!}\frac{1}{s^3} + \frac{1}{5!}\frac{1}{s^5} \ldots$$

Step 2 : $\therefore$

$$\phi(s) = \frac{1}{s} \sin \frac{1}{s} = \frac{1}{s^2} - \frac{1}{3!}\frac{1}{s^4} + \frac{1}{5!}\frac{1}{s^6} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{s^{2n+2}}$$

Step 3 : $\therefore$

$$L^{-1}\phi(s) = L^{-1}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} L^{-1}\frac{1}{s^{2n+2}}$$

Step 4 : Using the formula $L^{-1}\frac{1}{s^{n+1}} = \frac{t^n}{n!}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{t^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{\{(2n+1)!\}^2}$$

Exercise 7.7

Obtain inverse Laplace transforms of the following using series.

1. $\frac{1}{s^2+1}$ **[Ans. : sin t]**

2. $\frac{s}{s^2+1}$ **[Ans. : cos t]**

3. $\frac{1}{s^3+1}$ **[Ans. :** $\frac{t^2}{2!} - \frac{t^5}{5!} + \frac{t^8}{8!} - \frac{t^{11}}{11!} \ldots$**]**

4. $\frac{\sqrt{s}}{s-1}$ **[Ans. :** $\frac{1}{\sqrt{\pi}}\left[t^{1/2} + 2t^{1/2} + \frac{4}{3}t^{3/2} \ldots\right]$**]**

5. $\frac{1}{s+e^{-s}}$ **[Ans. :** $\sum_{a=0}^{\infty} \frac{(-1)^a}{a!}(t-a)^a U(t-a)$**]**

7.10 Laplace Transform of Special Functions

Heaviside's Unit Step Function : The name of the function itself indicates that the function involves, a step of unit height.

Definition :

$$U(t) \text{ or } H(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

Thus the function U (t) is zero when $t < 0$ and one when $t \geq 0$.

i.e.

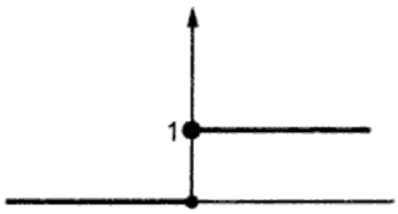


Fig. 7.1

Also the displaced unit step function is denoted by H (t – a) or U (t – a) and is defined by

$$U(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases}$$

i.e. the function is one after $t \geq a$.

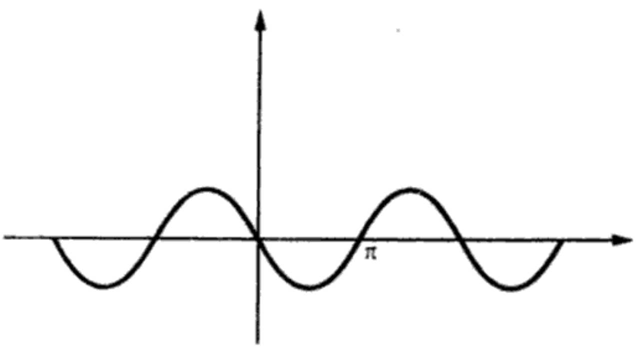
Note : Use of the function

a) Suppose we need certain function f (t) after the point t = a, then it can be obtained by f (t). U (t – a) i.e. by multiplying f (t) by U (t – a). This concept will be more clear by using the figures below.

Let f (t) = sin t

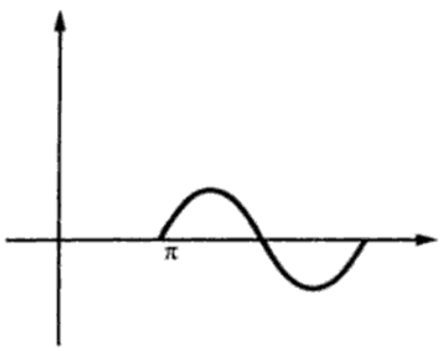
∴ graph of f (t) is as follows

The graph of U (t–π) is as follows.



f(t) = sin t
sine wave

Fig. 7.2



g(t) = sin t U(t – π)
sine wave starting from x = π

Fig. 7.3

$$U(t-\pi) = \begin{cases} 0 & \text{for } t < \pi \\ 1 & \text{for } t \geq \pi \end{cases}$$

When we consider g (t) = sin t U (t − π). As U (t − π) is zero for t < π, thus the value of g (t) will be zero for t < π and as U (t − π) is one for t ≥ π, thus the value of g (t) will be $\sin t \times (1)$ for t ≥ π

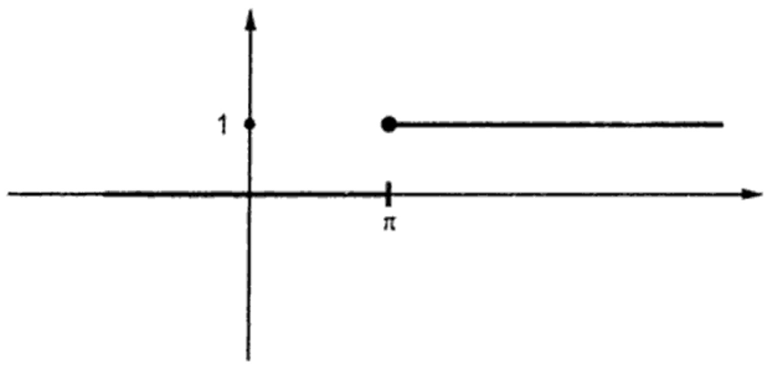


Fig. 7.4

∴ we can represent g (t) as $g(t) = \begin{cases} 0 & \text{for } t < \pi \\ \sin t & \text{for } t \geq \pi \end{cases}$

Thus g (t) = sin t U (t − π) exists only for t ≥ π. Generalising this the multiplication of f (t) with U (t − a) i.e. delayed or displaced unit step function gives f (t) after point a i.e. the portion of f (t) for t < a becomes zero.

b) **The displaced function f (t − a) U (t − a)**

Let $g(t) = f(t-a)\, U(t-a)$

i.e. $$g(t) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$$

which is a displaced function of f (t) by a. Observe the following figures.

Thus if f (t) is the original function then f (t − a) U (t − a) is its displaced function.

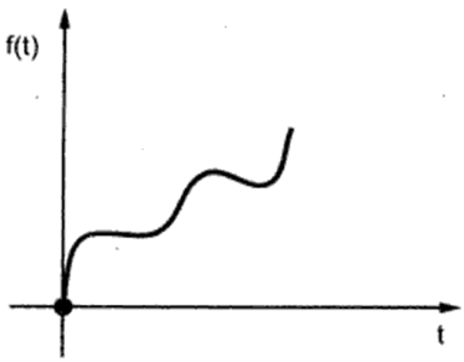


Fig. 7.5 Graph of f (t)

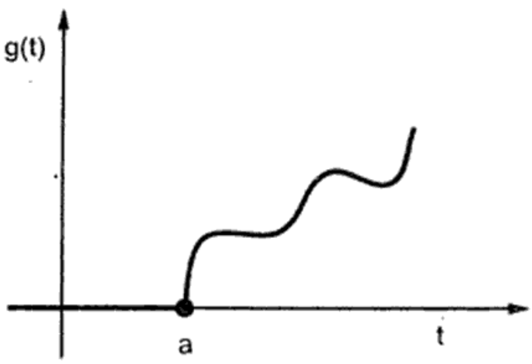


Fig. 7.6 Graph of displaced function g (t) = f (t–a) U (t–a)

c) The function f (t) within the interval $a \le t \le b$. Observe the following figures.

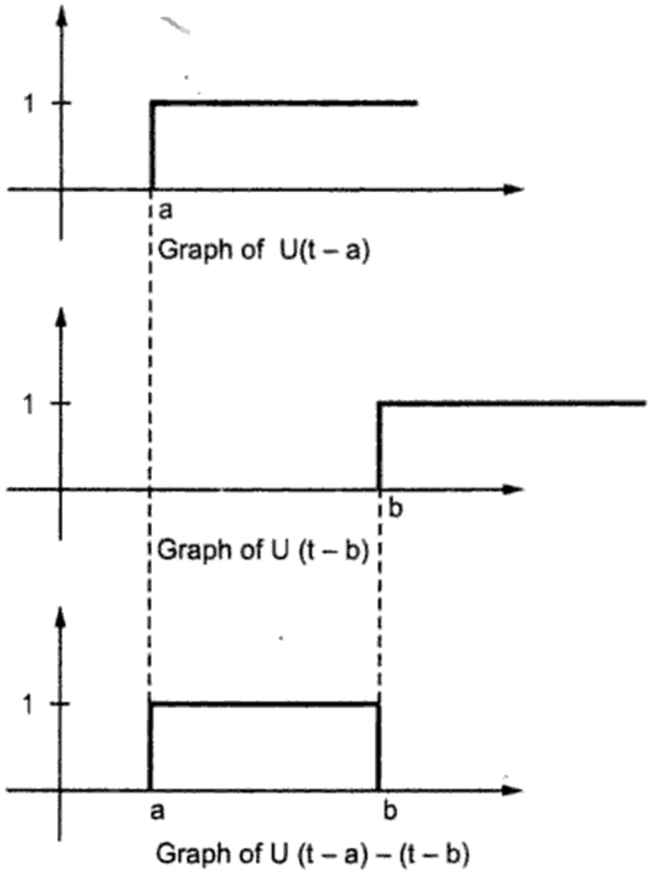


Fig. 7.7

What will happen if we subtract U (t – b) from U (t – a). We get the portion of U(t – a) after deleting the portion of U (t – b).

Thus U (t – a) – U (t – b) gives the non zero function "1" within the interval $a \le t \le b$.

Thus we have succeeded in finding "1" within the interval [a, b].

Now to find any function within this interval just multiply by that function to [U (t – a) – U (t – b)].

Thus if f (t) = cos t for $\pi/2 \le t \le \pi$ we can express f (t) in unit step function as

$$f(t) = \cos t \left[U\left(t - \frac{\pi}{2}\right) - U(t - \pi) \right]$$

7.11 Laplace Transforms using Heaviside Unit Step Function

a) $L[H(t-a)] = \frac{e^{-as}}{s}$

Proof : Step 1 : By definition of Laplace transforms

$$L[H(t-a)] = \int_0^{\infty} e^{-st} H(t-a)\, dt$$

Step 2 : As there are two values of H (t – a) split the integral into two integrals.

$$= \int_0^a e^{-st}\,(0)\,dt + \int_a^\infty e^{-st}\,(1)\,dt$$

Step 3 : Integrate and substitute the limits

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= 0 - \frac{e^{-as}}{-s} \qquad (\text{as } e^{-\infty} = 0)$$

$$= \frac{e^{-as}}{s}$$

b) Substituting a = 0 we get

$$L\,(H\,(t)) = \frac{1}{s}$$

c) $L\,[f\,(t-a)\,U\,(t-a)] = e^{-as}\,\phi\,(s)$ where $L\,f\,(t) = \phi(s)$

Proof : Step 1 : We know that

$$f\,(t-a)\;U\,(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$$

Step 2 : By definition of Laplace transforms

$$L\,f\,(t-a)\;U\,(t-a) = \int_0^\infty e^{-st}\,f\,(t-a)\cdot U\,(t-a)\,dt$$

Step 3 : Split the integral in two parts.

$$= \int_0^a e^{-st}\,0\,dt + \int_a^\infty e^{-st}\,f\,(t-a)\,dt$$

Step 4 : Put (t – a) = u, dt = du

t	a	∞
u	0	∞

$$= 0 + \int_0^\infty e^{-s\,(a+u)}\,f\,(u)\,du$$

$$= e^{-as}\int_0^\infty e^{-su}\,f\,(u)\,du$$

$$= e^{-as} \, L f(u) \qquad \text{By definition of LT.}$$

$$= e^{-as} \, \phi(s)$$

i.e. $\boxed{L f(t-a) \, U(t-a) = e^{-as} \, L f(t)}$

which is second shifting property from Laplace transforms.

d) $L f(t) \, U(t-a) = e^{-as} \, L f(t+a)$

Proof : Step 1 : By definition of Laplace transforms

$$L f(t) \, U(t-a) = \int_0^{\infty} e^{-st} \, f(t) \, U(t-a) \, dt$$

Step 2 : Split the integral

$$= \int_0^{a} e^{-st} f(t) \cdot 0 \, dt + \int_a^{\infty} e^{-st} \, f(t) \cdot 1 \, dt$$

Step 3 : Put $t - a = u$, $t = a + u$, $dt = du$

t	a	∞
u	0	∞

$$= 0 + \int_0^{\infty} e^{-s(a+u)} \cdot f(u+a) \, du$$

$$= e^{-as} \int_0^{\infty} e^{-su} \, f(u+a) \, du$$

$$= e^{-as} \, L f(u+a) \qquad \text{By definition}$$

As the variable is not important in definite integrals.

$$= e^{-as} \, L f(t+a)$$

Thus

$$\boxed{L \, f(t) \, U(t-a) = e^{-as} \, L f(t+a)}$$

e) By substituting $a = 0$ we get

$$L \, [f(t) \, U(t)] = e^{-0s} \, L f(t+0)$$

$$= L \, f(t)$$

$\therefore \quad \boxed{L \, f(t) \, U(t) = L \, f(t)}$

7.12 Type VIII: Problems on Laplace Transforms using Unit Step Function

Illustrations

Example 7.66 : *Find* $L\,e^{-3t}\,H(t-2)$

Solution : We know that,

$$L\,H(t-a) = \frac{e^{-as}}{s}$$

$$\therefore \quad L\,H(t-2) = \frac{e^{-2s}}{s}$$

By first shifting property $L\,e^{at}\,[f(t)] = \phi(s-a)$ using this property we get,

$$L\,e^{-3t}\,H(t-2) = \frac{e^{-2(s+3)}}{(s+3)}$$

Example 7.67 : *Find* $L\{\sin t\,u(t-\pi)\}$

Solution . We know that

$$L\,f(t)\,u(t-a) = e^{-as}\,L\,[f(t+a)]$$

$$\therefore \quad L\sin t\,u(t-\pi) = e^{-\pi s}\,L\,[\sin(t+\pi)]$$

$$= e^{-\pi s}\,L(-\sin t)$$

$$= e^{-\pi s}\cdot\frac{-1}{(s^2+1)}$$

$$= \frac{-e^{-\pi s}}{s^2+1}$$

Example 7.68 : *Find Laplace of* $e^{-t}\sin t\,u(t-\pi)$

Solution :

$$L\sin t\,u(t-\pi) = e^{-\pi s}\,L\,[\sin(t+\pi)]$$

$$= e^{-\pi s}\,L(-\sin t)$$

$$= e^{-\pi s}\cdot\frac{-1}{s^2+1}$$

$$= \frac{-e^{-\pi s}}{s^2+1}$$

By first shifting property $L\,e^{at}\,[f(t)] = \phi(s - a)$ using this property we get,

$$L\,e^{-t}\sin t\,u(t - \pi) = \frac{-e^{\pi(s+1)}}{(s+1)^2 + 1}$$

Example 7.69 : *Find* $L\left(1 + 2t + 3t^2 + 4t^3\right)H(t - 2)$

Solution : $L\,f(t)\,u(t - a) = e^{-as}\,L\,[f(t + a)]$

$$\therefore\ L\left(1 + 2t + 3t^2 + 4 + 3\right)H(t - 2)$$

$$= e^{-2s}\left[L(1) + 2\,(t + 2) + 3\,(t + 2)^2 + 4\,(t + 2)^3\right]$$

$$= e^{-2s}\left\{L\left[1 + 2t + 4 + 3\left(t^2 + 2t + 4\right) + 4\,(t + 2)\,(t + 2)\right]\right\}$$

$$= e^{-2s}\left\{L\left[5 + 2t + 3t^2 + 6t + 12 + 4\,(t + 2)\left(t^2 + 2t + 4\right)\right]\right\}$$

$$= e^{-2s}\left\{L\left[17 + 8t + 3t^2 + 4\left(t^3 + 2t^2 + 4t + 2t^2 + 4t + 8\right)\right]\right\}$$

$$= e^{-2s}\left\{L\left[49 + 40t + 19t^2 + 4t^3\right]\right\}$$

Now we find the transform using formula $L\left(t^n\right) = \dfrac{n!}{s^{n+1}}$

$$= e^{-2s}\left[\frac{49}{s} + \frac{40}{s^2} + \frac{19 \times 2!}{s^3} + \frac{4 \times 3!}{s^4}\right]$$

$$= e^{-2s}\left[\frac{49}{s} + \frac{40}{s^2} + \frac{38}{s^3} + \frac{24}{s^4}\right]$$

Example 7.70 : *Find L f (t) from the following figure :*

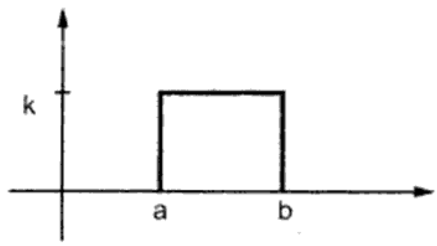


Fig. 7.8

Solution : Here the function f (t) is given by,

Thus $\quad f(t) = k\,[u(t - a) - u(t - b)]$

Using $L\,[f(t)\,u(t - a)] = e^{-as}\,L\,[f(t + a)]$ we get

$$L\,f(t) = k\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right] \qquad \text{As k is constant}$$

Example 7.71 : *Find Laplace transform of* $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$

Solution : Here f(t) in terms of unit step function is given by

$$f(t) = \cos t\,[u(t) - u(t-\pi)] + \sin t\,[u(t-\pi)]$$

$$f(t) = \cos t\, u(t) - \cos t\, u(t-\pi) + \sin t\, u(t-\pi)$$

$$L\,f(t) = \frac{s}{s^2+1} e^{0s} - e^{-\pi s} L \cos(t+\pi) + e^{\pi s} L \sin(t+\pi)$$

$(\sin(\pi + t) = -\sin t,\ \cos(\pi + t) = -\cos t)$

$$L\,f(t) = \frac{s}{s^2+1} + e^{-\pi s} \frac{s}{s^2+1} - \frac{\pi}{s^2+\pi^2} e^{\pi s}$$

Example 7.72 : *Find* $L f(t)$ *from the following Fig. 7.9.*

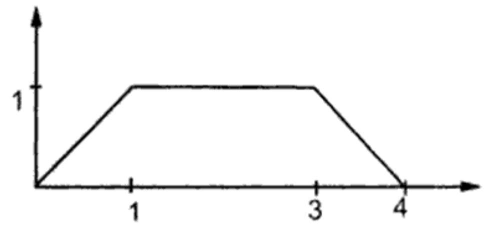


Fig. 7.9

Note : In such a problem firstly find equations of all the line segments. i.e. Here find equations of lines joining (0,0) to (1,1), (1, 1) to (3, 1), (3, 1) to (4, 0) these equations are $y = x$, $y = 1$, $y = 4 - x$ respectively then replace y by f(t) and x by t to write f(t) in terms of t.

Solution : Here

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 1 & 1 < t < 3 \\ 4 - t & 3 < t < 4 \end{cases}$$

Here f(t) in terms of unit step function is given by

$$\therefore \quad f(t) = t[u(t) - u(t-1)] + 1[u(t-1) - u(t-3)] + (4-t)[u(t-3) - u(t-4)]$$

$$\therefore \quad f(t) = t\,u(t) - (t-1)\,u(t-1) - (t-3)\,u(t-3) + (t-4)\,u(t-4)$$

$$L\,f(t) = e^{0s}\frac{1}{s^2} - e^{-s}\frac{1}{s^2} - e^{3s}\frac{1}{s^2} + e^{-4s}\frac{1}{s^2}$$

$$= \frac{1}{s^2}\left(1 - e^{-s} - e^{-3s} + e^{-4s}\right)$$

Example 7.73 : *Find $L f(t)$ from the following Fig. 7.10.*

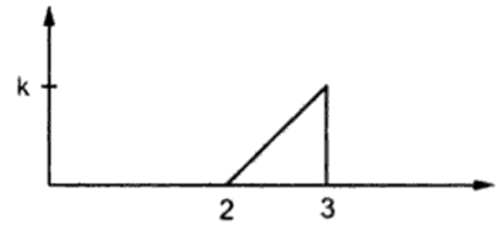


Fig. 7.10

Solution : To find the equation of line joining (2, 0) and (3, k).

i.e. $$\frac{y-0}{(k-0)} = \frac{x-2}{3-2}$$

$$y = k(x-2) \quad \text{for} \quad 2 < x < 3$$

Thus $f(t) = k(t-2) \quad 2 < t < 3 \quad \therefore$ f(t) in terms of unit step function is

$$f(t) = k(t-2)[u(t-2) - u(t-3)]$$

$$= k[(t-2)\,u(t-2) - (t-2)\,u(t-3)]$$

$$L\,f(t) = k\left[e^{-2s} L(t+2-2) - e^{-3s} L(t+3-2)\right]$$

$$= k\left[e^{-2s} L(t) - e^{-3s} L(t+1)\right]$$

$$= k\left[e^{-2s}\frac{1}{s^2} - e^{-3s}\left[\frac{1}{s^2}+\frac{1}{s}\right]\right]$$

$$= \frac{k}{s^2}\left[e^{-2s} - e^{-3s}(s+1)\right]$$

Example 7.74 : *Find $L f(t)$ from the following Fig. 7.11.*

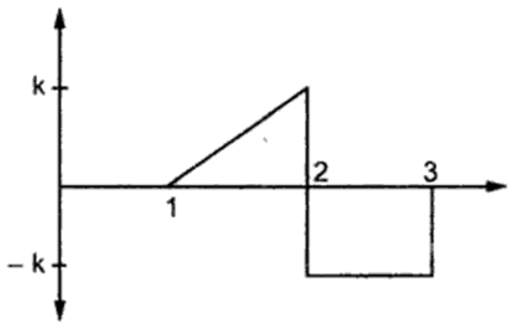


Fig. 7.11

Solution : Here the equations of lines joining (1, 0) to (2, k) and (2, – k) to (3, – k) are

$$y = k(x-1) \quad \text{and} \quad y = -k$$

Here $f(t) = k(t-1) \quad 1 < t < 2$

$= -k \quad 2 < t < 3$

Here f(t) in terms of unit step function is given by

$$f(t) = k(t-1)[u(t-1) - u(t-2)] - k[u(t-2) - u(t-3)]$$

$$= k\left[(t-1)u(t-1) - (t-1)u(t-2) - u(t-2) + u(t-3)\right]$$

$$L\, f(t) = k\left[e^{-s} L(t+1-1) - e^{-2s} L(t+2-1) - \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}\right]$$

$$= k\left[\frac{e^{-s}}{s^2} - e^{-2s}\frac{(s+1)}{s^2} - \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}\right]$$

$$= k\left\{\frac{e^{-s}}{s^2} - e^{-2s}\frac{(s+2)}{s^2} + \frac{e^{-3s}}{s}\right\}$$

Example 7.75 : $f(t) = \begin{cases} t^2 & 0 < t < 2 \\ 2t & t \geq 2 \end{cases}$ *Find* $L\, f(t)$

Solution : Here f(t) in terms of unit step function is given by

$$f(t) = t^2\left[u(t) - u(t-2)\right] + 2t\left[u(t-2)\right]$$

$$= t^2 u(t) - t^2 u(t-2) + 2tu(t-2)$$

$$L\, f(t) = e^{-0s} L(t^2) - e^{-2s} L(t+2)^2 + 2e^{-2s} L(t+2)$$

$$= \frac{2!}{s^3} - e^{-2s} L\left(t^2 + 4t + 4\right) + 2e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$$

$$= \frac{2}{s^3} - e^{-2s}\left[\frac{2}{s^3} + 4\frac{1}{s^2} + \frac{4}{s}\right] + e^{-2s}\left[\frac{s+2}{s^2}\right]$$

$$= \frac{2}{s^3} - e^{-2s}\left[\frac{2 + 4s + 4s^2}{s^3}\right] + 2e^{-2s}\left[\frac{s+2}{s^2}\right]$$

$$= \frac{2}{s^3} - e^{-2s}\left[\frac{2 + 4s + 4s^2 + 2s^2 + 4s}{s^3}\right]$$

$$= \frac{2}{s^3} - e^{-2s}\left[\frac{6s^2 + 8s + 2}{s^3}\right]$$

Example 7.76 : $f(t) = \begin{cases} \sin 2t & 0 < t < \pi \\ 0 & t > \pi \end{cases}$ *Find* $L\, f(t)$

Solution : Here f(t) in terms of unit step function is given by

$$f(t) = \sin 2t\left[u(t) - u(t-\pi)\right]$$

$$= \sin 2t\, u(t) - \sin 2t\, u(t-\pi)$$

$$L[f(t)] = L\left[\sin 2t\, u(t) - \sin 2t\, u(t-\pi)\right]$$

As $L[f(t)\, u(t)] = L[f(t)]$, ...

$$L[f(t)\,u(t-a)] = e^{-as} L[f(t+a)]$$

$$\therefore \quad L[f(t)] = L[\sin 2t] - e^{-\pi s} L[\sin 2(t+\pi)]$$

$$= L[\sin 2t] - e^{-\pi s} L[\sin 2t]$$

$$L[f(t)] = \left(1 - e^{-\pi s}\right)\frac{2}{s^2+4}$$

Exercise 7.8

Find Laplace transforms of the following using unit step function.

1. $f(t) = \begin{cases} t^2 & 0 < t < 1 \\ 4t & t > 1 \end{cases}$ **[Ans. : $\frac{2}{s^3} + e^{-s}\left[\frac{2}{s^2} + \frac{3}{s} - \frac{2}{s^3}\right]$]**

2. $f(t) = t^4 U(t-2)$ **[Ans. : $e^{-2s}\left[\frac{24}{s^5} + \frac{48}{s^4} + \frac{48}{s^3} + \frac{32}{s^2} + \frac{16}{s}\right]$]**

3. $f(t) = [2 + \cosh(t-1)]\, U(t-1)$ **[Ans. : $e^{-s}\left[\frac{2}{s} + \frac{s}{s^2-1}\right]$]**

4. $f(t) = \begin{cases} t^2 & 0 < t < 2 \\ t-1 & 2 < t < 3 \\ t^3 & t > 3 \end{cases}$ **[Ans. : $\frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{3}{s}\right) + e^{-3s}\left(\frac{6}{s^4} + \frac{6}{s^3} + \frac{2}{s^2} + \frac{25}{s}\right)$]**

5. $f(t) = \sin t \quad \frac{\pi}{2} < t < \frac{3\pi}{2}$ **[Ans. : $\frac{s}{s^2+1}\left[e^{-\pi s/2} + e^{-3\pi s/2}\right]$]**

6. $f(t) = \begin{cases} t+1 & 0 \le t \le 2 \\ 3 & t > 2 \end{cases}$ *also find* $L\, f'(t)$ **[Ans. : $L\, f(t) = \frac{1}{s} + \frac{1}{s^2}\left(1 - e^{-2s}\right) = \phi(s)$ say**
$L\, f'(t) = -f(0) + s\,\phi(s)$]

7. $\left(t^2 + 3t + 2\right) U(t-1)$ **[Ans. : $e^{-s}\left[\frac{2}{s^3} + \frac{5}{s^2} + \frac{6}{s}\right]$]**

8. $f(t) = \begin{cases} (t-2)^4 & t > 2 \\ 0 & t < 2 \end{cases}$ **[Ans. : $e^{-2s} \cdot \frac{24}{s^3}$]**

9. $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ t & t > \pi \end{cases}$ **[Ans. : $\frac{1 + e^{-\pi s}}{s^2+1} + \frac{e^{-\pi s}(1+\pi s)}{s^2}$]**

10. $f(t) = t\,e^{-2t} U(t-1)$ **[Ans. : $e^{-(s+2)}\frac{s+3}{(s+2)^2}$]**

11. $f(t) = \left(1 + 2t - t^2 + t^3\right) U(t-1)$ **[Ans. : $e^{-s}\left(\frac{6}{s^4} + \frac{4}{s^3} + \frac{3}{s^2} + \frac{3}{s}\right)$]**

12. $f(t) = \cos^2 t\; U(t-\pi)$ **[Ans. : $\frac{e^{-\pi s}}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)$]**

13. *Find the Laplace transform of the staircase function* $f(t) = K(n-1)$ *for* $(n-1)\,T < t < nT$, $n = 1, 2, 3 \ldots$ *where T is period.*

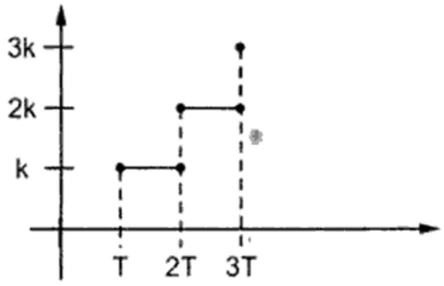


Fig. 7.12

Hint : Given $f(t) = K \qquad T < t < 2T$

$= 2K \qquad 2T < t < 3T$

$= 3K \qquad 3T < t < 4T$

$\vdots \qquad\qquad \vdots$

i.e. $f(t) = K\,[U(t-T) - U(t-2T)]$

$+ 2K\,[U(t-2t) - U(t-3t)]$

$+ 3K\,[U(t-3t) - U(t-4t)]$ .... and so on.

i.e. $f(t) = K\{U(t-T) + U(t-2T) + U(t-3T) \ldots\}$

$$L\, f(t) = K\left\{\frac{e^{-Ts}}{s} + \frac{e^{-2Ts}}{s} + \frac{e^{-3Ts}}{s} \ldots\right\}$$

$$= \frac{K}{s}\left\{e^{-Ts} + e^{-2Ts} + e^{-3Ts} \ldots\right\}$$

then use Geometric series $a + ar + ar^2 \ldots = \frac{a}{1-r}$ if $|r| < 1 \; = \; \frac{k}{s}\,\frac{e^{-Ts}}{1-e^{-Ts}}$

14. Represents the following functions in terms of unit step function and hence find their Laplace transforms.

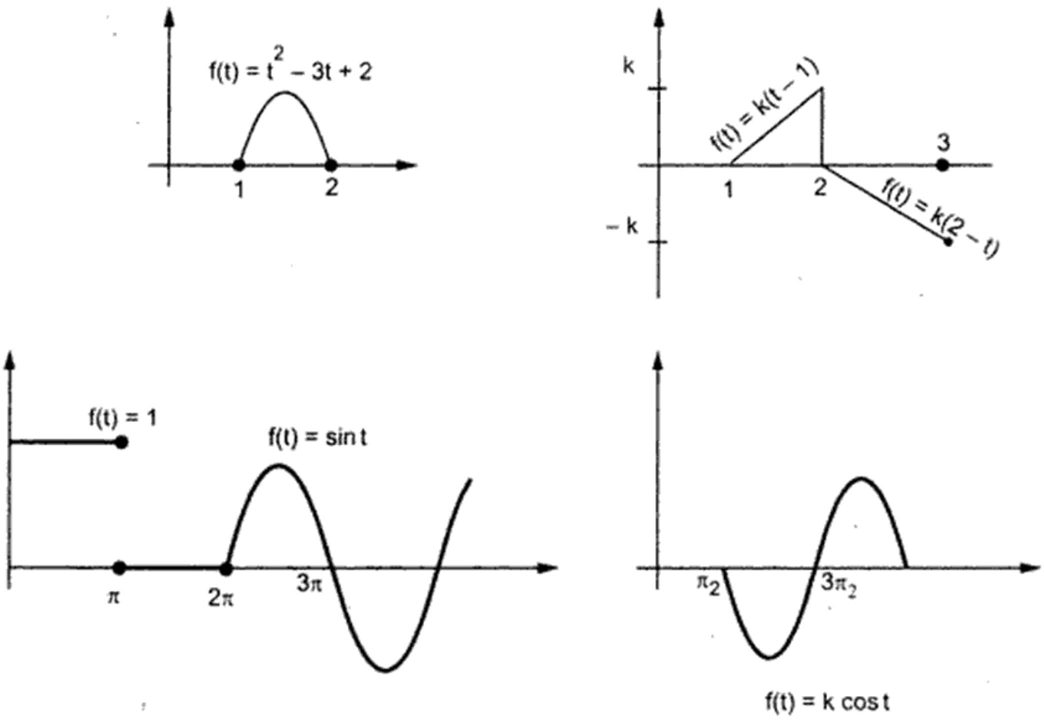


Fig. 7.13

[Ans. : i) $e^{-s}\left(\frac{2}{s^3} - \frac{1}{s^2}\right) - e^{-2s}\left(\frac{2}{s^3} + \frac{1}{s^2}\right)$

ii) $\frac{k}{s^2}\left\{e^{-s} + (3s - 2)\, e^{-2s} + (s + 1)\, e^{-3s}\right\}$

iii) $\frac{1}{s} - \frac{e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2 + 1}$

iv) $\frac{-k}{s^2 + 1}\left\{e^{-\pi s/2} + e^{-2\pi s}\right\}$]

15. *Find Laplace transform of the following*

i) $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ t & t > \pi \end{cases}$ [Ans. : $\frac{1 + e^{-\pi s}}{s^2 + 1} + e^{-\pi s}\left(\frac{\pi s + 1}{s^2}\right)$]

ii) $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ \sin 2t & \pi < t < 2\pi \\ \sin 3t & t > 2\pi \end{cases}$

[Ans. : $\frac{1}{s^2 + 1} + e^{-\pi s}\left(\frac{2}{s^2 + 4} + \frac{1}{s^2 + 1}\right) - e^{-2\pi s}\left(\frac{3}{s^2 + 9} + \frac{2}{s^2 + 4}\right)$]

iii) $f(t) = \begin{cases} 0 & 0 < t < \pi/2 \\ \sin t & \pi/2 < t < 3\pi/2 \\ \sin 3t & t > 2\pi \end{cases}$ [Ans. : $\frac{s}{s^2 + 1}\left\{e^{-\pi s/2} + e^{-3\pi s/2}\right\}$

7.13 Type IX Illustrations on Laplace inverse using $L^{-1}\, e^{-as}\, \phi(s) = f\,(t - a)\; U\,(t - a)$

Example 7.77 : *Find* $L^{-1}\, \frac{e^{-as}}{(s + b)^{5/2}}$

Solution : Let

$$\phi(s) = \frac{1}{(s + b)^{5/2}}$$

$$L^{-1}\,\phi(s) = e^{-bt}\, L^{-1}\,\frac{1}{s^{5/2}}$$

$$= e^{-bt}\,\frac{t^{3/2}}{\Gamma(5/2)}$$

$$f(t) = \frac{e^{-bt}\, t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{4e^{-bt}\, t^{3/2}}{3\sqrt{\pi}}$$

$$L^{-1}\, e^{-as}\,\frac{1}{(s + b)^{5/2}} = \left(\frac{4}{3\sqrt{\pi}}\, e^{-b(t-a)} (t-a)^{3/2}\right) u\,(t-a)$$

Example 7.78 : *Find* $L^{-1} \dfrac{e^{-2s}}{\sqrt{s+5}}$

Solution :

$$\phi(s) = \frac{1}{\sqrt{s+5}}$$

$$L^{-1}\,\phi(s) = e^{-5t}\, L^{-1} \frac{1}{\sqrt{s}}$$

$$= e^{-5t} \frac{t^{-1/2}}{\Gamma(1/2)}$$

$$= e^{-5t} \frac{1}{\sqrt{\pi t}}$$

$$L^{-1}\, e^{-2s}\,\phi(s) = \frac{e^{-5(t-2)}}{\sqrt{\pi(t-2)}}\, u(t-2)$$

Example 7.79 : *Find* $L^{-1} \dfrac{\left(1-\sqrt{s}\right) e^{-s}}{s^{3/2}}$

Solution :

$$\phi(s) = \frac{1-\sqrt{s}}{s^{3/2}}$$

$$= \frac{1}{s^{3/2}} - \frac{s^{1/2}}{s^{3/2}}$$

$$= \frac{1}{s^{3/2}} - \frac{1}{s}$$

$$L^{-1}\,\phi(s) = L^{-1} \frac{1}{s^{3/2}} - L^{-1} \frac{1}{s}$$

$$= \frac{t^{3/2-1}}{\Gamma(3/2)} - 1$$

$$= \frac{\sqrt{t}}{\frac{1}{2}\sqrt{\pi}} - 1$$

$$= \frac{2\sqrt{t}}{\sqrt{\pi}} - 1$$

$$L^{-1}\, e^{-s}\,\phi(s) = \left\{\frac{2\sqrt{t-1}}{\sqrt{\pi}} - 1\right\} u(t-1)$$

Example 7.80 : *Find* $L^{-1}\left[\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right]$

Solution : $L^{-1}\left[\frac{se^{-s/2}}{s^2+\pi^2}+\frac{\pi e^{-s}}{s^2+\pi^2}\right] = L^{-1}\left[\frac{s\,e^{-s/2}}{s^2+\pi^2}\right] + L^{-1}\left[\frac{\pi\,e^{-s}}{s^2+\pi^2}\right]$

Consider (i) $L^{-1}\, e^{-s/2}\frac{s}{s^2+\pi^2}$

Consider
$$\phi(s) = \frac{s}{s^2+\pi^2}$$
$$L^{-1}\phi(s) = L^{-1}\frac{s}{s^2+\pi^2}$$
$$= \cos\pi t$$
$$L^{-1}\, e^{-s/2}\phi(s) = \left[\cos\pi\left(t-\frac{1}{2}\right)\right] u\left(t-\frac{1}{2}\right)$$
$$= \left[\cos\left(\pi t-\frac{\pi}{2}\right)\right] u\left(t-\frac{1}{2}\right)$$
$$= \sin\pi t\; u\left(t-\frac{1}{2}\right) \qquad \dots (1)$$

To find (ii) $L^{-1}\, e^{-s}\frac{\pi}{s^2+\pi^2}$

Consider
$$\phi(s) = \frac{\pi}{s^2+\pi^2}$$
$$L^{-1}\phi(s) = L^{-1}\frac{\pi}{s^2+\pi^2}$$
$$= \pi L^{-1}\frac{1}{s^2+\pi^2}$$
$$= \pi\frac{1}{\pi}\sin\pi t$$
$$= \sin\pi t$$
$$L^{-1}\, e^{-s}\phi(s) = [\sin\pi(t-1)]\, u(t-1)$$
$$= [\sin(\pi t-\pi)]\, u(t-1)$$
$$= -(\sin\pi t)\, u(t-1) \qquad \dots (2)$$

$\therefore$ from (i) and (ii)
$$\therefore L^{-1}\left[\frac{se^{-s/2}}{s^2+\pi^2}+\frac{\pi e^{-s}}{s^2+\pi^2}\right] = (\sin\pi t)\, u(t-1/2) - [\sin\pi t\;\, u(t-1)]$$
$$= \sin\pi t\left[u\left(t-\frac{1}{2}\right) - u(t-1)\right]$$

Example 7.81 : *Find* $L^{-1}\dfrac{\left(1-e^{-s}\right)^2}{(s-1)\,(s-2)}$

Solution : Let

$$\phi(s) = \frac{1}{(s-1)(s-2)}$$

$$= \frac{1}{(s-2)} - \frac{1}{(s-1)}$$

$$L^{-1}\,\phi(s) = L^{-1}\frac{1}{s-2} - L^{-1}\frac{1}{s-1}$$

$$L^{-1}\,\phi(s) = e^{2t} - e^{t}$$

$$L^{-1}\left(1-e^{-s}\right)^2\phi(s) = L^{-1}\left(1-2e^{-s}+e^{-2s}\right)\phi(s)$$

$$= L^{-1}\,\phi(s) - 2L^{-1}\,e^{-s}\,\phi(s) + L^{-1}\,e^{-2s}\,\phi(s)$$

$$= \left[e^{2t} - e^{t}\right] - 2\left[e^{2(t-1)} - e^{(t-1)}\right]u(t-1) + \left[e^{2(t-2)} - e^{(t-2)}\right]u(t-2)$$

Exercise 7.9

Find inverse Laplace transforms of the following

1. $\dfrac{e^{-\pi s}}{s+a}$ **[Ans. : $e^{-a(t-\pi)}\,U(t-\pi)$]**

2. $\dfrac{e^{-3s}}{(s-2)^4}$ **[Ans. : $\dfrac{(t-3)^3}{6}\,e^{2(t-3)}\,U(t-3)$]**

3. $\dfrac{e^{-\pi s/2}+e^{-3\pi s/2}}{s^2+1}$ **[Ans. : $\cos t\left[U\left(t-\dfrac{3\pi}{2}\right) - U\left(t-\dfrac{\pi}{2}\right)\right]$]**

4. $\dfrac{e^{3-2s}}{(s+4)^{5/2}}$ **[Ans. : $e^{3}\cdot\dfrac{4}{3\sqrt{\pi}}\,e^{-4(t-2)}\,(t-2)^{3/2}\cdot U(t-2)$]**

5. $\dfrac{(s+2)}{s^2(s+3)}e^{-s}$ **[Ans. : $\left[\dfrac{2}{3}(t-1) - \dfrac{e^{-3(t-1)}}{9} + \dfrac{1}{9}\right]U(t-1)$]**

6. $\dfrac{e^{-s}\left(1-e^{-s}\right)}{s\left(s^2+1\right)}$ **[Ans. : $[1-\cos(t-1)]\,U(t-1) - [1-\cos(t-2)]\,U(t-2)$]**

7. $\dfrac{e^{-4s}(s+2)}{s^2+4s+5}$ **[Ans. : $\left\{e^{-2(t-4)}\cos(t-4)\right\}U(t-4)$]**

8. $\dfrac{s\,e^{-\pi s}}{s^2+4s+29}$ **[Ans. : $e^{2(t-\pi)}\left\{\cos 5(t-\pi) + \dfrac{2}{5}\sin 5(t-\pi)\right\}U(t-\pi)$**

9. $\dfrac{e^{-s}+e^{-2s}}{s^2-3s+2}$ **[Ans. : $\left[e^{2(t-1)} - e^{(t-1)}\right]U(t-1) + \left[e^{2(t-2)} - e^{(t-2)}\right]U(t-2)$]**

10. $\dfrac{(s+1)e^{-s}}{s^2+s+1}$ [Ans. : $\dfrac{e^{-(t-1)/2}}{\sqrt{3}}\left\{\sqrt{3}\cos\dfrac{\sqrt{3}}{2}(t-1)+\sin\dfrac{\sqrt{3}}{2}(t-1)\right\}U(t-1)$]

11. $\dfrac{e^{4-3s}}{(s+4)^{5/2}}$ [Ans. : $\dfrac{4(t-3)^{3/2}\,e^{-4(t-4)}}{3\sqrt{\pi}}U(t-3)$]

12. $\dfrac{e^{-\pi s}}{s\left(s^2-4s+5\right)}$ [Ans. : $\dfrac{1}{5}\left[1+e^{2(t-\pi)}(\cos t-2\sin t)\right]U(t-\pi)$]

13. $\dfrac{e^{-s}}{(s-2)\left(s^2+2s+2\right)}$ [Ans. : $\left\{\dfrac{1}{2}e^{2(t-1)}-\dfrac{1}{2}e^{-(t-1)}[\cos(t-1)-\sin(t-1)\right\}U(t-1)$]

14. $\dfrac{e^{-2s}}{s\sqrt{s+1}}$ [Ans. : $\left\{\operatorname{erf}\sqrt{t-2}\right\}U(t-2)$]

15. $\dfrac{\left(e^{-s}-1\right)^2}{\sqrt{s+4}}$ [Ans. : $\dfrac{e^{-4(t-2)}}{\sqrt{\pi(t-2)}}U(t-2)-\dfrac{2e^{-4(t-1)}}{\sqrt{\pi(t-1)}}U(t-1)+\dfrac{e^{-4t}}{\sqrt{\pi t}}$]

7.14 Type X

Dirac Delta or Unit Impulse Function

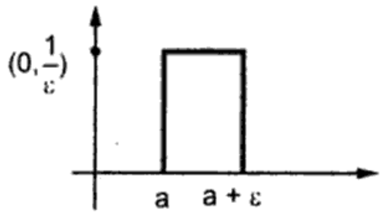


Fig. 7.14

Consider the function

$$f(t)=\begin{cases}0 & t<a\\ 1/\varepsilon & a\le t\le a+\varepsilon \quad \text{where } \varepsilon>0\\ 0 & t>a+\varepsilon\end{cases}$$

The area under the graph is always unity.

$$\delta(t-a)=\lim_{\varepsilon\to 0}f(t)$$

7.15 Properties and Theorems on Dirac Delta Function

a) Shifting property

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$$ Provided f (t) is continuous

Proof : Step 1 : Consider $\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt$

Step 2 : Split the integral

$$=\int_{-\infty}^{a} f(t)\,\delta(t-a)\,dt+\int_{a}^{a+t} f(t)\,\delta(t-a)\,dt+\int_{a+t}^{\infty} f(t)\,\delta(t-a)\,dt$$

Step 3 : Using the definition of $\delta(t-a)$ we get

$$= 0 + \lim_{t \to 0} \int_{a}^{a+t} f(t) \cdot \frac{1}{t} \, dt + 0$$

Step 4 : Let $\int f(t) \, dt = G(t)$ say

$$= \lim_{t \to 0} \frac{[G(t)]_a^{a+t}}{\varepsilon}$$

Step 5 : Substitute the limits

$$= \lim_{t \to 0} \frac{G(a+t) - G(a)}{a}$$

Step 6 : By the definition of derivatives by 1^{st} principal

$$= G'(a)$$

$$= f(a)$$

As $\int f(t) \, dt = G(t)$

$\therefore f(t) = G'(t)]$

b) Similarly we can show

$$\boxed{\int_{0}^{\infty} f(t) \, \delta(t-a) \, dt = f(a)} \quad \text{... (i)}$$

Also $$\boxed{\int_{-\infty}^{\infty} f(t) \, \delta'(t-a) \, dt = -f'(a)} \quad \text{... (ii)}$$

c) Laplace transform of $\delta(t-a)$ by definition of Laplace

$$L \, \delta(t-a) = \int_{0}^{\infty} e^{-st} \, \delta(t-a) \, dt$$

$$= e^{-as} \quad \text{using (i)}$$

$$\boxed{L \, \delta(t-a) = e^{-as}}$$

Substituting $a = 0$ we get

$$\boxed{L(\delta(t)) = 1}$$

d) Laplace transform of $f(t) \, \delta(t-a)$ by definition of Laplace

$$L \, f(t) \, \delta(t-a) = \int_{0}^{\infty} e^{-st} \, f(t) \, \delta(t-a) \, dt$$

$$= e^{-as} f(a) \quad \text{using (i)}$$

Thus $$\boxed{L \, f(t) \, \delta(t-a) = e^{-as} f(a)}$$

Substituting a = 0 we get

$$\boxed{L\ f(t)\,\delta(t) = f(0)}$$

e) Relation between Laplace's of Dirac Delta and unit step function

$$L\,\delta(t-a) = L\,[U'(t-a)]$$

Proof : We know that $L\,f'(t) = -f(0) + s\phi(s)$

$$\therefore \quad L\,U'(t-a) = -[U(t-a)]_{t=0} + s\,L\,U(t-a)$$

$$= 0 + s\cdot\frac{e^{-as}}{s}$$

$$= e^{-as}$$

$$= L\,\delta(t-a)$$

Thus $\boxed{L\ U'(t-a) = L\,\delta(t-a)}$

Similarly $\boxed{L\ U''(t-a) = L\,\delta'(t-a)}$

7.16 Illustrations

Find the Laplace transform of the following.

Example 7.82 : $L\ \sin t\,\delta(t-4)$

Solution :

$$= \int_0^\infty e^{-st}\sin t\,\delta(t-4)\cdot dt \quad \{\text{Using the definition of Laplace}\}$$

$$= e^{-4s}\sin 4$$

Example 7.83 : $L\ \sin 2t\,\delta(t-\pi/4) - t^2\,\delta(t-4)$

Solution : $L\sin 2t\ \delta(t-\pi/4)$

$$= \int_0^\infty e^{-st}\sin 2t\,\delta(t-\pi/4)\cdot dt$$

$$= e^{-\frac{\pi}{4}s}\sin\frac{\pi}{2}$$

$$= e^{-\frac{\pi}{4}s}$$

Also $$L\,t^2\,\delta(t-4) = \int_0^\infty e^{-st}\,t^2\,(t-4)\cdot dt$$

$$= e^{-4s}\,16$$

$$\therefore \quad L\left[\sin 2t\,\delta\left(t-\frac{\pi}{4}\right) - t^2\,\delta(t-4)\right] = e^{-\frac{\pi s}{4}} - 16\,e^{-4s}$$

Example 7.84 : $t\,u(t-4) - t^3\,\delta(t-2)$

Solution : $L\left[t\,u\,(t-4) - t^3\,\delta\,(t-2)\right]$

$$\text{Consider} \quad L\,t\,u\,(t-4) = e^{-4s}\,L\,(t+4)$$

$$= e^{-4s}\left[\frac{1}{s^2} + \frac{4}{s}\right]$$

$$\text{Also} \quad L\,t^3\,\delta\,(t-2) = \int_0^\infty e^{-st}\,t^3\,\delta\,(t-2) = e^{-2s}\,8$$

$$\therefore\ L\left[t\,U(t-4) - t^3\delta(t-2)\right] = e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) - 8e^{-2s}$$

Example 7.85 : $L\,te^{-2t}\,\delta(t-2)$

Solution : $L\,te^{-2t}\,\delta\,(t-2)$

$$= \int_0^\infty e^{-st}\,t\cdot e^{-2t}\,\delta\,(t-2)\cdot dt$$

$$= e^{-2s}\,e^{-4}\cdot(2)$$

$$= 2e^{-(2s+4)}$$

$$= 2e^{-2(s+2)}$$

Example 7.86 : *Evaluate* $\int_0^\infty t\,e^{-at}\,\delta'\,(t-b)\,dt$ *(a, b are positive)*

Solution : Step 1 : As $\delta'(t-b) = 0$ when $t \neq b$

$\therefore$ The integral

$$\int_0^\infty t\,e^{-at}\,\delta'\,(t-b)\,dt = \int_{-\infty}^\infty t\,e^{-at}\,\delta'\,(t-b)\,dt$$

Step 2 : Using $\int_{-\infty}^\infty f\,(t)\,\delta'\,(t-a)\,dt = -f'\,(a)$

We get

$$= -\frac{d}{dt}\left[t\,e^{-at}\right]_{t=b}$$

$$= -\left[e^{-at} - at\,e^{-at}\right]_{t=b}$$

$$= e^{-bt}\,[bt-1]$$

Exercise 7.10

Find the Laplace transform of the following

1. $\delta(t-\pi) + e^{-t}\,\delta(t)$ **[Ans. : $1 + e^{-\pi s}$]**

2. $t^2\,U(t-2) - \cosh t\;\delta(t-4)$ **[Ans. : $e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right) - e^{-4s}\cosh 4$]**

3. $\cos(t-1)\,U(t-1) + t^4\,\delta(t-5)$ **[Ans. : $\frac{s\,e^{-s}}{s^2+1} + 625\,e^{-5s}$]**

4. $\sin t\,\cos 2t\;\delta(t-2)$ **[Ans. : $e^{-2s}\sin 2\cos 4$]**

5. $\sin 2t\,\cos 4t\;\delta\left(t - \frac{\pi}{4}\right)$ **[Ans. : $-e^{-\pi s/4}$]**

6. $\left(t^2 + 3t + 2\right)U(t-1) + \frac{\sin t}{t}\,\delta(t-3)$ **[Ans. : $e^{-s}\left(\frac{2}{s^3} + \frac{\text{[illegible]}}{s^2} + \frac{6}{s}\right) + \frac{\sin 3}{3}\,e^{-3s}$]**

Evaluate the following integrals

1. $\int_{-\infty}^{\infty} \sin 2t\;\delta\left(t - \frac{\pi}{4}\right) dt$ **[Ans. : 1]**

2. $\int_{0}^{\infty} e^{-t}\sin t\;\delta'(t-a)\,dt$ **[Ans. : $e^{-a}(\sin a - \cos a)$]**

3. $\int_{0}^{\infty} t^2 e^{-t}\sin t\;\delta(t-2)\,dt$ **[Ans. : $4\,e^{-2}\sin 2$]**

4. $\int_{0}^{\infty} \sin \omega t\left[\delta\left(t - \frac{\pi}{2}\right) + \delta'(t-\pi)\right] dt$ **[Ans. : $\frac{\sin \pi\omega}{2} - \omega\cos \pi\omega$]**

5. $\int_{0}^{\infty} e^{-4t}\,\delta'(t-2)\,dt$ **[Ans. : $\left[-\frac{d}{dt}e^{-4t}\right]_{t \to 2} = 4e^{-8}$]**

7.17 Applications of Laplace Transforms

Let $\quad L\,y(t) = \overline{Y}(s)$

$$\therefore \quad L\frac{dy}{dt} = -y(0) + s\,\overline{Y}(s)$$

$$L\frac{d^2y}{dt^2} = -y'(0) - s\,y(0) + s^2\,\overline{Y}(s)$$

$$L\frac{d^3y}{dt^3} = -y''(0) - s\,y'(0) - s^2\,y(0) + s^3\,\overline{Y}(s)$$

$$L\int_0^t y\,dt = \frac{1}{s}\,\overline{Y}(s)$$

Illustrations

Example 7.87 : *Solve* $\frac{dy}{dt} + 2y(t) + \int_0^t y(t)dt = \sin t$ *given* $y(0) = 1$.

Solution : **Step 1 :** Let $L\,y(t) = Y(s)$

$$\therefore \quad L\frac{dy}{dt} = -y(0) + s\,Y(s) = -1 + s\,Y(s)$$

$$L\int_0^t y\,dt = \frac{1}{s}Y(s), \quad L\sin t = \frac{1}{s^2+1}$$

Step 2 : Take Laplace transform of given equation.

$$\therefore \quad L\left[\frac{dy}{dt}\right] + L[2\,y(t)] + L\int_0^t y\,dt = \frac{1}{s^2+1}$$

Step 3 : Substituting the values.

$$-1 + s\,Y(s) + 2\,Y(s) + \frac{1}{s}Y(s) = \frac{1}{s^2+1}$$

Step 4 : Collecting the terms of Y(s) on one side

$$Y(s)\left[s + 2 + \frac{1}{s}\right] = 1 + \frac{1}{s^2+1}$$

$$Y(s)\left(\frac{s^2+2s+1}{s}\right) = 1 + \frac{1}{s^2+1}$$

$$Y(s) = \frac{s}{s^2+2s+1} + \frac{s}{(s^2+2s+1)(s^2+1)}$$

Step 5 : To find inverse use partial fractions or adjustment

$$Y(s) = \frac{s+1-1}{(s+1)^2} + \frac{1}{2}\left[\frac{1}{s^2+1} - \frac{1}{s^2+2s+1}\right]$$

$$Y(s) = \frac{1}{s+1} - \frac{1}{(s+1)^2} + \frac{1/2}{s^2+1} - \frac{1/2}{(s+1)^2}$$

$$Y(s) = \frac{1}{s+1} + \frac{1/2}{s^2+1} - \frac{3/2}{(s+1)^2}$$

Step 6 : Take inverse Laplace

$$y(t) = e^{-t} + \frac{1}{2}\sin t - \frac{3}{2}\,t\,e^{-t}$$

Example 7.88 : *Solve* $y''' - y = e^t$ *given* $y(0) = y'(0) = y''(0) = 0$

Solution : Step 1 : Let $L\,y(t) = Y(s)$

$$\therefore \quad L\,y'''(t) = -y''(0) - s\,y'(0) - s^2\,y(0) + s^3\,Y(s)$$

$$= 0 - 0 - 0 + s^3\,Y(s)$$

Step 2 : Take Laplace transform of given equation

$$L(y''') - L(y) = L(e^t)$$

Step 3 : Substituting the values we get

$$s^3\,Y(s) - Y(s) = \frac{1}{s-1}$$

Step 4 : Collecting the terms of Y(s) on one side

$$Y(s) = \frac{1}{(s-1)(s^3-1)}$$

$$Y(s) = \frac{1}{(s-1)(s-1)(s^2+s+1)}$$

$$Y(s) = \frac{1}{(s^2-2s+1)(s^2+s+1)}$$

Step 5 : To find inverse separate the terms (adjustment)

$$Y(s) = \frac{1}{3s}\left[\frac{1}{s^2-2s+1} - \frac{1}{s^2+s+1}\right]$$

$$Y(s) = \frac{1}{3s}\left[\frac{1}{(s-1)^2} - \frac{1}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}\right]$$

Step 6 : Take inverse Laplace transform

$$y(t) = \frac{1}{3}\left\{\int_0^t t\,e^t\,dt - \frac{2}{\sqrt{3}}\int_0^t e^{-t/2}\sin\frac{\sqrt{3}\,t}{2}\,dt\right\}$$

$$= \frac{1}{3}\left\{\left[t\,e^t - e^t\right]_0^t - \frac{2}{\sqrt{3}}\left[\frac{e^{-t/2}}{\frac{1}{4}+\frac{3}{4}}\left(-\frac{1}{2}\sin\frac{\sqrt{3}\,t}{2} - \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}t}{2}\right)\right]_0^t\right\}$$

$$= \frac{1}{3}\left[t\,e^{t} - e^{t} + 1\right] - \frac{1}{3\sqrt{3}}\left[e^{-t/2}\left(-\sin\frac{\sqrt{3}\,t}{2} - \sqrt{3}\cos\frac{\sqrt{3}\,t}{2}\right) + \sqrt{3}\right]$$

$$= \frac{1}{3}\,t\,e^{t} - \frac{e^{t}}{3} + \frac{e^{-t/2}}{3}\left(\sin\frac{\sqrt{3}\,t}{2} + \sqrt{3}\cos\frac{\sqrt{3}\,t}{2}\right)$$

Example 7.89 : $y'' - 3y' + 2y = 12e^{-2t}$, $y(0) = 2$, $y'(0) = 6$. **[May-2005, Dec.-2005]**

Solution : Step 1 : Let $L(y(t)) = Y(s)$

$$\therefore \quad L\,y'(t) = -y(0) + s\,Y(s)$$

$$= -2 + s\,Y(s)$$

$$L\,y''(t) = -y'(0) - s\,y(0) + s^2\,Y(s)$$

$$= -6 - 2s + s^2\,Y(s)$$

Step 2 : Take Laplace transform of given equation

$$L(y'') - L(3y') + L(2y) = L(12e^{-2t})$$

Step 3 : Substituting the values we get

$$[-6 - 2s + s^2\,Y(s)] - 3[-2 + s\,Y(s)] + 2\,Y(s) = \frac{12}{s+2}$$

Step 4 : Collecting the terms of Y(s) on one side

$$(s^2 - 3s + 2)\,Y(s) = \frac{12}{s+2} + 2s$$

$$Y(s) = \frac{2s^2 + 4s + 12}{(s+2)(s^2 - 3s + 2)}$$

$$Y(s) = \frac{2s^2 + 4s + 12}{(s+2)(s-1)(s-2)}$$

Step 5 : Use partial fractions

$$Y(s) = \frac{1}{s+2} - \frac{6}{s-1} + \frac{7}{s-2}$$

Step 6 : Take inverse

$$y(t) = e^{-2t} - 6e^{t} + 7e^{2t}$$

Example 7.90 : $y'' + 2y' + y = t\,e^{-t}$, $y(0) = 1$, $y'(0) = -2$ **[May-2001]**

Solution :

Step 1 : Let $L\,y(t) = Y(s)$

$$L\,y'(t) = -y(0)+s\,Y(s)$$
$$= -1+s\,Y(s)$$
$$L\,y''(t) = -y'(0)-s\,y(0)+s^2\,Y(s)$$
$$= +2-s+s^2\,Y(s)$$

Step 2 : Taking Laplace of given equation.

$$L[y'']+2\,L[y']+L[y] = L[t\,e^{-t}]$$

Step 3 : Substituting the values

$$[2-s+s^2\,Y(s)]+2[-1+s\,Y(s)]+Y(s) = \frac{1}{(s+1)^2}$$

Step 4 : Collecting the terms of Y(s) on one side

$$\left(s^2+2s+1\right)Y(s) = s+\frac{1}{(s+1)^2}$$

$$\therefore \qquad Y(s) = \frac{s}{(s+1)^2}+\frac{1}{(s+1)^4}$$

Step 5 : Adjustment

$$= \frac{s+1-1}{(s+1)^2}+\frac{1}{(s+1)^4}$$

$$= \frac{1}{s+1}-\frac{1}{(s+1)^2}+\frac{1}{(s+1)^4}$$

Step 6 : Take inverse

$$y(t) = e^{-t}-t\,e^{-t}+e^{-t}\cdot\frac{t^3}{3!}$$

Example 7.91 : *Solve* $y(t)+\int_0^t y(t)\,dt = 1-e^{-t}$

Solution : **Step 1 :** Let $L\,y(t) = Y(s)$

$$\therefore \qquad L\int_0^t y(t)\,dt = \frac{1}{s}\,Y(s)$$

Step 2 : Taking Laplace transform of given equation

$$Y(s)+\frac{1}{s}\,Y(s) = \frac{1}{s}-\frac{1}{s+1}$$

Step 3 : Collecting Y(s) on one side

$$Y(s)\left[\frac{s+1}{s}\right] = \frac{1}{s}-\frac{1}{s+1}$$

$$Y(s) = \frac{s}{s+1}\left[\frac{1}{s}-\frac{1}{s+1}\right]$$

$$= \frac{s}{s+1}\left[\frac{s+1-s}{s(s+1)}\right]$$

$$Y(s) = \frac{1}{(s+1)^2}$$

Step 4 : Take inverse

$$y(t) = e^{-t}\,t$$

Example 7.92 : *Solve* $\frac{dy}{dt}+4y+5\int_0^t y\,dt=e^{-t}\ ,\ y(0)=0$

Solution : Step 1 : Let L y = Y(s)

$$L\,y' = -y(0)+s\,Y(s)$$

$$= 0+s\ Y(s)$$

$$L\int_0^t y\,dt = \frac{1}{s}\ Y(s)$$

Step 2 : Take Laplace of given equation

$$s\,Y(s)+4\,Y(s)+\frac{5}{s}\ Y(s) = \frac{1}{s+1}$$

Step 3 : Collecting the terms of Y(s)

$$Y(s)\left[s+4+\frac{5}{s}\right] = \frac{1}{s+1}$$

$$Y(s) = \frac{s}{s^2+4s+5}\,\frac{1}{s+1}$$

$$Y(s) = \frac{s}{(s+1)(s^2+4s+5)}$$

Step 4 : Use partial fractions

$$Y(s) = \frac{A}{s+1}+\frac{B\,s+C}{s^2+4s+5}$$

Thus $$s = A(s^2+4s+5)+(s+1)(B\,s+C)$$

Put $s = 0$ $\quad 0 = 5A + C$

$s = 1$ $\quad 1 = 10A + 2B + 2C$

$s = -1$ $\quad -1 = 2A$

$\therefore$ We get $A = -\frac{1}{2}$, $C = \frac{5}{2}$, $B = \frac{1}{2}$

Step 5 : Substituting A, B, C

$$Y(s) = \frac{-1/2}{s+1} + \frac{\frac{1}{2}s + \frac{5}{2}}{s^2+4s+5}$$

$$Y(s) = \frac{1}{2}\left\{\frac{-1}{s+1} + \frac{s+5}{(s+2)^2+1}\right\}$$

$$= \frac{1}{2}\left\{\frac{-1}{s+1} + \frac{s+2+3}{(s+2)^2+1}\right\}$$

Step 6 : Take inverse

$$y(t) = \frac{1}{2}\left\{-e^{-t} + e^{-2t}\, L^{-1}\frac{s+3}{s^2+1}\right\}$$

$$= \frac{1}{2}\left\{-e^{-t} + e^{-2t}(5\cos t - 9\sin t)\right\}$$

Example 7.93 : *Solve* $t\,y'' + 2y' + t\,y = \cos t$ *given that* $y(0) = 1$.

Solution : Step 1 : Take Laplace of given equation

$$t\,y'' + 2\,y' + t\,y = \cos t$$

$$\therefore L[t\,y''] + 2\,L[y'] + L[t\,y] = L\,[\cos t]$$

$$\therefore -\frac{d}{ds}\,L[y''] + 2\,L[y'] - \frac{d}{ds}\,L[y] = \frac{s}{s^2+1}$$

$$\therefore -\frac{d}{ds}[s^2\,Y(s) - s\,y(0) - y'(0)] + 2[s\,Y(s) - y(0)] - \frac{d}{ds}\,Y(s) = \frac{s}{s^2+1}$$

$$\therefore -\left[s^2\frac{d}{ds}\,Y(s) + 2s\,Y(s) - y(0) - 0\right] + 2[s\,Y(s) - y(0)] - \frac{d}{ds}\,Y(s) = \frac{s}{s^2+1}$$

Put $y(0) = 1$ and $y'(0) =$ constant.

$$\therefore -\left[s^2\frac{d}{ds}\,Y(s) + 2s\,Y(s) - 1\right] + 2s\,Y(s) - 2(1) - \frac{d}{ds}\,Y(s) = \frac{s}{s^2+1}$$

Step 2 : Collect the terms of Y(s) on one side

$$\therefore\ -(s^2+1)\frac{d}{ds}Y(s)+(-2s+2s)Y(s)-1=\frac{s}{s^2+1}$$

$$\therefore\ (s^2+1)\frac{d}{ds}Y(s) = -\frac{s}{s^2+1}-1$$

$$\therefore\ \frac{d}{ds}Y(s) = -\frac{s}{(s^2+1)^2}-\frac{1}{s^2+1}$$

Step 3 : Taking inverse Laplace transform we get,

$$L^{-1}\left[\frac{d}{ds}Y(s)\right] = -L^{-1}\left[\frac{s}{(s^2+1)^2}\right]-L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$\therefore\ -t\,L^{-1}[Y(s)] = -\frac{1}{2(1)}\,t\sin(1)\,t-\sin t \quad \ldots\ \because\ L^{-1}\left[\frac{d}{ds}F(s)\right]=-t\,f(t)$$

and using the result $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]=\frac{1}{2a}\,t\sin at$

$$\therefore\ t\,y(t) = \sin t+\frac{1}{2}\,t\sin t$$

$$\therefore\ y(t) = \frac{\sin t}{t}+\frac{1}{2}\sin t$$

Example 7.94 : *Solve using Laplace transforms*

$$\frac{dy}{dt}+3y(t)+2\int_0^t y(t)dt = t \text{ given } y(0)=0$$

Solution : Step 1 : Take Laplace transform of given equation

$$\frac{dy}{dt}+3y(t)+2\int_0^t y(t)\,dt = t$$

$$\therefore\ L\left[\frac{dy}{dt}\right]+3L[y(t)]+2L\left[\int_0^t y(t)\,dt\right] = L[t]$$

i.e. $[s\,Y(s)-y(0)]+3\,Y(s)+2\,\frac{Y(s)}{s}=\frac{1}{s^2}$

Step 2 : Collect the terms of Y(s) on one side

i.e. $s\,Y(s)+3\,Y(s)+\frac{2}{s}Y(s)=\frac{1}{s^2}$

$$\therefore \quad \left(s+3+\frac{2}{s}\right) Y(s) = \frac{1}{s^2}$$

$$\text{i.e.} \quad \frac{s^2+3s+2}{s} Y(s) = \frac{1}{s^2}$$

$$\therefore \quad Y(s) = \frac{s}{s^2\left(s^2+3s+2\right)} = \frac{1}{s(s+1)(s+2)}$$

$$= \frac{1}{2}\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{(s+2)}$$

Step 3 : Taking inverse Laplace transform we get

$$y(t) = \frac{1}{2} L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{2}(1) - e^{-t} + \frac{1}{2} e^{-2t} = \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$$

Example 7.95 : *Solve using Laplace transforms* $\left(D^2+n^2\right)x = a\sin(nt+a)$, $x(0)=x'(0)=0$

Solution : **Step 1 :** Take Laplace of given equation

$$x''+n^2 x = a\sin(nt+\alpha) = a[\sin nt\cos\alpha + \cos nt\sin\alpha]$$

$$\therefore \quad L[x''] + n^2 L[x] = a\cos\alpha\, L[\sin nt] + a\sin\alpha\, L[\cos nt]$$

$$\text{i.e.} \quad \left[s^2 X(s) - s\,x(0) - x'(0)\right] + n^2 X(s) = a\cos\alpha\left[\frac{n}{s^2+n^2}\right] + a\sin\alpha\left[\frac{s}{s^2+n^2}\right]$$

Step 2 : Collect the terms of X(s)

$$\therefore \quad \left(s^2+n^2\right) X(s) = a\cos\alpha\left(\frac{n}{s^2+n^2}\right) + a\sin\alpha\left(\frac{s}{s^2+n^2}\right) \quad \ldots \because x(0)=x'(0)=0$$

$$\therefore \quad X(s) = a\cos\alpha\left[\frac{n}{\left(s^2+n^2\right)^2}\right] + a\sin\alpha\left[\frac{s}{\left(s^2+n^2\right)^2}\right]$$

Step 3 : Taking inverse Laplace transform we get

$$\therefore \quad x(t) = L^{-1}[X(s)] = a(\cos\alpha)\, n\, L^{-1}\left[\frac{1}{\left(s^2+n^2\right)^2}\right] + a(\sin\alpha)\, L^{-1}\left[\frac{s}{\left(s^2+n^2\right)^2}\right] \quad \ldots (1)$$

We know

$$L^{-1}\frac{1}{\left(s^2+a^2\right)^2} = \frac{1}{2a^3}(\sin at - at\cos at)$$

$$L^{-1}\frac{s}{\left(s^2+a^2\right)^2} = \frac{t}{2a}\sin at$$

Thus
$$x(t) = a(\cos\alpha)\,n\frac{1}{2n^3}(\sin nt - nt\cos nt) + a\sin\alpha\,\frac{t\sin nt}{2n}$$
$$= \frac{a\cos\alpha}{2n^2}(\sin nt - nt\cos nt) + \frac{a\sin\alpha}{2n}(t\sin nt)$$

Example 7.96 : *Solve the following differential equation by Laplace Transform method :* $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$ *for which* $y(0) = y'(0) = 0$ *and* $y''(0) = 6$.

Solution : We have, $y''' + 2y'' - y' - 2y = 0$

Step 1 : Taking Laplace transform of both the sides,

$L[y'''] + 2L[y''] - L[y'] - 2L[y] = L[0] = 0$

i.e. $[s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0)] + 2[s^2 Y(s) - s y(0) - y'(0)]$

$-[s Y(s) - y(0)] - 2Y(s) = 0$

Step 2 : Substituting $y(0) = y'(0) = 0$ and $y''(0) = 6$, we get

$[s^3 Y(s) - 0 - 0 - 6] + 2[s^2 Y(s) - 0 - 0] - [s Y(s) - 0] - 2Y(s) = 0$

i.e. $(s^3 + 2s^2 - s - 2)\,Y(s) = 6$

Step 3 : Collect the terms of Y(s)

$$Y(s) = \frac{6}{s^3+2s^2-s-2} = \frac{6}{s^2(s+2)-(s+2)}$$
$$= \frac{6}{(s+2)(s^2-1)} = \frac{6}{(s+2)(s+1)(s-1)}$$
$$= 6\left[\frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s-1}\right]$$

Step 4 : Use partial fractions for finding A, B, C.

where $A = \frac{1}{(-2-1)(-2-1)} = \frac{1}{(-1)(-3)} = \frac{1}{3}$, $B = \frac{1}{(1)(-2)} = -\frac{1}{2}$, $C = \frac{1}{(3)(2)} = \frac{1}{6}$

$$\therefore \quad Y(s) = 6\left[\frac{1}{3(s+2)} - \frac{1}{2(s+1)} + \frac{1}{6(s-1)}\right] = \frac{2}{s+2} - \frac{3}{s+1} + \frac{1}{s-1}$$

Step 5 : Taking inverse Laplace transform we get

$$L^{-1}[Y(s)] = 2L^{-1}\left[\frac{1}{s+2}\right] - 3L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{1}{s-1}\right]$$

i.e. $y(t) = 2e^{-2t} - 3e^{-t} + e^{t}$ which is the required solution.

Example 7.97 : *Solve by using Laplace transform method :*

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}=8\cos t \quad \text{given : } y(\pi)=-1,\ \frac{dy}{dt}=-1 \text{ when } t=0.$$ **[May-2001]**

Solution : We have $y''+y' = 8\cos t$

Step 1 : Take Laplace transform of given equation

$$\therefore \qquad L[y'']+L[y'] = 8\,L[\cos t]$$

i.e. $$[s^2\,Y(s)-s\,y(0)-y'(0)]+[s\,Y(s)-y(0)] = 8\,\frac{s}{s^2+1}$$

Step 2 : Substituting $y'(0)=-1$ and $y(0)=k$ (say)

i.e. $$[s^2\,Y(s)-s(k)-(-1)]+[s\,Y(s)-(k)] = 8\,\frac{s}{s^2+1}$$

$$\therefore\ (s^2+s)\,Y(s)-(s+1)\,k+1 = \frac{8\,s}{s^2+1}$$

Step 3 : Collect the terms of Y(s)

$$\therefore \qquad s(s+1)\,Y(s) = \frac{8\,s}{s^2+1}+(s+1)\,k-1$$

$$\therefore \qquad Y(s) = \frac{8\,s}{s(s+1)(s^2+1)}+\frac{(s+1)\,k}{s(s+1)}-\frac{1}{s(s+1)}$$

$$= \frac{8}{(s+1)(s^2+1)}+\frac{k}{s}-\frac{1}{s(s+1)}$$

Step 4 : Use partial fractions

$$= 8\left[\frac{1/2}{s+1}+\frac{(-1/2)\,s+1/2}{s^2+1}\right]+\frac{k}{s}-\left[\frac{1}{s}-\frac{1}{s+1}\right]$$

Step 5 : Take inverse Laplace transform

$$= 4\,L^{-1}\left[\frac{1}{s+1}+\frac{-s+1}{s^2+1}\right]+k\,L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{1}{s}\right]+L^{-1}\left[\frac{1}{s+1}\right]$$

$$= 4\,L^{-1}\left[\frac{1}{s+1}-\frac{s}{s^2+1}+\frac{1}{s^2+1}\right]+k(1)-1+e^{-t}$$

$$= 4\left[e^{-t}-\cos t+\sin t\right]+(k-1)+e^{-t}$$

$$\therefore \qquad y(t) = 5\,e^{-t}-4\cos t+4\sin t+k-1 \qquad \ldots (1)$$

Now putting $t=\pi$ where $y(\pi)=-1$ (given) we get

$$-1 = 5\,e^{-\pi}-4\cos\pi+4\sin\pi+k-1$$

$$\therefore \quad 0 = 5e^{-\pi} + 4 + k$$

$$\therefore \quad k = -4 - 5e^{-\pi}$$

Substituting in (1) we get

$$y(t) = 5e^{-t} - 4\cos t + 4\sin t - 4 - 5e^{-\pi} - 1$$

$$= 5e^{-t} - 4\cos t + 4\sin t - 5e^{-\pi} - 5$$

Example 7.98 : *Solve by using Laplace transform method* $y''' - 3y'' + 3y' - y = t^2 e^t$ *given* $y(0) = 1,\ y'(0) = 0,\ y''(0) = -2.$ **[May-2002]**

Solution : We have, $y''' - 3y'' + 3y' - y = t^2 e^t$

Step 1 : Taking Laplace of given equation

$$\therefore L[y'''] - 3L[y''] + 3L[y'] - L[y] = L[t^2 e^t]$$

i.e. $[s^3 Y(s) - s^2 y(0) - s\,y'(0) - y''(0)] - 3[s^2 Y(s) - s\,y(0) - y'(0)]$

$$+ 3[s\,Y(s) - y(0)] - Y(s) = \frac{2}{(s-1)^3}$$

Step 2 : Substituting

$$y(0) = 1$$

$$y'(0) = 0$$

$$y''(0) = -2 \quad \text{we get}$$

i.e. $[s^3 Y(s) - s^2(1) - 0 - (-2)] - 3[s^2 Y(s) - s(1) - 0] + 3[s\,Y(s) - 1] - Y(s) = \dfrac{2}{(s-1)^3}$

Step 3 : Collecting the terms of Y(s)

$$\therefore (s^3 - 3s^2 + 3s - 1)\,Y(s) - s^2 + 2 + 3s - 3 = \frac{d}{ds}\left[\frac{-1}{(s-1)^2}\right] = +\left[\frac{+2}{(s-1)^3}\right]$$

$$(s-1)^3\,Y(s) = s^2 - 3s + 1 + \frac{2}{(s-1)^3} = (s^2 - s - 2s + 2 - 1) + \frac{2}{(s-1)^3}$$

$$Y(s) = \frac{2}{(s-1)^6} + \frac{s(s-1) - 2(s-1) - 1}{(s-1)^3}$$

Step 4 : Adjustment

$$= \frac{2}{(s-1)^6} + \frac{s}{(s-1)^2} - \frac{2}{(s-1)^2} - \frac{1}{(s-1)^3}$$

$$= \frac{2}{(s-1)^6} + \frac{(s-1)-1}{(s-1)^2} - \frac{1}{(s-1)^3} = \frac{2}{(s-1)^6} + \frac{1}{(s-1)} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3}$$

Step 5 : Take inverse Laplace transform

$$y(t) = 2\,L^{-1}\left[\frac{1}{(s-1)^6}\right]+L^{-1}\left[\frac{1}{s-1}\right]-L^{-1}\left[\frac{1}{(s-1)^2}\right]-L^{-1}\left[\frac{1}{(s-1)^3}\right]$$

$$= 2\,e^t\,L^{-1}\left[\frac{1}{s^6}\right]+e^t-e^t\,L^{-1}\left[\frac{1}{s^2}\right]-e^t\,L^{-1}\left[\frac{1}{s^3}\right]$$

$$= 2\,e^t\,\frac{t^5}{5!}+e^t-e^t\,t-e^t\,\frac{t^2}{2!}=e^t\left(\frac{t^5}{60}+1-t-\frac{t^2}{2}\right)$$

Example 7.99 : *Solve the following differential equation by Laplace transform method : $y''+3y'+2y=\cos t\;\delta(t-\pi)$, where $y(0)=y'(0)=0$.* **[May-2004]**

Solution : We have $y''+3y'+2y=\cos t\cdot\delta(t-\pi)$

Step 1 : Take Laplace of given equation

$\therefore\; L[y'']+3\,L[y']+2\,L[y]=L[\cos t\cdot\delta(t-\pi)]$

i.e. $[s^2\,Y(s)-s\,y(0)-y'(0)]+3[s\,Y(s)-y(0)]+2\,Y(s)=e^{-\pi s}\cos\pi$

Step 2 : Substituting $y(0)=y'(0)=0$

i.e. $s^2\,Y(s)-0-0+3\,s\,Y(s)-0+2\,Y(s)=-e^{-\pi s}$

Step 3 : Collecting the terms of Y(s)

i.e. $(s^2+3s+2)\,Y(s)=-e^{-\pi s}$

$$\therefore\qquad Y(s) = -\frac{e^{-\pi s}}{(s^2+3s+2)}=-\frac{e^{-\pi s}}{(s+1)(s+2)}$$

Step 4 : Take inverse Laplace transform

$$\therefore\qquad L^{-1}[Y(s)] = y(t)=-L^{-1}\left[\frac{e^{-\pi s}}{(s+1)(s+2)}\right]$$

$$= -L^{-1}\left\{e^{-\pi s}\left[\frac{1}{s+1}-\frac{1}{s+2}\right]\right\}$$

$$= \left[e^{-2(t-\pi)}-e^{-(t-\pi)}\right]U(t-\pi)$$

Example 7.100 : *Solve $x''+3x'+2x=t\,\delta(t-1)$ for which $x(0)=x'(0)=0$.*

Solution : Step 1 : Taking Laplace transform of given equation

$L(x'')+3\,L(x')+2\,L(x)=L\,t\delta(t-1)$

$[-x'(0)-s\,x(0)+s^2\,X(s)]+3[-x(0)+s\,X(s)]+2\,X(s)=e^{-s}$

Step 2 : Substituting $x(0)=x'(0)=0$

$(s^2+3s+2)\ X(s) = e^{-s}$

Step 3 : Take X(s) on one side

$$X(s) = \frac{e^{-s}}{(s+1)(s+2)}$$

$$= e^{-s}\left[\frac{1}{s+1}-\frac{1}{s+2}\right]$$

Step 4 : Taking inverse Laplace transform on both sides

$$x(t) = \left\{e^{-(t-1)}-e^{-2(t-1)}\right\}\ U(t-1)$$

Example 7.101 : *Using Laplace solve*

$y''+4y = f(t) \qquad y(0) = 0, \quad y'(0) = 1$

where $f(t)$ *is given by*

i) $f(t) = \sin t \qquad \pi < t < 2\pi$

ii) $f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases}$

iii) $f(t) = U(t-2)$

iv) $f(t) = \delta(t)$

v) $f(t) = \delta(t-2)$

Solution : Take Laplace transform of

$$y''+4y = f(t)$$

$$L(y'')+L(4y) = L\,f(t)$$

$$[-y'(0)-s\,y(0)+s^2\,y(s)]+4\,y(s) = \phi(s)$$

Substituting $y(0) = 0, \quad y'(0) = 1$

$$[-1-0+s^2\,Y(s)]+4\,Y(s) = \phi(s)$$

$$(s^2+4)\,Y(s) = 1+\phi(s)$$

$$Y(s) = \frac{1}{s^2+4}+\frac{\phi(s)}{s^2+4} \qquad \text{... (i)}$$

where $\quad L\,f(t) = \phi(s)$

i) Here $\quad f(t) = \sin t \quad \pi < t < 2\pi$

$\therefore \quad f(t) = \sin t\,[U(t-\pi)-U(t-2\pi)]$

$$L\ f(t) = e^{-\pi s} L \sin(t+\pi) - e^{-2\pi s} L \sin(t+2\pi)$$

$$= e^{-\pi s} L(-\sin t) - e^{-2\pi s} L(\sin t)$$

$$\phi(s) = e^{-\pi s}\left(\frac{-1}{s^2+1}\right) - e^{-2\pi s}\frac{1}{s^2+1}$$

Substituting $\phi(s)$ in (i)

$$Y(s) = \frac{1}{s^2+4} - \frac{e^{-\pi s}}{(s^2+1)(s^2+4)} - \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}$$

$$Y(s) = \frac{1}{s^2+4} - \frac{e^{-\pi s}}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right] - \frac{e^{-2\pi s}}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]$$

Take inverse

$$y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\left[\sin(t-\pi) - \frac{1}{2}\sin 2(t-\pi)\right] U(t-\pi)$$

$$- \frac{1}{3}\left[\sin(t-2\pi) - \frac{1}{2}\sin 2(t-2\pi)\right] U(t-2\pi)$$

$$y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\left[-\sin t - \frac{1}{2}\sin 2t\right] U(t-\pi)$$

$$- \frac{1}{3}\left[+\sin t - \frac{1}{2}\sin 2t\right] U(t-2\pi)$$

ii) Here $f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases}$

$\therefore$ $$f(t) = 1\,[U(t-0) - U(t-1)]$$

$$L\ f(t) = \frac{1}{s} - \frac{e^{-s}}{s}$$

Substituting in (i)

$$Y(s) = \frac{1}{s^2+4} + \frac{1}{s^2+4}\left[\frac{1}{s} - \frac{e^{-s}}{s}\right]$$

$$= \frac{1}{s^2+4} + \frac{1}{s}\cdot\frac{1}{s^2+4} - \frac{1}{s(s^2+4)}\ e^{-s}$$

$$= \frac{1}{s^2+4} + \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] - \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] e^{-s}$$

Take inverse Laplace transform

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}[1 - \cos 2t] - \frac{1}{4}[1 - \cos 2(t-1)]\,U(t-1)$$

iii) Here $f(t) = U(t-2)$

$$\therefore \quad L\,f(t) = \frac{e^{-2s}}{s}$$

Substituting in (i)

$$Y(s) = \frac{1}{s^2+4} + \frac{1}{s(s^2+4)}\,e^{-2s}$$

$$= \frac{1}{s^2+4} + \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] e^{-2s}$$

Take inverse Laplace transform.

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}[1 - \cos 2\,(t-2)]\,U\,(t-2)$$

(iv) Here $f(t) = \delta\,(t)$

$$L\,f(t) = 1$$

Substituting in (i)

$$Y(s) = \frac{1}{s^2+4} + \frac{1}{s^2+4}$$

$$= \frac{2}{s^2+4}$$

Taking inverse Laplace we get,

$$y(t) = \sin 2t$$

(v) Here $f(t) = \delta\,(t-2)$

$$L\,f(t) = e^{-2s}$$

Substituting in (i)

$$Y(s) = \frac{1}{s^2+4} + \frac{e^{-2s}}{s^2+4}$$

Taking inverse Laplace transform

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{2}\sin 2\,(t-2)\;U\,(t-2)$$

Example 7.102 : $\dfrac{d^2y}{dt^2} + 9y = 18t \qquad y(0) = 0,\; y\,(\pi/2) = 0$

Solution : Step 1 : Taking Laplace transform of given equation,

$$[-y'(0) - s\,y(0) + s^2\,Y(s)] + 9[Y\,(s)] = 18\frac{1}{s^2}$$

Step 2 : Substituting $y(0) = 0, \; y'(0) = C_1$ (say)

$$\left[-C_1 - 0 + s^2 Y(s)\right] + 9Y(s) = \frac{18}{s^2}$$

$$(s^2+9)\, Y(s) = C_1 + \frac{18}{s^2}$$

$$Y(s) = \frac{C_1}{s^2+9} + \frac{18}{s^2(s^2+9)}$$

$$= \frac{C_1}{s^2+9} + \frac{18}{9}\left[\frac{1}{s^2} - \frac{1}{s^2+9}\right]$$

$$= \frac{C_1}{s^2+9} + \frac{2}{s^2} - \frac{2}{s^2+9}$$

$$Y(s) = \frac{(C_1-2)}{s^2+9} + \frac{2}{s^2}$$

Step 3 : Taking inverse,

$$y(t) = \left(\frac{C_1-2}{3}\right)\sin 3t + 2t \qquad \text{... (I)}$$

Put $t = \frac{\pi}{2}$,

$$y\left(\frac{\pi}{2}\right) = \left(\frac{C_1-2}{3}\right)\sin\frac{3\pi}{2} + 2\left(\frac{\pi}{2}\right)$$

$$0 = \left(\frac{C_1-2}{3}\right)(-1) + \pi$$

$$\Rightarrow \qquad \frac{C_1-2}{3} = \pi$$

Substituting in (I),

$$y(t) = \pi\sin 3t + 2t$$

Examaple 7.103 : $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 5x = e^{-t}\sin t, \;\; x(0) = 0$

$x'(0) = 1$

Solution : Step 1 : Taking Laplace transform of given equation

$$L(x'') + 2L(x') + 5L(x) = L(e^{-t}\sin t)$$

$$-x'(0) - sx(0) + s^2 X(s) + 2[-x(0) + sX(s)] + 5X(s) = \frac{1}{(s+1)^2+1}$$

Step 2 : Substituting $x(0) = 0, \; x'(0) = 1$

$$-1 - 0 + s^2 X(s) + 0 + 2sX(s) + 5X(s) = \frac{1}{s^2+2s+2}$$

$$\left[s^2+2s+5\right] X(s) = 1 + \frac{1}{s^2+2s+2}$$

$$X(s) = \frac{1}{s^2+2s+5} + \frac{1}{(s^2+2s+5)\;(s^2+2s+2)}$$

$$X(s) = \frac{s^2+2s+3}{(s^2+2s+5)\;(s^2+2s+2)}$$

Step 3 : For partial fraction put $s^2 + 2s = u$

$$X(s) = \frac{u+3}{(u+5)\,(u+2)}$$

$$= \frac{A}{u+5} + \frac{B}{u+2}$$

$$= \frac{2/3}{u+5} + \frac{1/3}{u+2}$$

Again substitute $u = s^2 + 2s$

$$X(s) = \frac{2/3}{s^2+2s+5} + \frac{1/3}{s^2+2s+2}$$

$$X(s) = \frac{1}{3}\left[\frac{2}{(s+1)^2+4} + \frac{1}{(s+1)^2+1}\right]$$

Step 4 : Taking inverse Laplace transform

$$x(t) = \frac{e^{-t}}{3}\, L^{-1}\left(\frac{2}{s^2+4} + \frac{1}{s^2+1}\right)$$

$$x(t) = \frac{e^{-t}}{3}\,[\sin 2t + \sin t]$$

Example 7.104 : $y'' + 4y' + 13y = \frac{1}{3}\, e^{-2t} \sin 3t\,,\;\; y(0) = 1$

$y'(0) = -2$

Solution : Step 1 : Taking Laplace transforms of given equation.

$$L\,[y''(t)] + 4L\,[y'(t)] + 13L\,[y(t)] = \frac{1}{3}\, L\left[e^{-2t}\sin 3t\right]$$

$$\left[-y'(0) - sy(0) + s^2Y(s)\right] + 4\left[-y(0) + Y(s)\right] + 13Y(s) = \frac{1}{3}\cdot\frac{3}{(s+2)^2+9}$$

Step 2 : Substituting y(0) = 1, y'(0) = – 2 and collecting the terms of Y(s) on one side.

$$+2-s+s^2\,Y(s)-4+4s\,Y(s)+13\,Y(s) = \frac{1}{s^2+4s+13}$$

$$\left(s^2+4s+13\right)Y(s) = s-2+\frac{1}{s^2+4s+13}$$

$$Y(s) = \frac{s-2}{s^2+4s+13}+\frac{1}{(s^2+4s+13)^2}$$

$$Y(s) = \frac{s+2-4}{(s+2)^2+9}+\frac{1}{\left[(s+2)^2+9\right]^2}$$

Step 3 : Taking inverse Laplace transform we get

$$y(t) = e^{-2t}\left\{\frac{s}{s^2+9}+\frac{1}{(s^2+9)^2}\right\}$$

We know,

$$L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}\,(\sin at - at\cos at)$$

$$L^{-1}\frac{s}{(s^2+a^2)} = \cos at$$

$$\therefore \quad y(t) = e^{-2t}\left[\cos 3t+\frac{1}{54}\,(\sin 3t - 3t\cos 3t)\right]$$

Example 7.105 : *Use Laplace transform to solve LC circuit with e.m.f. $E = E_0\,U(t-1)$ initially $i = 0$, $q = 0$ at $t = 0$ and find current at any time t in the circuit.* **(Dec.-2001)**

Solution : Let q be the charge and i be the current in the circuit at any time t.

∴ The differential equation of the given circuit is

$$L\frac{d^2q}{dt^2}+\frac{1}{C}q = E_0\,U(t-1)$$

Step 1 : Taking Laplace transform we get

$$L\cdot L\left[\frac{d^2q}{dt^2}\right]+\frac{1}{C}L[q] = E_0\,L[U(t-1)]$$

Step 2 : Let L q (t) = Q (s)

$$\text{i.e.} \therefore\ L\left[s^2\,Q(s)-s\,q(0)-q'(0)\right]+\frac{1}{C}Q(s) = E_0\frac{e^{-s}}{s}$$

Step 3 : Substituting q (0) = 0, q' (0) = 0

$$\therefore \quad \left[L s^2 + \frac{1}{C}\right] Q(s) - 0 - 0 = E_0 \frac{e^{-s}}{s}$$

$$\therefore \quad Q(s) = \frac{E_0}{\left(Ls^2 + \frac{1}{C}\right)} \frac{e^{-s}}{s} = \frac{E_0}{L} \frac{1}{\left(s^2 + \frac{1}{LC}\right)} \frac{e^{-s}}{s}$$

$$\therefore \quad Q(s) = \frac{E_0}{L}\left[\frac{e^{-s}}{s\left(s^2+\omega^2\right)}\right] \quad \text{... (1) where } \omega^2 = \frac{1}{LC}$$

Step 4 : Consider

Now

$$L^{-1}\left[\frac{1}{s\left(s^2+\omega^2\right)}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2+\omega^2}\right] dt = \frac{1}{\omega}\int_0^t \sin \omega t \, dt$$

$$= \frac{1}{\omega}\left[-\frac{\cos \omega t}{\omega}\right]_0^t = -\frac{1}{\omega^2}[\cos \omega t - 1]$$

$$= \frac{1}{\omega^2}(1 - \cos \omega t)$$

Step 5 : Taking inverse Laplace transform of (1)

$$\therefore \quad q(t) = \frac{E_0}{L\omega^2}[1 - \cos \omega (t-1)] \; U(t-1)$$

$$= \frac{E_0}{L}(L\,C)\left[1 - \cos \frac{(t-1)}{\sqrt{LC}}\right] U(t-1) \quad \text{... } \because \omega^2 = \frac{1}{LC}$$

$$\therefore \quad q(t) = E_0 \, C\left[1 - \cos \frac{(t-1)}{\sqrt{LC}}\right] U(t-1)$$

Step 6 : $\therefore$ The current i at any time t is $i = \frac{dq}{dt}$ $\therefore$ differentiating w.r.t. t we get

$$i(t) = \frac{d}{dt}\left\{E_0 \, C\left[1 - \cos \frac{1}{\sqrt{LC}}(t-1)\right] U(t-1)\right\}$$

$$= E_0 C\left\{\left[0 + \frac{1}{\sqrt{LC}} \sin \frac{(t-1)}{\sqrt{LC}}\right] U(t-1) + \left[1 - \cos \frac{1}{\sqrt{LC}}(t-1)\right] U'(t-1)\right\}$$

$$\therefore \quad i(t) = E_0 C\left\{\left[\frac{1}{\sqrt{LC}} \sin \frac{1}{\sqrt{LC}}(t-1)\right] U(t-1) + \left[1 - \cos \frac{1}{\sqrt{LC}}(t-1)\right] \delta(t-1)\right\}$$

$$\text{... } U'(t-a) = \delta(t-a)$$

Example 7.106 : *A constant electromotive force E is applied at t = 0 to an electrical circuit consisting of an inductance of L Henry, resistance R ohm and capacitor C farad in series. The initial value of the current and charge are zero. Find current i at any time t.*

Solution : Let q be the charge and i be the current in the circuit at any time t.

$\therefore$ Applying Kirchhoff's law we get,

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = E \quad \text{and we have } i(0) = 0$$

Step 1 : Taking Laplace transform of given equation

$$L \cdot L\left[\frac{di}{dt}\right] + R\,L[i(t)] + \frac{1}{C}L\left[\int_0^t i\,dt\right] = E\,L[1]$$

i.e. $$L[sI(s) - i(0)] + R\,I(s) + \frac{1}{C}\frac{1}{s}I(s) = \frac{E}{s} \qquad \text{where } L[i(t)] = I(s)$$

Step 2 : Substituting i (0) = 0

i.e. $$\left(Ls + R + \frac{1}{Cs}\right)I(s) = \frac{E}{s}$$

$$\therefore \quad I(s) = \frac{E}{Ls^2 + Rs + \frac{1}{C}} = \frac{E}{L}\frac{1}{\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

Step 3 : Adjusting the perfect square.

$$= \frac{E}{L}\frac{1}{\left[\left(s^2 + 2\frac{Rs}{2L} + \frac{R^2}{4L^2}\right) + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)\right]} = \frac{E}{L}\frac{1}{\left(s + \frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

Step 4 : Let $\frac{R}{2L} = a$ and $\frac{1}{LC} - \frac{R^2}{4L^2} = \omega^2$

$$\therefore \quad I(s) = \frac{E}{L}\frac{1}{(s+a)^2 + \omega^2} \qquad \text{if } \omega^2 > 0$$

$$= \frac{E}{L}\frac{1}{(s+a)^2} \qquad \text{if } \omega = 0$$

$$I(s) = \frac{E}{L}\frac{1}{(s+a)^2 - \omega^2} \qquad \text{if } \omega^2 < 0$$

Step 5 : Taking inverse Laplace transform

Case 1 : $\omega^2 > 0$

$$i(t) = \frac{E}{L} L^{-1}\left[\frac{1}{(s+a)^2+\omega^2}\right] = \frac{E}{L} e^{-at} L^{-1}\left[\frac{1}{s^2+\omega^2}\right]$$

$$= \frac{E}{L\omega} e^{-at} \sin \omega t \qquad \text{if } \omega^2 > 0$$

Case 2 : $\omega^2 = 0$

$$= \frac{E}{L} L^{-1}\left[\frac{1}{(s+a)^2}\right] = \frac{E}{L} e^{-at} L^{-1}\left[\frac{1}{s^2}\right] = \frac{E}{L} e^{-at}\, t \qquad \omega^2 = 0$$

Case 3 : $\omega^2 < 0$

$$= \frac{E}{L} L^{-1}\left[\frac{1}{(s+a)^2-\omega^2}\right] = \frac{E}{L} e^{-at} L^{-1}\left[\frac{1}{s^2-\omega^2}\right]$$

$$= \frac{E}{L\omega} e^{-at} \sinh \omega t, \qquad \omega^2 < 0, \text{ where } \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Example 7.107 : *Solve the following differential equation by Laplace transform method* $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = t\,\delta(t-1)$ *for which* $y(0) = y'(0) = 1$. **(May-2001)**

Solution : We have, $y'' + 3y' + 2y = t\,\delta(t-1)$

Step 1 : Taking Laplace transforms we get

$$L[y''] + 3L[y'] + 2L[y] = L[t\,\delta(t-1)]$$

i.e. $[s^2 Y(s) - s\,y(0) - y'(0)] + 3[s\,Y(s) - y(0)] + 2Y(s) = e^{-s}$ (1)

$$\ldots \because L[f(t)\,\delta(t-a)] = e^{-as} f(a),\ f(t) = t$$

Step 2 : Substituting

$$y(0) = y'(0) = 1$$

$$\therefore \quad [s^2 Y(s) - s - 1] + 3[s\,Y(s) - 1] + 2Y(s) = e^{-s}$$

i.e. $$(s^2 + 3s + 2)\,Y(s) - s - 4 = e^{-s}$$

$$\therefore \quad (s^2 + 3s + 2)\,Y(s) = e^{-s} + s + 4$$

$$\therefore \quad Y(s) = \frac{e^{-s}}{s^2+3s+2} + \frac{s+4}{s^2+3s+2} = \frac{e^{-s}}{(s+1)(s+2)} + \frac{s+4}{(s+1)(s+2)}$$

$$= e^{-s}\left[\frac{1}{s+1} - \frac{1}{s+2}\right] + \left[\frac{3}{s+1} - \frac{2}{s+2}\right]$$

Step 3 : Taking inverse Laplace transform we get

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[e^{-s}\left(\frac{1}{s+1}-\frac{1}{s+2}\right)\right] + 3\,L^{-1}\left[\frac{1}{s+1}\right] - 2\,L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \left[e^{-(t-1)} - e^{-2(t-1)}\right] U(t-1) + 3e^{-t} - 2e^{-2t}$$

7.18 Solution of Simultaneous Differential Equations using Laplace Transforms

Example 7.108 : *Solve the following simultaneous equations using Laplace transform method,* $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$ *given* $x = y = 0$ *when* $t = 0$.

Solution : Step 1 : Taking Laplace transform of both the equations we get

$$L\left[\frac{dx}{dt}\right] + 5\,L[x] - 2\,L[y] = L[t]$$

i.e. $$[s\,X(s) - x(0)] + 5\,X(s) - 2\,Y(s) = \frac{1}{s^2} \quad \text{and} \because x(0) = 0$$

$\therefore$ $$s\,X(s) - 0 + 5\,X(s) - 2\,Y(s) = \frac{1}{s^2}$$

i.e. $$(s+5)\,X(s) - 2\,Y(s) = \frac{1}{s^2} \qquad \dots (1)$$

and from second equation

$$L[2X] + L\left[\frac{dy}{dt}\right] + L[y] = L[0]$$

i.e. $$2\,X(s) + [s\,Y(s) - y(0)] + Y(s) = 0 \quad \text{and} \because y(0) = 0$$

$\therefore$ $$2\,X(s) + s\,Y(s) + Y(s) = 0$$

i.e. $$2\,X(s) + (s+1)\,Y(s) = 0 \qquad \dots (2)$$

Step 2 : Now, solving equations (1) and (2) by Cramer's rule we get

$$X(s) = \frac{\begin{vmatrix} \frac{1}{s^2} & -2 \\ 0 & s+1 \end{vmatrix}}{\begin{vmatrix} s+5 & -2 \\ 2 & s+1 \end{vmatrix}} = \frac{\frac{1}{s^2}(s+1)}{(s^2+6s+5)+4} = \frac{1}{s^2}\,\frac{(s+1)}{(s+3)^2}$$

Step 3 : Find partial fraction for X (s).

Now, $$\frac{s+1}{s^2\,(s+3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{(s+3)^2} \qquad \dots (3)$$

Using partial fraction we get $A = \frac{1}{27}$, $B = \frac{1}{9}$, $C = -\frac{1}{27}$, $D = \frac{-2}{9}$

Step 4 : Substituting A, B, C, D we get

$$X(s) = \frac{s+1}{s^2 (s+3)^2} = \frac{1}{27}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2} - \frac{1}{27}\frac{1}{(s+3)} - \frac{2}{9}\frac{1}{(s+3)^2}$$

Step 5 : $\therefore$ Taking Inverse Laplace transform we have

$$L^{-1} X(s) = x(t) = \frac{1}{27} L^{-1}\left[\frac{1}{s}\right] + \frac{1}{9} L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{27} L^{-1}\left[\frac{1}{s+3}\right] - \frac{2}{9} L^{-1}\left[\frac{1}{(s+3)^2}\right]$$

$$= \frac{1}{27}(1) + \frac{1}{9} t - \frac{1}{27} e^{-3t} - \frac{2}{9} e^{-3t} L^{-1}\left[\frac{1}{s^2}\right]$$

i.e.
$$x(t) = \frac{1}{27} + \frac{t}{9} - \frac{1}{27} e^{-3t} - \frac{2}{9} e^{-3t}\, t \qquad \ldots (4)$$

Step 6 : Put $X(s) = \frac{s+1}{s^2 (s+3)^2}$ in (2) to find Y (s).

$$(s+1)\, Y(s) = -2\, X(s) = \frac{-2(s+1)}{s^2 (s+3)^2}$$

$$\therefore \quad Y(s) = -\frac{2}{s^2 (s+3)^2} = \frac{4}{27}\frac{1}{s} - \frac{2}{9}\frac{1}{s^2} - \frac{4}{27(s+3)} - \frac{2}{9(s+3)^2}$$

... Using partial fractions method as for X (s)

Step 7 : Taking inverse Laplace transform we have,

$$L^{-1} Y(s) = y(t) = \frac{4}{27} L^{-1}\left[\frac{1}{s}\right] - \frac{2}{9} L^{-1}\left[\frac{1}{s^2}\right] - \frac{4}{27} L^{-1}\left[\frac{1}{s+3}\right] - \frac{2}{9} L^{-1}\left[\frac{1}{(s+3)^2}\right]$$

i.e.
$$y(t) = \frac{4}{27} - \frac{2}{9} t - \frac{4}{27} e^{-3t} - \frac{2}{9} e^{-3t} L^{-1}\left[\frac{1}{s^2}\right]$$

i.e.
$$y(t) = \frac{4}{27} - \frac{2}{9} t - \frac{4}{27} e^{-3t} - \frac{2}{9} e^{-3t}\, t \qquad \ldots (5)$$

Thus (4) and (5) give the required solutions.

Example 7.109 : *Use Laplace transforms to solve the following simultaneous differential equations* $\frac{dx}{dt} + \frac{dy}{dt} + x = -e^{-t}$, $\frac{dx}{dt} + 2\frac{dy}{dt} + 2x + 2y = 0$ *subject to* $x(0) = -1$, $y(0) = 1$.

Solution : Step 1 : Taking Laplace transforms of the given equations we get,

$$L\left[\frac{dx}{dt}\right] + L\left[\frac{dy}{dt}\right] + L[x(t)] = L[-e^{-t}]$$

i.e. $[s\, X(s) - x(0)] + [s\, Y(s) - y(0)] + X(s) = -\frac{1}{s+1}$

Step 2 : Substituting x (0) = – 1, y (0) = 1

i.e. $$s\,X(s) + 1 + s\,Y(s) - 1 + X(s) = -\frac{1}{s+1}$$

i.e. $$(s+1)\,X(s) + s\,Y(s) = -\frac{1}{s+1} \qquad \text{... (1)}$$

Step 3 : From the second equation

$$L\left[\frac{dx}{dt}\right] + 2L\left[\frac{dy}{dt}\right] + 2L\,[x] + 2\,L[y] = L\,[0]$$

i.e. $[s\,X(s) - x(0)] + 2\,[s\,Y(s) - y(0)] + 2\,X(s) + 2\,Y(s) = 0$

Step 4 : Substituting x (0) = – 1, y (0) = 1

i.e. $[s\,X(s) + 1] + 2\,[s\,Y(s) - 1] + 2\,X(s) + 2\,Y(s) = 0$

i.e. $$(s+2)\,X(s) + 2\,(s+1)\,Y(s) = 1 \qquad \text{... (2)}$$

Step 5 : Solving equations (1) and (2) by Cramer's rule we get,

$$X(s) = \frac{\begin{vmatrix} -\frac{1}{s+1} & s \\ 1 & 2\,(s+1) \end{vmatrix}}{\begin{vmatrix} s+1 & s \\ s+2 & 2\,(s+1) \end{vmatrix}} = \frac{-2-s}{2\left(s^2+2s+1\right)-\left(s^2+2s\right)} = \frac{-(s+2)}{s^2+2s+2}$$

$$\therefore \quad X(s) = -\left[\frac{(s+1)+1}{(s+1)^2+1}\right] = -\frac{(s+1)}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

Step 6 : Taking inverse Laplace transforms

$$\therefore \quad x(t) = L^{-1}\,[X(s)] = -L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= -e^{-t}L^{-1}\left[\frac{s}{s^2+1}\right] - e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] \qquad \text{... by first shifting theorem}$$

$$= -e^{-t}\cos t - e^{-t}\sin t = -e^{-t}\,(\cos t + \sin t) \qquad \text{... (3)}$$

Step 7 : Similarly using Cramers rule we get

$$Y(s) = \frac{\begin{vmatrix} s+1 & -\frac{1}{s+1} \\ s+2 & 1 \end{vmatrix}}{\begin{vmatrix} s+1 & s \\ s+2 & 2\,(s+1) \end{vmatrix}} = \frac{s+1+\frac{s+2}{s+1}}{2\left(s^2+2s+1\right)-\left(s^2+2s\right)} = \frac{\left(s^2+2s+1\right)+s+2}{(s+1)\left(s^2+2s+2\right)}$$

$$= \frac{s^2+3s+3}{(s+1)\left(s^2+2s+2\right)} = \frac{(s+1)^2+(s+1)+1}{(s+1)\left[(s+1)^2+1\right]}$$

Step 8 : Taking inverse Laplace transforms we get

$$\therefore \quad y(t) = L^{-1}\left[\frac{(s+1)^2+(s+1)+1}{(s+1)\left[(s+1)^2+1\right]}\right] = e^{-t}L^{-1}\left[\frac{s^2+s+1}{s\left(s^2+1\right)}\right]$$

... by first shifting theorem

$$= e^{-t}\ L^{-1}\left[\frac{\left(s^2+1\right)+s}{s\left(s^2+1\right)}\right] = e^{-t}L^{-1}\left[\frac{1}{s}+\frac{1}{s^2+1}\right]$$

i.e. $$y(t) = e^{-t}\ (1+\sin t) \qquad \dots (4)$$

Thus (3) and (4) gives the required solution.

Exercise 7.11

(I) Find the solution of each of the following differential equation using Laplace transforms

1. $y''+y=\sin 3t, \quad y(0)=0,\ y'(0)=0$ **[Ans. : $\frac{1}{8}$ [3 sin t − sin 3t]]**
2. $\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=0 \quad y(0)=1,\ y'(0)=2,\ y''(0)=2$ **[Ans. : $\frac{5}{3}e^{x}-e^{-x}+\frac{1}{3}e^{-2x}$]**
3. $y''+2y'+2y=5\sin t \quad y(0)=y'(0)=0$ **[Ans. : $2e^{-t}\cos t+e^{-t}\sin t-2\cos t+\sin t$]**
4. $y''+4y=H(t-2)+\delta(t) \quad y(0)=0,\ y'(0)=1$ **[Ans. : $\frac{1}{4}$ [1 − cos 2(t − 2)] H (t − 2) + sin 2t]**
5. $y''+y'-2y=1-2t, \quad y(0)=0,\ y'(0)=4$ **[Ans. : $y=e^{t}-e^{-2t}+t$]**
6. $y''+y=t\cos 2t \quad y(0)=y'(0)=0$ **[Ans. : $y=\frac{4}{9}\sin 2t-\frac{5}{9}\sin t-\frac{t}{3}\cos 2t$]**
7. $y'+4y+5\int_0^t y\,dt=e^{-t}, \quad y(0)=0$ **[Ans. : $\frac{1}{2}\left(1+e^{-2t}-2e^{-t}\right)$]**
8. $y'+y-2\int_0^t y\,dt=\frac{1}{2}t^2, \quad y(0)=1$ **[Ans. : $y(t)=\frac{e^t}{3}+\frac{11}{12}e^{-2t}-\frac{t}{2}-\frac{1}{4}$]**
9. $y''+y'-2y=2\left(1+t-t^2\right), \quad y(0)=0,\ y'(0)=3$ **[Ans. : $y(t)=t^2-e^{-2t}+e^{t}$]**
10. $y''+y'-2y=3\cos 3t-11\sin 3t \quad y(0)=0,\ y'(0)=6$ **[Ans. : $y(t)=e^{t}-e^{-2t}+\sin 3t$]**
11. $y''+4y'+3y=10\sin t \quad y(0)=y'(0)=0$ **[Ans. : $y(t)=\frac{5e^{-t}}{2}-\frac{1}{2}e^{-3t}+\sin t-2\cos t$]**
12. $y'''+y'=2,\ y(0)=3, \quad y'(0)=1,\ y''(0)=2$ **[Ans. : $y(t)=5+2t-\sin t-2\cos t$]**
13. $y''+4y'+3y=10\sin t \quad y(0)=y'(0)=0$ **[Ans. : $y(t)=\frac{5}{2}e^{-t}-\frac{1}{2}e^{-3t}+\sin t-2\cos t$]**
14. $y''-2y'+y=e^{-2t} \quad y(0)=y'(0)=0$ **[Ans. : $y(t)=-\frac{1}{9}e^{t}+\frac{1}{3}t\,e^{3t}+\frac{1}{9}e^{-2t}$]**
15. $y''+4y'+8y=1 \quad y(0)=0,\ y'(0)=1$ **[Ans. : $y(t)=\frac{1}{8}-\frac{1}{8}e^{2t}(\cos 2t-3\sin 2t)$]**

16. $y'' + 3y' + 2y = f(t) \quad y(0) = y'(0) = 0$

where $f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases}$ **[Ans. :** $\frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2} - \left\{\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right\} U(t-1)$**]**

17. $y'' - 3y' + 2y = 4e^{2t} \quad y(0) = -3, \ y'(0) = 5$ **[Ans. :** $y(t) = -7e^{t} + 4e^{2t} + 4t\, e^{2t}$**]**

18. $y'' - 3y' + 2y = 4t + e^{3t}, \quad y(0) = 1, \ y'(0) = -1$ **[Ans. :** $y(t) = 3 + 2t + \frac{1}{2}e^{3t} - 2e^{2t} - \frac{1}{2}e^{t}$**]**

(II) *Solve the system of equations using Laplace transforms*

1. $\left.\begin{array}{l} \frac{dx}{dt} + y = \sin t \\ \frac{dy}{dt} + x = \cos t \end{array}\right\}$ *Given* $x(0) = 2, \ y(0) = 0$ **[Ans.** $x = e^{t} + e^{-t}, \ y = e^{-t} - e^{t} + \sin t$**]**

2. $\left.\begin{array}{l} \frac{dx}{dt} - y = e^{t} \\ \frac{dy}{dt} + x = \sin t \end{array}\right\}$ *Given* $x(0) = 1, \ y(0) = 0$ **[Ans. :** $x = \frac{e^{t}}{2} + \frac{1}{2}\cos t + \sin t - \frac{t}{2}\cos t$

$y = -\frac{e^{t}}{2} - \frac{1}{2}\sin t + \frac{1}{2}\cos t - \frac{t}{2}\sin t$**]**

3. $\left.\begin{array}{l} \frac{dx}{dt} + \frac{dy}{dt} + x = -e^{-t} \\ \frac{dx}{dt} + 2\frac{dy}{dt} + 2x + 2y = 0 \end{array}\right\}$ *Given* $x(0) = -1, \ y(0) = 1$ **[Ans. :** $x = -e^{-t}(\cos t + \sin t)$

$y = e^{-t}(1 + \sin t)$**]**

4. $\frac{dx}{dt} + x(t) + 3\int_0^t y(t)\, dt = \cos t + 3\sin t$

$\frac{2dx}{dt} + \frac{3dy}{dt} + 6y = 0, \ x(0) = -3 \quad y(0) = 2$ **[Ans. :** $x(t) = \sin t - 2\cos t - e^{-3t}$,

$y(t) = \frac{-2}{3}\sin t + 2e^{-3t}$**]**

5. $\left.\begin{array}{l} \frac{dx}{dt} = 2x - 3y \\ \frac{dy}{dt} = y - 2x \end{array}\right\}$ $x(0) = 8, \ y(0) = 3$ **[Ans. :** $x = 3e^{4t} + 5e^{4t} + 5e^{-t}, \ y = 5e^{-t} - 2e^{4t}$**]**

6. $\left.\begin{array}{l} x' + y' = t \\ x'' - y = e^{-t} \end{array}\right\}$ $x(0) = 3, \ y(0) = 0 \ x'(0) = -2$ **[Ans. :** $x(t) = \frac{1}{2}\cos t - \frac{3}{2}\sin t + \frac{1}{2}e^{-t} + \frac{t^2}{2} + 2$

$y(t) = -\frac{1}{2}\cos t + \frac{3}{2}\sin t - \frac{1}{2}e^{-t} + 1$**]**

7. $\left.\begin{array}{l} \frac{dx}{dt} + 3x + y = 0 \\ \frac{dy}{dt} + x + y = 0 \end{array}\right\}$ $x(0) = 1, \ y(0) = 1$ **[Ans. :** $x = (1 - 2t)\, e^{-2t}, \ y = (1 + 2t)\, e^{-2t}$**]**

8. $\left.\begin{array}{l}\dfrac{dx}{dt}-\dfrac{dy}{dt}+2y=\cos 2t\\ \dfrac{dx}{dt}+\dfrac{dy}{dt}-2x=\sin 2t\end{array}\right\}$ *Given* $x(0)=0,\ y(0)=-1$ **[Ans. : $x=\dfrac{e^t}{2}(\cos t+\sin t)-\dfrac{1}{2}\cos 2t$**

$y=-e^t(\cos t-\sin t)-\sin 2t$]

9. $\left.\begin{array}{l}\left(D^3-3\right)x-4y=0\\ x+\left(D^2+1\right)y=0\end{array}\right\}$ $x(0)=y(0)=y'(0)=0,\ x'(0)=2$ **[Ans. : $x=2t\cosh t$,**

$y=(1-t)\sinh t$]

(III) *The current flowing in an electrical circuit is given by* $L\dfrac{di}{dt}+Ri=E$ *where L, R, E are constants. Using Laplace transform find i in terms of t, given that at* $t=0,\ i=0$. **[Ans. : $i=\dfrac{E}{R}\left(1-e^{-Rt/L}\right)$]**

University Questions

Dec. - 98

1. *Solve any two :*

   i) *Obtain* $L^{-1}\left[\tan^{-1}\dfrac{1}{s}\right]$

   ii) *Obtain* $L^{-1}\left[\dfrac{s-2}{s^2+2s+9}\right]$

   iii) *Evaluate* $\int_0^{\infty} e^{-3t}\, t\cos 2t\, dt$ *by using the definition of Laplace transform.*

[6 Marks]

2. *Obtain* $L^{-1}\left[\dfrac{s}{(s+1)\left(s^2+1\right)}\right]$ *by using convolution theorem.*

[4 Marks]

May - 99

1. *Solve (any two) :*

   i) *Obtain* $L^{-1}\left[\log\dfrac{s^2+a^2}{s^2+b^2}\right]$

   ii) *Obtain* $L^{-1}\left[\dfrac{s-1}{s^2+6s+25}\right]$

   iii) *Evaluate* $\int_0^{\infty}\dfrac{\cos 6t-\cos 4t}{t}\,dt$

[6 Marks]

2. *Solve by using Laplace transform method :*

   $\dfrac{d^2y}{dt^2}-3\dfrac{dy}{dt}+2y=12e^{-2t}$ *given* $y(0)=2,\ y(0)=6.$

[5 Marks]

Dec. - 99

1. *Solve :* $y'' - 3y' + 2y = 4e^{2t}$ *using Laplace transform, given that* $y(0) = y'(0) = 0.$ **[7 Marks]**

May - 2000

1. *Find inverse laplace transform of any two of the following :*

   i) $\dfrac{s}{s^4 + 4a^4}$

   ii) $\log\left(1 + \dfrac{a^2}{s^2}\right)$

   iii) $\dfrac{s+7}{s^2 + 2s + 2}$ **[6 Marks]**

2. *Solve by using Laplace transform :*

   $\left(D^2 + n^2\right)x = a\sin(nt + \alpha)\left(D \equiv \dfrac{d}{dt}\right)$

   [Given $x(0) = x^1 / x'(0) = 0$*]* **[5 Marks]**

3. *Use convolution theorem to find :* $L^{-1}\left\{\dfrac{1}{s\sqrt{s+4}}\right\}$ **[5 Marks]**

Dec. - 2000

1. *Solve by using Laplace transform method :*

   $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 5y = e^{-t}\cdot\sin t$.

   Given that $y(0) = 0;\ \ y'(0) = 1.$ **[6 Marks]**

May - 2001

1. *Show that :* $L^{-1}\left\{\dfrac{s\cdot\cos\alpha + \omega\cdot\sin\alpha}{s^2 + \omega^2}\right\} = \cos(\omega t - \alpha)$

   (ω*,* α *being constants).* **[3 Marks]**

2. *Solve by using Laplace transform method :*

   $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = te^{-t}$ *with* $y(0) = 1,\ \ y'(0) = -2.$ **[6 Marks]**

Dec. - 2001

1. *Find the inverse Laplace transform of* $\dfrac{3s+5}{s^2 + 2s + 1}$ **[4 Marks]**

2. *Solve by using Laplace transform method*

   $y'' + 3y' + 2y = U(t-2)$ *with* $y(0) = y'(0) = 0.$ **[5 Marks]**

May - 2002

1. *Find the inverse Laplace transforms of any two of the following :*

   i) $\dfrac{4s+3}{s^3 - 3s^2 + 2s}$

ii) $\dfrac{s}{(s-6)^4}$

iii) $\tan^{-1}\left(\dfrac{1}{s}\right)$. **[6 Marks]**

2. *Solve the following differential equation by using the Laplace transform method :*

$\dfrac{d^2 y}{dt^2} - 3\dfrac{dy}{dt} + 2y = 12\, e^{-2t}$ *provided* $y(0) = 2,\ y'(0) = 6.$ **[6 Marks]**

Dec. - 2002

1. *Find the inverse Laplace transform of any two of the following :*

i) $\log\left(\dfrac{s+b}{s+a}\right)$ ii) $\dfrac{s^2}{(s^2+4)^2}$

iii) *Using convolution theorem* $\dfrac{s}{(s^2+4)(s-2)}$ **[6 Marks]**

May - 2003

1. *Find inverse Laplace transform (any two) :*

i) $\dfrac{1}{\sqrt{7s+6}}$ ii) $\dfrac{3s+1}{(s+1)^4}$ iii) $\dfrac{1}{s^2(s^2+a^2)}$

(Use convolution theorem). **[6 Marks]**

Dec. - 2003

1. *Find inverse Laplace transform (any two) :* **[6 Marks]**

i) $\dfrac{3s+1}{(s+1)^4}$

ii) $\dfrac{e^{-3s}}{(s-2)^4}$

iii) $\cot^{-1}s$

May - 2004

1. *Find Laplace inverse of (any two) :* **[8 Marks]**

i) $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

ii) $\dfrac{s}{s^4+4a^4}$

iii) $\dfrac{s e^{-2s}}{s^2-5s+6}$

2. *Evaluate :*

$$\int_0^{\infty} e^{-3t}\left[\frac{\cos 4t - \cos 2t}{t}\right] dt$$

[3 Marks]

Dec. - 2004

1. *Find inverse Laplace transform (any two) :*

   i) $\log\left(1+\dfrac{25}{s^2}\right)$

   ii) $\dfrac{(s+2)^2}{\left(s^2+4s+8\right)^2}$

   iii) $\dfrac{e^{-2s}(3s+1)}{(s-1)^3}$ **[6 Marks]**

2. *Solve by Laplace transform method :*

   $$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=e^{-t}\sin t$$

   $y(0)=0,\ y(0)=1$ **[5 Marks]**

May - 2005

1. *Solve the differential equation by using Laplace transform method :*

   $y''+2y'+y=te^{-t}$, *with* $y(0)=1,\ y'(0)=-2$ **[6 Marks]**

2. *Find inverse Laplace transform of the following (any two) :*

   i) $\dfrac{1}{(s+1)\left(s^2+1\right)}$

   ii) $\dfrac{1}{s}\log\left(\dfrac{s^2+9}{s^2+16}\right)$

   iii) $\dfrac{s+7}{s^2+2s+2}$ **[6 Marks]**

Dec. - 2005

1. *Solve the differential equation :*

   $$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=12e^{-2t}$$

   subject to the conditions $y(0)=2,\ y'(0)=6.$ **[5 Marks]**

2. *Find inverse Laplace transforms of the following (any two) :*

   i) $\dfrac{s}{\left(s^4+a^4\right)}$

   ii) $\tan^{-1}\dfrac{1}{s}$

   iii) $\dfrac{s^2}{\left(s^2+1\right)^2}$ **[6 Marks]**

May - 2006

1. *Solve by using Laplace transform :*

$$y'(t) + \int_0^t y(t)\,dt + 2y(t) = 1$$

where $y(0) = 0$. **[5 Marks]**

2. *Find the inverse Laplace transform :*

$F(s) = \log\left(\dfrac{1-s}{1+s}\right).$ **[5 Marks]**

❑❑❑

8

Fourier Transforms

8.1 Introduction

Necessity is the origin of every research. In day today life we deal with money. Suppose there are 10 coins of one rupee and one note of 10 rupees. i.e. there are two forms of money with same market value. If we need to do a telephone call then 10 Rs. note will not work. i.e. we need different forms of money for our individual use (i.e. application).

In electronics we deal with functions, naturally we need different forms of functions without changing their value.

In our previous studies we have studied the following transformations.

1) Matrices : for solving system of equations.

2) Logarithm : for solving the problems involving multiplication and division.

3) Fourier series : for the representation of periodic functions.

There are many non-periodic functions such as voltage, isolated pulse, decaying exponential unit step, which are non-periodic, and suitable representation of these functions can be obtained by considering the limiting form of Fourier series of a periodic function, as the period becomes infinite.

Thus Fourier integral arises from Fourier series of a function with period ∞ .

Thus Fourier integral is a transformation which transforms a non-periodic function of time t into a function of continuous frequency variable λ

Fourier transforms are very useful in solving boundary value problems in partial differential equations.

8.2 Dirichlet's Conditions

i) A function f(x) is defined and single valued except at finite points in $(-l, l)$.

ii) f(x) is periodic outside $(-l, l)$ with period $2l$.

iii) f(x) and $f'(x)$ are sectionally continuous.

iv) f(x) is integrable in $(-l, l)$.

8.3 Complex Form of Fourier Series (without proof)

If f(x) is a periodic function with period 2L defined in the interval $-L<x<L$ and satisfies Dirichlet's conditions then the Fourier series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \quad \text{... (1)}$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(u)\, du$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(u)\cos\left(\frac{n\pi u}{L}\right) du$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(u)\sin\left(\frac{n\pi u}{L}\right) du$$

Using $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

and rearranging the constants we can express f(x) in equation (1)

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{i\left(\frac{n\pi x}{L}\right)}} \quad \text{... (2)}$$

where

$$\boxed{C_n = \frac{1}{2L}\int_{-L}^{L} f(u)\, e^{-i\left(\frac{n\pi u}{L}\right)} du} \quad \text{... (3)}$$

which is called as complex exponential form of Fourier series.

8.4 Fourier Integral Theorem (without proof)

If a function f(x) is such that

i) f(x) satisfies Dirichlet's conditions.

ii) f(x) is absolutely integrable i.e.

$\int_{-\infty}^{\infty} |f(x)|\, dx$ converges.

Then we can represent f(x) as a integral

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda} \quad \text{... (4)}$$

i.e.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\left[\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du\right] d\lambda$$

8.5 Equivalent Forms of Fourier Integral

From equation (4) (Fourier integral theorem)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda$$

We know that $e^{-i\theta} = \cos\theta - i\sin\theta$

$$\therefore \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)[\cos\lambda(u-x) - i\sin\lambda(u-x)]\, du\, d\lambda$$

We know that $\int_{-a}^{a} F(\lambda)\, d\lambda = 0$ if $F(\lambda)$ is an odd function of λ.

$$= 2\int_{0}^{a} F(\lambda)\, d\lambda \text{ if } F(\lambda) \text{ is an even function of } \lambda.$$

As $\sin\lambda(u-x)$ is an odd function of λ

and $\cos\lambda(u-x)$ is an even function of λ

$\therefore \int_{-\infty}^{\infty} \sin\lambda(u-x)\, d\lambda = 0$ and

$$\int_{-\infty}^{\infty} \cos\lambda(u-x)\, d\lambda = 2\int_{0}^{\infty} \cos\lambda(u-x)\, d\lambda$$

$\therefore$ The integral becomes,

$$f(x) = \frac{2}{2\pi}\int_{\lambda=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\lambda(u-x)\, du\, d\lambda$$

As $\cos(A-B) = \cos A\cos B + \sin A \sin B$

$$\therefore \quad f(x) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\,[\cos\lambda u\cos\lambda x + \sin\lambda u\sin\lambda x]\, du\, d\lambda$$

Thus

$$f(x) = \int_{0}^{\infty} [F_1(\lambda)\cos\lambda x + F_2(\lambda)\sin\lambda x]\, d\lambda \qquad \text{... (5)}$$

is another equivalent form of Fourier integral

where

$$F_1(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\lambda u\, du \qquad \text{... (6)}$$

$$F_2(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\lambda u\, du \qquad \text{... (7)}$$

8.6 Fourier Sine Integral

If f(x) is an odd function of x then from equation (6) and (7)

$$F_1(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\lambda u\, du = 0$$

and

$$F_2(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\lambda u\, du$$

$$= \frac{2}{\pi}\int_{0}^{\infty} f(u)\sin\lambda u\, du$$

Substituting $F_1(\lambda)$ and $F_2(\lambda)$ in equation (5) we get

$$\boxed{f(x) = \frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(u)\sin\lambda u\cdot\sin\lambda x\, du\, d\lambda} \quad \dots (8)$$

Equation (8) is called as Fourier sine integral of f(x).

Similarly if f(x) is an even function of x then $F_1(\lambda) = \frac{2}{\pi}\int_{0}^{\infty} f(u)\cos\lambda u du$ and $F_2(\lambda) = 0$.

Substituting in (5) we get the Fourier cosine integral.

$$\boxed{f(x) = \frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(u)\cos\lambda u\cdot\cos\lambda x\, du\, d\lambda} \quad \dots (9)$$

8.7 Fourier Transforms

From Fourier integral theorem i.e. equation (4)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda$$

i.e.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du\right] e^{i\lambda x}\, d\lambda$$

If

$$\boxed{F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du} \quad \dots (10)$$

then

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda} \quad \dots (11)$$

The function $F(\lambda)$ is called as Fourier transform of f(x) (equation (10) gives the expression for Fourier Transform) while f(x) is called as inverse Fourier transform (equation (11) gives the expression for inverse).

8.8 Fourier Sine Transform

If a function defined in $-\infty < x < \infty$ is an odd function of x then from the expression of Fourier sine integral (equation (8))

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(u)\sin\lambda u \cdot \sin\lambda x \, du\, d\lambda$$

i.e.
$$f(x) = \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty f(u)\sin\lambda u\, du\right]\sin\lambda x\, d\lambda$$

If we denote the integral as

$$\boxed{F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du} \qquad \text{... (12)}$$

then
$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\, d\lambda} \qquad \text{... (13)}$$

$F_s(\lambda)$ gives the Fourier sine transform of f(x)

and f(x) gives the inverse Fourier sine transform of $F_s(\lambda)$.

8.9 Fourier Cosine Transform

If a function f(x) defined in $-\infty < x < \infty$ is an even function of x then from the expression of Fourier cosine integral i.e. equation no. (9) we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(u)\cos\lambda u \cdot \cos\lambda x \, du\, d\lambda$$

i.e.
$$f(x) = \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty f(u)\cos\lambda u\, du\right]\cos\lambda x\, d\lambda$$

We denote the integral as

$$\boxed{F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du} \qquad \text{... (14)}$$

then $$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x \, d\lambda$$ ... (15)

$F_c(\lambda)$ gives Fourier cosine transform of f(x) and

f(x) gives inverse Fourier cosine transform of f(x).

Note

1) The expressions f(x) and $F(\lambda)$ form a Fourier transform pair.

2) Even and odd functions : If f(x) is defined in $-\infty < x < \infty$ and $f(-x) = f(x)$ then f(x) is said to be an even function of x.

If $f(-x) = -f(x)$ then it is said to be an odd function of x.

3) $\int_{-a}^{a} f(x)\,dx = 0$ if f(x) is an odd function.

$= 2\int_0^a f(x)\,dx$ if f(x) is an even function.

4) if f(x) is an even function of x then the graph of f(x) is symmetric about y axis. If f(x) is an odd function of x then the graph of f(x) is symmetric about opposite quadrants.

5)

Odd function	×	Odd function	= even
odd function	÷	odd function	= even
odd	±	odd	= odd
odd	×	even	= odd
odd	÷	even	= odd
even	±	even	= even
even	±	odd	= neither odd nor even

6) $\int_0^\infty \frac{\sin ax}{x}\,dx = \frac{\pi}{2}$ if a is positive.

$= -\frac{\pi}{2}$ if a is negative.

7) $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b\cos bx]$

$\int e^{ax} .\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b \sin bx]$

8) $\int_0^{\infty} e^{-au} \sin\lambda u \, du = \left\{ \frac{e^{-au}}{a^2+\lambda^2} [-a \sin\lambda u - \lambda \cos\lambda u] \right\}_0^{\infty}$

$$= \left\{ 0 - \frac{e^0}{a^2+\lambda^2} (0-\lambda) \right\}$$

$$= \frac{\lambda}{a^2+\lambda^2} \quad \left\{ \begin{array}{r} \text{as} \quad e^{-\infty} = 0 \\ e^0 = 1 \\ \sin 0 = 0 \\ \cos 0 = 1 \end{array} \right\}$$

9) $\int_0^{\infty} e^{-au} \cos\lambda u \, du = \left\{ \frac{e^{-au}}{a^2+\lambda^2} [-a \cos\lambda u + \lambda \sin\lambda u] \right\}_0^{\infty}$

$$= 0 - \frac{e^{-0}}{a^2+\lambda^2} [-a+0]$$

$$= \frac{a}{a^2+\lambda^2}$$

10) Integration by parts

$$\int uv \, dx = u \int v \, dx - \int \left(\frac{du}{dx} \int v \, dx \right) dx$$

11) Generalized rule

$$\int uv \, dx = u v_1 - u' v_2 + u'' v_3 \ldots$$

where dashes indicate derivatives and suffixes indicate integrals.

Generalised rule is useful for evaluating the integral of product of two functions where the first function is a function of positive powers of x like x^2, x^3+1, ... and the second function of the product must be sine, cosine or exponential function.

The following table gives the Fourier transform pairs for ready reference.

Sr. No.	Name of the transform	Expression for the transform $F(\lambda)$ =	Inverse transform f(x) =
1.	Fourier transform	$\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\lambda u} du$	$\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda$
2.	Fourier cosine transform Fourier transform for even f(x)	$\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \cos\lambda u \, du$	$\sqrt{\frac{2}{\pi}} \int_0^{\infty} F_C(\lambda) \cos\lambda x \, d\lambda$
3.	Fourier sine transform Fourier transform for odd f(x)	$\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \sin\lambda u \, du$	$\sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\lambda) \sin\lambda u \, du$

We may use the following table also (Note the constant multiples of the integrals)

Sr. No.	Name of the transform	Expression for the transform	Inverse transform
1.	Fourier transform	$F(\lambda)=\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$	$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$
2.	Fourier cosine transform Fourier transform for even f(x)	$F(\lambda)=\int_{0}^{\infty} f(u)\cos\lambda u\, du$	$f(x)=\frac{2}{\pi}\int_{0}^{\infty} F_c(\lambda)\cos\lambda x\, d\lambda$
3.	Fourier sine transform Fourier transform for odd f(x)	$F(\lambda)=\int_{0}^{\infty} f(u)\sin\lambda u\, du$	$f(x)=\frac{2}{\pi}\int_{0}^{\infty} F_s(\lambda)\sin\lambda x\, d\lambda$

Note

$$\int_{-\infty}^{\infty} f(x)\, dx = 0 \qquad \text{if } f(x) \text{ is odd}$$

$$= 2\int_{0}^{\infty} f(x)\, dx \qquad \text{if } f(x) \text{ is even.}$$

8.10 Illustrative Examples : Type I

Example 8.1 : *Find Fourier cosine transform of*

$$f(x) = \begin{cases} \cos x & 0 < x < a \\ 0 & x \geq a \end{cases}$$

Solution : Step 1 : Consider F.C.T. formula

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(u)\cos\lambda u\, du$$

Step 2 : Split the integral and put the value of f(u)

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{2}\left\{\int_{0}^{a} 2\cos u\cdot\cos\lambda u\, du + \int_{a}^{\infty} 0\right\}$$

Step 3 : Use $2\cos\lambda u\cos u = \cos(\lambda-1)\, u + \cos(\lambda+1)\, u$

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_{0}^{a} \cos(\lambda-1)\, u + \cos(\lambda+1)\, u\, du$$

Step 4 : Integrate

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(\lambda-1)\, u}{\lambda-1} + \frac{\sin(\lambda+1)\, u}{\lambda+1}\right]_{0}^{a}$$

Step 5 : Put the limits of u

$$F_c(\lambda) = \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(\lambda-1)\, a}{\lambda-1} + \frac{\sin(\lambda+1)\, a}{\lambda+1}\right]$$

Example 8.2 : *Find Fourier sine transform of*

$$f(x)=\begin{cases} \sin x & 0<x<a \\ 0 & x \geq a \end{cases}$$

Solution : Step 1 : Consider F.S.T. formula

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \sin \lambda u \, du$$

Step 2 : Split the integral and put the value of f(u)

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \left\{ \int_0^a 2 \sin u \cdot \sin \lambda u \, du + \int_a^{\infty} 0 \right\}$$

Step 3 : Use $2 \sin \lambda u \sin u = \cos(\lambda - 1) u - \cos(\lambda + 1) u$

$$\therefore \quad f_s(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^a \cos(\lambda-1) u - \cos(\lambda+1) u \, du$$

Step 4 : Integrate

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(\lambda-1) u}{\lambda-1} - \frac{\sin(\lambda+1) u}{\lambda+1} \right]_0^a$$

Step 5 : Put the limits for u

$$F_s(\lambda) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sin(1-\lambda) a}{1-\lambda} - \frac{\sin(1+\lambda) a}{1+\lambda} \right)$$

Example 8.3 : *Find Fourier sine and cosine transform of* $f(x) = e^{-x} + e^{-2x}$

Solution : Step 1 : Consider F.S.T. formula

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \sin \lambda u \, du$$

Step 2 : Put the value of f(u)

$$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^{\infty} \left(e^{-u} + e^{-2u}\right) \cdot \sin \lambda u \, du \right\}$$

Step 3 : Integrate

$$\therefore \quad F_s(\lambda) = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-u}}{\lambda^2+1} (-\sin \lambda u - \lambda \cos \lambda u) \right]_0^{\infty} +$$

$$\left[\frac{e^{-2u}}{\lambda^2+4} (-2 \sin \lambda u - \lambda \cos \lambda u) \right]_0^{\infty}$$

Step 4 : Put the limits for u

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\lambda}{\lambda^2+1}+\frac{\lambda}{\lambda^2+4}\right]$$

Step 5 : Simplify

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}}\,\frac{\lambda\left(2\lambda^2+5\right)}{\lambda^4+5\lambda^2+4}$$

For finding cosine transform

Step 1 : Consider F.C.T. formula

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(u)\cos\lambda u\,du$$

Step 2 : Put the value of f(u)

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\left\{\int_0^{\infty}\left(e^{-u}+e^{-2u}\right)\cos\lambda u\,du\right\}$$

Step 3 : Integrate

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{2}{\pi}}\left[\frac{e^{-u}}{\lambda^2+1}\left(-\cos\lambda u+\lambda\sin\lambda u\right)\right]_0^{\infty} +$$

$$\left[\frac{e^{-2u}}{\lambda^2+4}\left(-2\cos\lambda u+\lambda\sin\lambda u\right)\right]_0^{\infty}$$

Step 4 : Put the limits for u

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{\lambda^2+1}+\frac{2}{\lambda^2+4}\right]$$

Step 5 : Simplify

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\left[\frac{3\lambda^2+6}{\lambda^4+5\lambda^2+4}\right]$$

Example 8.4 : *Find Fourier sine and cosine transform of*

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & x > 2 \end{cases}$$

Solution : **Step 1 :** Consider F.S.T. formula

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(u)\sin\lambda u\,du$$

Step 2 : Split the integral and substitute the value of f(u)

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}}\left[\int_0^1 u \sin \lambda u \, du + \int_1^2 (2-u)\sin \lambda u \, du\right]$$

Step 3 : Integrating by parts

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}}\left\{\left[u\frac{-\cos \lambda u}{\lambda}\right]_0^1 - \int_0^1 \frac{-\cos \lambda u}{\lambda} du + \left[(2-u)\frac{-\cos \lambda u}{\lambda}\right]_1^2 - \int_1^2 -1\frac{-\cos \lambda u}{\lambda} du\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-\cos \lambda}{\lambda} + \frac{\sin \lambda}{\lambda^2} + \frac{\cos \lambda}{\lambda} - \frac{\sin 2\lambda}{\lambda^2} + \frac{\sin \lambda}{\lambda^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin \lambda}{\lambda^2} - \frac{\sin 2\lambda}{\lambda^2}\right]$$

Step 4 : As $\sin 2\lambda = 2 \sin \lambda \cos \lambda$ we get

$$\therefore \quad F_s(\lambda) = \sqrt{\frac{2}{\pi}}\left(\frac{2 \sin \lambda(1-\cos \lambda)}{\lambda^2}\right)$$

Now for finding cosine transform

Step 5 : Consider F.C.T. formula

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos \lambda u \, du$$

Step 6 : Split the integral and substitute the value of f(u)

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\left[\int_0^1 u \cos \lambda u \, du + \int_1^2 (2-u)\cos \lambda u \, du\right]$$

Step 7 : Integrating by parts,

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\left\{\left[u\frac{\sin \lambda u}{\lambda}\right]_0^1 - \int_0^1 \frac{\sin \lambda u}{\lambda} du + \left[(2-u)\frac{\sin \lambda u}{\lambda}\right]_1^2 - \int_1^2 -1\frac{\sin \lambda u}{\lambda} du\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin \lambda}{\lambda} + \frac{\cos \lambda}{\lambda^2} - \frac{1}{\lambda^2} - \frac{\sin \lambda}{\lambda} - \frac{\cos 2\lambda}{\lambda^2} + \frac{\cos \lambda}{\lambda^2}\right]$$

Step 8 : Thus $F_c(\lambda) = \sqrt{\frac{2}{\pi}}\left(\frac{2\cos \lambda - \cos 2\lambda - 1}{\lambda^2}\right)$

Example 8.5 : *Find Fourier transform of*

$$f(x) = \begin{cases} 1 & |x| \le a \\ 0 & |x| > a \end{cases}$$

Solution : **Step 1 :** As f(x) is an even function of x

$$\because \quad f(-x) = f(x)$$

i.e. by changing x by – x, f(x) remains unchanged.

$\therefore$ We find F.C.T.

$$F(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \cos \lambda u \, du$$

Step 2 : As $\quad f(x) = 1, \quad -a < x < a$

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_{-a}^{a} 1 \cdot \cos \lambda u \, du$$

Step 3 : Integrate

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \cdot \left[\frac{\sin \lambda u}{\lambda}\right]_0^a$$

Step 4 : Put the limits

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin \lambda a}{\lambda}\right)$$

Example 8.6 : *Find Fourier integral representation of*

$$f(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

Hence evaluate $\int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$ *and* $\int_0^{\infty} \frac{\sin \lambda}{\lambda} d\lambda$

Solution : **Step 1 :** As f(x) is an even function of x we find F.C.T.

$$F(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \cos \lambda u \, du$$

Step 2 : As $\quad f(x) = 1, \quad -1 < x < 1$

$$\therefore \quad F(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{1} 1 \cdot \cos \lambda u \, du$$

Step 3 :

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^1 \cos \lambda u \, du$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{\sin \lambda u}{\lambda}\right]_0^1$$

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin \lambda}{\lambda}\right)$$

Step 4 : Now to prove next part consider inverse Fourier cosine transform.

$$\therefore \quad f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\lambda) \cos \lambda x \, d\lambda$$

Step 5 : Put the value of $F(\lambda)$.

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \left(\frac{\sin \lambda}{\lambda}\right) \cos \lambda x \, d\lambda$$

$$\frac{\pi}{2} f(x) = \int_0^\infty \frac{\sin \lambda}{\lambda} \cos \lambda x \, d\lambda$$

$\therefore$ The value of the integral $\int_0^\infty \frac{\sin \lambda}{\lambda} \cos \lambda x \, d\lambda$ is $\frac{\pi}{2} f(x)$

$$\therefore \quad \int_0^\infty \frac{\sin \lambda}{\lambda} \cos \lambda x \, d\lambda = \frac{\pi}{2} f(x)$$

Step 6 : Again put $x = 0$

$$\therefore \quad \int_0^\infty \frac{\sin \lambda}{\lambda} d\lambda = \frac{\pi}{2} f(0)$$

Now from the definition of $f(x)$, $f(0) = 1$

$$\therefore \quad \int_0^\infty \frac{\sin \lambda}{\lambda} d\lambda = \frac{\pi}{2}$$

Example 8.7 : *Find Fourier transform of*

$$f(x) = \begin{cases} 1 - x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

Hence evaluate $\int_0^\infty \left(\frac{x \cos x - \sin x}{x^3}\right) \cos \frac{x}{2} \, dx$

Solution : **Step 1 :** As f(x) is an even function of x we find F.C.T. of f(x)

$$\text{F.C.T. } F(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(u) \cos \lambda u \, du$$

Step 2 : Split the integral into two parts as substitute values of f(u)

$$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^1 \left(1-u^2\right) \cos \lambda u \, du + \int_1^\infty 0 \right\}$$

Step 3 : Use generalized formula for integration by parts

i.e. $$\int uv \, dx = u v_1 - u' v_2 + u'' v_3 \ldots$$

dashes indicates derivatives and suffixes indicates integrals.

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \left(1-u^2\right) \cos \lambda u \, du$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(1-u^2\right) \left(\frac{\sin \lambda u}{\lambda} \right) - (-2u) \left(\frac{-\cos \lambda u}{\lambda^2} \right) + (-2) \left(\frac{-\sin \lambda u}{\lambda^3} \right) \right]_0^1$$

Step 4 : Substitute upper and lower limits

$$= \sqrt{\frac{2}{\pi}} \left\{ 0 - \frac{2 \cos \lambda}{\lambda^2} + \frac{2 \sin \lambda}{\lambda^3} - (0-0-0) \right\}$$

Step 5 : Simplify

$$F(\lambda) = \sqrt{\frac{2}{\pi}} \left[\frac{-2(\lambda \cos \lambda - \sin \lambda)}{\lambda^3} \right]$$

Step 6 : As it is asked to find the integral therefore we get must find inverse.

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\lambda) \cos \lambda x \, d\lambda$$

Step 7 : Substitute the value of F(λ)

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{-2(\lambda \cos \lambda - \sin \lambda)}{\lambda^3} \cos \lambda x \, d\lambda$$

Step 8 : To get the required integral in terms of the parameter λ put x = 1/2

$$f\left(\frac{1}{2}\right) = \frac{-4}{\pi} \int_0^\infty \frac{\lambda \cos \lambda - \sin \lambda}{\lambda^3} \left(\cos \frac{\lambda}{2} \right) d\lambda$$

Step 9 : As $$f(x) = \begin{cases} 1-x^2 & |x|<1 \\ 0 & |x|>1 \end{cases}$$

$\therefore$ $$f\left(\frac{1}{2}\right) = 1-\left(\frac{1}{2}\right)^2 = \frac{3}{4}$$ substituting we get

$$\left(\frac{3}{4}\right)\left(\frac{-\pi}{4}\right) = \int_0^\infty \left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\frac{\lambda}{2}\,d\lambda$$

Step 10 : $\therefore$ As the variable is not important in definite integrals here replace λ by x

$\therefore$ $$\frac{-3\pi}{16} = \int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx$$

Example 8.8 : *Find Fourier sine transform of*

$$f(x) = \begin{cases} 1 & -2 \le x < 0 \\ -1 & 0 < x \le 2 \end{cases}$$

Hence show that $$\int_0^\infty \frac{(\cos 2w - 1)\sin 2w}{w}\,dw = -\frac{\pi}{2}$$

Solution : Step 1 : Consider F.S.T. formula,

$$F(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\sin\lambda u\,du$$

Step 2 : Split the integral and put the value of f(u),

$$= \sqrt{\frac{2}{\pi}}\left\{\int_0^2 (-1)\sin\lambda u\,du + \int_2^\infty 0\right\}$$

Step 3 : Integrate,

$$= -\sqrt{\frac{2}{\pi}}\left[\frac{-\cos\lambda u}{\lambda}\right]_0^2$$

Step 4 : Put the limits for u,

$$F(\lambda) = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{\lambda}[\cos 2\lambda - 1]$$

Step 5 : Consider inverse F.T

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(\lambda)\sin\lambda x\,d\lambda$$

Step 6 : Put the value of $F(\lambda)$

$$f(x) = \sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos 2\lambda - 1}{\lambda}\sin\lambda x\, d\lambda$$

Step 7 : Put x = 2

$$f(2) = \frac{2}{\pi}\int_0^\infty \frac{(\cos 2\lambda - 1)\sin 2\lambda}{\lambda}\, d\lambda$$

Step 8 : Put f(2) = − 1

$$(-1)\frac{\pi}{2} = \int_0^\infty \frac{(\cos 2\lambda - 1)\sin 2\lambda}{\lambda}\, d\lambda$$

Step 9 : As the variable is not important replacing λ by w we get

$$\int_0^\infty \frac{(\cos 2w - 1)\sin 2w}{w}\, dw = \frac{-\pi}{2}$$

Example 8.9 : *Find Fourier transform of $e^{-|x|}$.*

Solution : Step 1 : $e^{-|x|}$ is an even function. Hence we find Fourier cosine transform

Consider F.C.T. formula,

$$\therefore \quad F(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos\lambda u\, du$$

Step 2 : Substitute the value of f(u)

$$\therefore \quad F(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-u}\cos\lambda u\, du$$

Step 3 : Integrate

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{2}{\pi}}\left[\frac{e^{-u}}{\lambda^2+1}(-\cos\lambda u + \lambda\sin\lambda u)\right]_0^\infty$$

Step 4 : Put the limits

$$\therefore \quad F_c(\lambda) = \frac{1}{\lambda^2+1}$$

Example 8.10 : *Find Fourier sine transform of $e^{-|x|}$. Hence evaluate*

$$\int_0^\infty \frac{x\sin mx}{1+x^2}\, dx$$

Solution : **Step 1 :** Consider F.S.T. formula

$$F(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \sin \lambda u \, du$$

Step 2 : Put the value of f(u)

$$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^\infty e^{-u} \sin \lambda u \, du \right\}$$

Step 3 : Integrate

$$\therefore \quad F_s(\lambda) = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-u}}{\lambda^2+1} (-\sin \lambda u + \lambda \cos \lambda u) \right]_0^\infty$$

Step 4 : Put the limits for u

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{\lambda^2+1}$$

Step 5 : Consider inverse Fourier transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda$$

$$\therefore \quad e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda$$

Step 6 : Put x = m and then λ = x we get

$$\therefore \quad e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{x \sin mx}{1+x^2} dx$$

$$\therefore \quad \int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$$

Example 8.11 : *Using Fourier integral representation show that*

$$\int_0^\infty \frac{\cos \lambda x}{1+\lambda^2} d\lambda = \frac{\pi}{2} e^{-x} \quad \text{and} \quad \int_0^\infty \frac{\lambda \sin \lambda x}{1+\lambda^2} d\lambda = \frac{\pi}{2} e^{-x}$$

Solution : **Step 1 :** As $\cos \lambda x$ is present in the equation we find Fourier cosine transform of $f(x) = \frac{\pi}{2} e^{-x}$

$$\therefore \quad F(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \cos \lambda u \, du$$

Step 2 : Substitute the value of f(u)

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\pi}{2} e^{-u} \cos \lambda u \, du$$

$$= \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-u} \cos \lambda u \, du$$

Step 3 : Integrate

$$= \sqrt{\frac{\pi}{2}} \left\{ \frac{e^{-u}}{1+\lambda^2} [-\cos \lambda u + \lambda \sin \lambda u] \right\}_0^\infty$$

Step 4 : Put the limits

$$= \sqrt{\frac{\pi}{2}} \frac{1}{1+\lambda^2}$$

Step 5 : Consider inverse Fourier cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\lambda) \cos \lambda u \, d\lambda$$

$$\therefore \quad \frac{\pi}{2} e^{-x} = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \sqrt{\frac{\pi}{2}} \cdot \frac{1}{1+\lambda^2} \cos \lambda x \, d\lambda$$

Thus $\displaystyle\int_0^\infty \frac{\cos \lambda x}{1+\lambda^2} d\lambda = \frac{\pi}{2} e^{-x}$

To prove next part

$$\int_0^\infty \frac{\lambda \sin \lambda x}{1+\lambda^2} d\lambda = \frac{\pi}{2} e^{-x}$$

Step 1 : As sin λx is present in the equation we find Fourier sine transform of $f(x) = \frac{\pi}{2} e^{-x}$.

Step 2 :

$$\therefore \quad F(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \sin \lambda u \, du$$

Step 3 : Substitute the value of f(u)

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\pi}{2} e^{-u} \sin \lambda u \, du$$

$$= \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-u} \sin \lambda u \, du$$

Step 4 : Integrate

$$= \sqrt{\frac{\pi}{2}} \left\{ \frac{e^{-u}}{1+\lambda^2} [-\sin\lambda u - \lambda \cos \lambda u] \right\}_0^\infty$$

Step 5 : Substitute the limits for u.

$$= \sqrt{\frac{\pi}{2}} \left(\frac{\lambda}{1+\lambda^2} \right)$$

Step 6 : Consider inverse Fourier sine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\lambda) \sin\lambda u \, du$$

$$\therefore \quad \frac{\pi}{2} e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{1+\lambda^2} \right) \sin\lambda x \, d\lambda$$

Thus $\int_0^\infty \frac{\lambda \sin\lambda x}{1+\lambda^2} d\lambda = \frac{\pi}{2} e^{-x}$

Example 8.12 : *Find Fourier sine and cosine transform of e^{-x} and hence show that* $\int_0^\infty \frac{\cos mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$ *and* $\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$

Solution : Step 1 : Consider F.S.T. formula

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \sin\lambda u \, du$$

Step 2 : Put the value of f(u), $f(u) = e^{-u}$

$$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^\infty e^{-u} \cdot \sin\lambda u \, du \right\}$$

Step 3 : Integrate

$$\therefore \quad F_s(\lambda) = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-u}}{\lambda^2+1} (-\sin\lambda u - \lambda \cos\lambda u) \right]_0^\infty$$

Step 4 : Put the limits for u

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{\lambda^2+1} \right)$$

Step 5 : Consider inverse Fourier transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \frac{\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda$$

$$e^{-x} = \frac{\pi}{2} \int_0^\infty \frac{\lambda}{1+\lambda^2} \sin \lambda x \, d\lambda$$

Step 6 : Put x = m

$$\frac{\pi}{2} \cdot e^{-m} = \int_0^\infty \frac{\lambda \sin m\lambda}{1+\lambda^2} \, d\lambda$$

Step 7 : As the variable is not important we can write

$$\frac{\pi}{2} e^{-m} = \int_0^\infty \frac{x \sin mx}{1+x^2} \, dx$$

To prove the 2^{nd} part

Step 1 : Consider F.C.T formula

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \cos \lambda u \, du$$

Step 2 : Put the value of f(u).

$$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^\infty e^{-u} \cdot \cos \lambda u \, du \right\}$$

Step 3 : Integrate

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-u}}{\lambda^2+1} (-\cos \lambda u + \lambda \sin \lambda u) \right]_0^\infty$$

Step 4 : Put the limits for u

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \left(\frac{1}{1+\lambda^2} \right)$$

Step 5 : Consider inverse Fourier cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\lambda^2+1} \cos \lambda x \, d\lambda$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{1}{\lambda^2+1} \cos \lambda x \, d\lambda$$

$$e^{-x} = \frac{\pi}{2}\int_0^\infty \frac{1}{1+\lambda^2}\cos\lambda x \, d\lambda$$

Step 6 : Put x = m

$$\frac{\pi}{2}\cdot e^{-m} = \int_0^\infty \frac{\cos m\lambda}{1+\lambda^2} d\lambda$$

Step 7 : As variable is not important we can write above equation as

$$\therefore \quad \frac{\pi}{2} e^{-m} = \int_0^\infty \frac{\cos mx}{1+x^2} dx$$

Example 8.13 : *Using Fourier integral representation show that*

$$\int_0^\infty \frac{2}{1+\lambda^2}\cos\lambda x \, d\lambda = \begin{cases} 0 & x<0 \\ \pi/2 & x=0 \\ \pi e^{-x} & x>0 \end{cases}$$

Solution : Step 1 : As $\cos\lambda x$ is present in the integral we find the cosine transform

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos\lambda u \, du$$

Step 2 : Substitute the value of f(u)

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty \pi e^{-u}\cos\lambda u \, du$$

Step 3 : Integrate

$$= \sqrt{2\pi}\left[\frac{e^{-u}}{1+\lambda^2}(-\cos\lambda u + \lambda\sin\lambda u)\right]_0^\infty$$

$$= \sqrt{2\pi}\,\frac{1}{1+\lambda^2}$$

Step 4 : Consider inverse Fourier cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{2\pi}\cdot\frac{1}{1+\lambda^2}\cos\lambda x \, d\lambda$$

$$f(x) = \int_0^\infty \frac{2}{1+\lambda^2}\cos\lambda x \, d\lambda$$

Thus

$$= \int_0^\infty \frac{2}{1+\lambda^2}\cos\lambda x \, d\lambda = \begin{cases} 0 & x<0 \\ \pi/2 & x=0 \\ \pi e^{-x} & x>0 \end{cases}$$

Example 8.14 : *Using Fourier integral representation show that*

$$\int_0^{\infty} \frac{2\lambda}{1+\lambda^2} \sin \lambda x \, d\lambda = \begin{cases} 0 & x<0 \\ \pi/2 & x=0 \\ \pi e^{-x} & x>0 \end{cases}$$

Solution : **Step 1 :** As $\sin \lambda x$ is present in the integral, consider F.S.T. formula

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \sin \lambda u \, du$$

Step 2 : Put the value of f(u)

$$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^{\infty} \pi e^{-u} \cdot \sin \lambda u \, du \right\}$$

Step 3 : Integrate

$$\therefore \quad F_s(\lambda) = \sqrt{\frac{2}{\pi}} \left[\frac{\pi e^{-u}}{\lambda^2+1} (-\sin \lambda u - \lambda \cos \lambda u) \right]_0^{\infty}$$

Step 4 : Put the limits for u

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \frac{\pi \lambda}{\lambda^2+1} = \sqrt{2\pi} \frac{\lambda}{1+\lambda^2}$$

Step 5 : Consider inverse Fourier sine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{2\pi} \cdot \frac{\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda$$

$$f(x) = 2\int_0^{\infty} \frac{\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda$$

Thus

$$2\int_0^{\infty} \frac{\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda = \begin{cases} 0 & x<0 \\ \pi/2 & x=0 \\ \pi . e^{-x} & x>0 \end{cases}$$

Example 8.15 : *Using Fourier integral representation show that*

$$\int_0^{\infty} \frac{\cos \lambda \pi/2}{1-\lambda^2} \cos \lambda x \, d\lambda = \begin{cases} \frac{\pi}{2} \cos x & |x| \le \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$$

Solution : **Step 1 :** As $\cos \lambda x$ is present in the integral

$\therefore$ We should find cosine transform of

$$f(x) = \begin{cases} \frac{\pi}{2} \cos x, & |x| \leq \frac{\pi}{2} \\ 0 \quad , & |x| > \frac{\pi}{2} \end{cases}$$

Step 2 : Consider F.C.T. formula

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \cdot \cos \lambda u \cdot du$$

Step 3 : Split the integral

$$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^{\pi/2} f(u) \cdot \cos \lambda u \cdot du + \sqrt{\frac{2}{\pi}} \int_{\pi/2}^{\infty} f(u) \cos \lambda u \right\}$$

Step 4 : Put the values of f(u)

$$= \sqrt{\frac{2}{\pi}} \int_0^{\pi/2} \frac{\pi}{2} \cos u \cdot \cos \lambda u \cdot du + 0$$

Step 5 : Simplify

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} \cdot \frac{1}{2} \int_0^{\pi/2} (2 \cos \lambda u \cdot \cos u) \, du$$

Step 6 : Use $2 \cos \lambda u \cdot \cos u = \cos(\lambda+1) u + \cos(\lambda-1) u$

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \int_0^{\pi/2} [\cos(\lambda+1) u + \cos(\lambda-1) u] \, du$$

Step 7 : Integrate

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \left[\frac{\sin(\lambda+1) u}{\lambda+1} + \frac{\sin(\lambda-1) u}{\lambda-1} \right]_0^{\pi/2}$$

Step 8 : Use $\sin\left(\frac{\pi}{2}+\theta\right) = \cos \theta$ and $\sin\left(\frac{\pi}{2}-\theta\right) = +\cos\theta$

$$\left\{ \sin(\lambda+1)\frac{\pi}{2} = \cos \frac{\pi\lambda}{2} \; , \; \sin(\lambda-1)\frac{\pi}{2} = -\cos \frac{\pi\lambda}{2} \right\}$$

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{\pi}{2}} \frac{1}{2} \left[\frac{\cos \frac{\pi\lambda}{2}}{\lambda+1} - \frac{\cos \frac{\pi\lambda}{2}}{\lambda-1} \right]$$

Step 9 : Take L.C.M

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2}\left[\frac{-2}{\lambda^2-1}\right] \cos\frac{\pi\lambda}{2}$$

Step 10 : Simplify

$$F_c(\lambda) = \sqrt{\frac{\pi}{2}} \frac{1}{(1-\lambda^2)} \cos\frac{\pi\lambda}{2}$$

Step 11 : Consider inverse cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_C(\lambda) \cdot \cos\lambda x \cdot d\lambda$$

Step 12 : Put the value of $F(\lambda)$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{\pi}{2}} \frac{1}{(1-\lambda^2)} \cos\frac{\pi\lambda}{2} \cdot \cos\lambda x \cdot d\lambda$$

Step 13 : Simplify

$$\therefore \qquad f(x) = \int_0^\infty \frac{\cos\frac{\pi\lambda}{2}\cos\lambda x}{1-\lambda^2} d\lambda$$

Example 8.16 : *Using Fourier integral representation show that*

$$\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda} \sin\lambda x \, d\lambda = \begin{cases} \pi/2 & 0<x<\pi \\ 0 & x>\pi \end{cases}$$

Solution : Step 1 : As $\sin\lambda x$ is present in the integral

$\therefore$ We should find sine transform of

$$f(x) = \begin{cases} \frac{\pi}{2}, & 0<x<\pi \\ 0, & x>\pi \end{cases}$$

Step 2 : Consider F.S.T. formula

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \sin\lambda u \, du$$

Step 3 : Split the integral

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\pi f(u) \sin\lambda u \, du + \sqrt{\frac{2}{\pi}} \int_\pi^\infty f(u) \sin\lambda u \, du$$

Step 4 : Put the values of f(u)

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \frac{\pi}{2} \sin \lambda u \, du + \sqrt{\frac{2}{\pi}} \int_{\pi}^{\infty} 0 \sin \lambda u \, du$$

Step 5 : Simplify

$$= \sqrt{\frac{\pi}{2}} \int_0^{\pi} \sin \lambda u \, du + 0$$

Step 6 : Integrate

$$= \sqrt{\frac{\pi}{2}} \left[\frac{-\cos \lambda u}{\lambda} \right]_0^{\pi}$$

Step 7 : Put the limits of u

$$= \sqrt{\frac{\pi}{2}} \left[\frac{-\cos \lambda\pi - (-1)}{\lambda} \right]$$

$$= \sqrt{\frac{\pi}{2}} \frac{(1 - \cos \lambda\pi)}{\lambda}$$

Step 8 : Consider inverse sine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\lambda) \sin \lambda x \, d\lambda$$

Step 9 : Put the value of $F(\lambda)$

$$\therefore \quad f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{\pi}{2}} \frac{1 - \cos \lambda\pi}{\lambda} \sin \lambda x \, d\lambda$$

Step 10 : Simplify

$$f(x) = \int_0^{\infty} \frac{1 - \cos \lambda\pi}{\lambda} \sin \lambda x \, d\lambda$$

Example 8.17 : *Using Fourier integral representation show that*

$$\int_0^{\infty} \frac{\lambda \sin \lambda\pi}{1 - \lambda^2} \cos \lambda x \, d\lambda = \begin{cases} \frac{\pi}{2} \cos x & |x| \leq \pi \\ 0 & |x| > \pi \end{cases}$$

Solution : **Step 1 :** As $\cos \lambda x$ is present in the integral

$\therefore$ We should find cosine transform of

$$f(x) = \begin{cases} \frac{\pi}{2} \cos x, & |x| \le \pi \\ 0 \quad , & |x| > \pi \end{cases}$$

Step 2 : Consider F.C.T. formula

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \cos \lambda u \, du$$

Step 3 : Split the integral

$$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^{\pi} f(u) \cos \lambda u \, du + \sqrt{\frac{2}{\pi}} \int_{\pi}^{\infty} f(u) \cos \lambda u \right\}$$

Step 4 : Put the values of f(u)

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^{\pi} \frac{\pi}{2} \, 2 \cos u \cos \lambda u \, du + 0$$

Step 5 : Use $2 \cos \lambda u \cos u = \cos(\lambda+1) u + \cos(\lambda-1) u$

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \int_0^{\pi} [\cos(\lambda+1) + \cos(\lambda-1) u] du$$

Step 6 : Integrate

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \left[\frac{\sin(\lambda+1) u}{\lambda+1} + \frac{\sin(\lambda-1) u}{\lambda-1} \right]_0^{\pi}$$

Step 7 : Use $\sin(\pi+\theta) = -\sin\theta$ and $\sin(\pi-\theta) = \sin\theta$

$\{\sin(\lambda+1)\pi = -\sin \pi\lambda, \ \sin(\lambda-1)\pi = -\sin \pi\lambda\}$

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{\pi}{2}} \frac{1}{2} \left[\frac{-\sin \pi\lambda}{\lambda+1} - \frac{\sin \pi\lambda}{\lambda-1} \right]$$

Step 8 : Take L.C.M

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \left[\frac{-2\lambda}{\lambda^2 - 1} \right] \sin \pi\lambda$$

Step 9 : Simplify

$$F(\lambda) = \sqrt{\frac{\pi}{2}} \frac{\lambda}{(1-\lambda^2)} \sin \pi\lambda$$

Step 10 : Consider inverse cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\lambda) \cos \lambda x \, d\lambda$$

Step 11 : Put the value of $F(\lambda)$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{\pi}{2}} \frac{\lambda}{(1-\lambda^2)} \sin \pi\lambda \cos \lambda x \, d\lambda$$

Step 12 : Simplify

$$\therefore \quad f(x) = \int_0^\infty \frac{\lambda \sin \pi\lambda}{(1-\lambda^2)} \cos \lambda x \, d\lambda$$

Example 8.18 : *Using Fourier integral representation show that*

$$\int_0^\infty \frac{\lambda \cos \lambda \pi / 2}{1-\lambda^2} \sin \lambda x \, d\lambda = \begin{cases} \frac{\pi}{2} \sin x & 0 \le x \le \frac{\pi}{2} \\ 0 & x > \frac{\pi}{2} \end{cases}$$

Solution : Step 1 : As $\sin \lambda x$ is present in the integral

$\therefore$ We should find sine transform of

$$f(x) = \begin{cases} \frac{\pi}{2} \sin x, & 0 \le x \le \frac{\pi}{2} \\ 0 & , \; x > \frac{\pi}{2} \end{cases}$$

Step 2 : Consider F.S.T. formula

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \sin \lambda u \, du$$

Step 3 : Split the integral

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\pi/2} f(u) \sin \lambda u \, du + \sqrt{\frac{2}{\pi}} \int_{\pi/2}^\infty f(u) \sin \lambda u \, du$$

Step 4 : Put the values of f(u)

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\pi/2} \frac{\pi}{2} \sin u \sin \lambda u \, du + \sqrt{\frac{2}{\pi}} \int_{\pi/2}^\infty 0 \sin \lambda u \, du$$

Step 5 : Use $2 \sin \lambda u \cdot \sin u = \cos(\lambda-1) u - \cos(\lambda+1) u$

$$\therefore \quad F_s(\lambda) = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \int_0^{\pi/2} [\cos(\lambda-1) u - \cos(\lambda+1) u] \, du$$

Step 6 : Integrate

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2}\left[\frac{\sin(\lambda-1)\,u}{\lambda-1} - \frac{\sin(\lambda+1)\,u}{\lambda+1}\right]_0^{\pi/2}$$

Step 7 : Use $\sin\left(\frac{\pi}{2}+\theta\right) = +\cos\theta$ and $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

Step 8 : Put the limits of u

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2}\left[\frac{-\cos\lambda\pi/2}{\lambda-1} - \frac{\cos\lambda\pi/2}{\lambda+1}\right]$$

$$= \sqrt{\frac{\pi}{2}}\, \frac{1}{2}\left[\frac{-2\lambda}{\lambda^2-1}\right]\cos\pi\lambda/2$$

$$= \sqrt{\frac{\pi}{2}}\, \frac{\lambda\cos\lambda\pi/2}{1-\lambda^2}$$

Step 9 : Consider inverse sine transform

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(\lambda)\,\sin\lambda x\,d\lambda$$

Step 10 : Put the value of $F(\lambda)$

$$\therefore \quad f(x) = \int_0^\infty \frac{\lambda\cos\lambda\pi/2}{1-\lambda^2}\,\sin\lambda x\,d\lambda$$

Example 8.19 : *Using Fourier integral representation show that*

$$\int_0^\infty \frac{\sin\lambda\pi}{1-\lambda^2}\,\sin\lambda x\,d\lambda = \begin{cases}\frac{\pi}{2}\sin x & 0\le x\le\pi\\ 0 & x>\pi\end{cases}$$

Solution : Step 1 : As $\sin\lambda x$ is present in the integral

$\therefore$ We should find sine transform of

$$f(x) = \begin{cases}\frac{\pi}{2}\sin x, & 0\le x\le\pi\\ 0, & x>\pi\end{cases}$$

Step 2 : Consider F.S.T. formula

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\,\sin\lambda u\,du$$

Step 3 : Split the integral

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(u) \sin \lambda u \, du + \sqrt{\frac{2}{\pi}} \int_{\pi}^{\infty} f(u) \sin \lambda u \, du$$

Step 4 : Put the values of f(u)

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^{\pi} \frac{\pi}{2} \, 2 \sin u \sin \lambda u \, du + \sqrt{\frac{2}{\pi}} \int_{\pi}^{\infty} 0 \sin \lambda u \, du$$

Step 5 : Use $2 \sin \lambda u \cdot \sin u = \cos(\lambda - 1) u - \cos(\lambda + 1) u$

$$\therefore \quad F_s(\lambda) = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \int_0^{\pi} [\cos(\lambda-1) - \cos(\lambda+1) u] \, du$$

Step 6 : Integrate

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \left[\frac{\sin(\lambda-1) u}{\lambda-1} - \frac{\sin(\lambda+1) u}{\lambda+1} \right]_0^{\pi}$$

Step 7 : Use $\sin(\pi + \theta) = -\sin \theta$ and $\sin(\pi - \theta) = \sin \theta$

$\{\sin(\lambda+1)\pi = -\sin \pi\lambda, \ \sin(\lambda-1)\pi = -\sin \pi\lambda\}$

Step 8 : Put the limits of u.

$$F_s(\lambda) = -\sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \left[\frac{\sin \lambda\pi}{\lambda-1} + \frac{+\sin \lambda\pi}{\lambda+1} \right]$$

$$F_s(\lambda) = -\sqrt{\frac{\pi}{2}} \, \frac{1}{2} \left[\frac{2\lambda}{\lambda^2 - 1} \right] \sin \pi\lambda$$

$$F_s(\lambda) = \frac{\lambda \sin \pi\lambda}{1 - \lambda^2}$$

Step 9 : Consider inverse sine transform.

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\lambda) \sin \lambda x \, d\lambda$$

Step 9 : Put the value of $F(\lambda)$

$$\therefore \quad f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\lambda \sin \pi\lambda}{1 - \lambda^2} \sin \lambda x \, d\lambda = \frac{\pi}{2} \sin x$$

Example 8.20 : *Using Fourier integral representation show that*

$$\int_0^\infty \left[\frac{1-\lambda \sin\frac{\lambda\pi}{2}}{1-\lambda^2}\right] \cos \lambda x \, d\lambda = \begin{cases} \frac{\pi}{2} \sin x & 0 \le x \le \frac{\pi}{2} \\ 0 & x > \frac{\pi}{2} \end{cases}$$

Solution : Step 1 : As cos λx is present in the integral

$\therefore$ We should find cosine transform of

$$f(x) = \begin{cases} \frac{\pi}{2} \sin x & 0 \le x \le \frac{\pi}{2} \\ 0 & x > \frac{\pi}{2} \end{cases}$$

Step 2 : Consider the F.C.T. formula

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \cdot \cos \lambda u \, du$$

Step 3 : Split the integral

$$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^{\pi/2} f(u) \cdot \cos \lambda u \cdot du + \sqrt{\frac{2}{\pi}} \int_{\pi/2}^\infty f(u) \cos \lambda u \right\}$$

Step 4 : Put the values of f (u).

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^\pi \frac{\pi}{2} 2\sin u \cdot \cos \lambda u \cdot du + 0$$

Step 5 : Use $2 \cos \lambda u \cdot \sin u = \sin (\lambda+1) u - \sin (\lambda-1) u$

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \int_0^{\pi/2} [\sin (\lambda+1) u - \sin (\lambda-1) u] du$$

Step 6 : Integrate

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \left[\frac{-\cos(\lambda+1) u}{\lambda+1} + \frac{\cos(\lambda-1) u}{\lambda-1}\right]_0^{\pi/2}$$

Step 7 : Use $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$ and $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

$$\left\{\cos(\lambda+1)\frac{\pi}{2} = -\sin\frac{\pi\lambda}{2}, \ \cos(\lambda-1)\frac{\pi}{2} = \sin\frac{\pi\lambda}{2}\right\}$$

$$\therefore \quad F_c(\lambda) = \sqrt{\frac{\pi}{2}} \frac{1}{2} \left[\frac{\sin \lambda\pi/2}{\lambda+1} + \frac{\sin \lambda\pi/2}{\lambda-1} + \frac{1}{\lambda+1} - \frac{1}{\lambda-1}\right]$$

Step 8 : Take L.C.M

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \left[\frac{2 - 2\lambda \sin \lambda\pi / 2}{1 - \lambda^2} \right]$$

$$\therefore \quad f(x) = \int_0^{\infty} \left[\frac{1 - \lambda \sin \frac{\lambda \pi}{2}}{1 - \lambda^2} \right] \cos \lambda x \, d\lambda$$

Example 8.21 : *Using Fourier integral representation show that* $\int_0^{\infty} \frac{\lambda^3 \sin \lambda x}{\lambda^4 + 4} d\lambda = \frac{\pi}{2} e^{-x} \cos x$

Solution : As '$\sin \lambda x$' is present in the expression

$\therefore$ We find Fourier sine transform of

$$f(x) = \frac{\pi}{2} \cdot e^{-x} \cos x$$

Step 2 : Consider F.S.T. formula

$$\therefore \quad F(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \sin \lambda u \, du$$

Step 3 : Put the value of f(u)

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\pi}{2} e^{-u} \cos u \sin \lambda u \, du$$

Step 4 : Take $\frac{\pi}{2}$ outside the integral.

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} \int_0^{\infty} e^{-u} \cos u \sin \lambda u \, du$$

Step 5 : Use $\{2 \sin A \cos B = \sin(A + B) + \sin(A - B)\}$

$$F(\lambda) = \sqrt{\frac{\pi}{2}} \int_0^{\infty} \frac{e^{-u}}{2} [\sin(\lambda+1) u + \sin(\lambda-1) u] du$$

Step 6 : Separate the integral

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \left[\int_0^{\infty} e^{-u} \sin(\lambda+1) u + \int_0^{\infty} e^{-u} \sin(\lambda-1) u \right] du$$

Step 7 : We know that $\int_0^{\infty} e^{-u} \sin \lambda u \, du = \frac{\lambda}{1 + \lambda^2}$ thus, replacing λ by $(\lambda+1)$ we get,

$$(1) \;=\; \int_0^\infty e^{-u}\sin(\lambda+1)\,u\,du = \frac{(\lambda+1)}{1+(\lambda+1)^2}$$

Step 8 : Replacing $\lambda+1$ by $\lambda-1$ we get

$$(2) \;=\; \int_0^\infty e^{-u}\sin(\lambda-1)\,u\,du = \frac{\lambda-1}{1+(\lambda-1)^2}$$

Step 9 : Substitute in $F(\lambda)$, we get

$$F(\lambda) \;=\; \sqrt{\frac{\pi}{2}}\cdot\frac{1}{2}\left[\frac{\lambda+1}{1+(\lambda+1)^2}+\frac{\lambda-1}{1+(\lambda-1)^2}\right]$$

Step 10 : Simplify

$$= \sqrt{\frac{\pi}{2}}\cdot\frac{1}{2}\left[\frac{2\lambda^3}{\lambda^4+4}\right] = \sqrt{\frac{\pi}{2}}\left[\frac{\lambda^3}{\lambda^4+4}\right]$$

Step 11 : Consider inverse Fourier sine transform

$$f(x) \;=\; \sqrt{\frac{2}{\pi}}\int_0^\infty f(\lambda)\sin\lambda x\,d\lambda$$

Step 12 : Put the value of $F(\lambda)$

$$f(x) \;=\; \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{\pi}{2}}\left(\frac{\lambda^3}{\lambda^4+4}\right)\sin\lambda x\cdot d\lambda$$

Step 13 : Put the value of f(x)

$$\therefore \quad \int_0^\infty \frac{\lambda^3\sin\lambda x}{\lambda^4+4}\,d\lambda \;=\; \frac{\pi}{2}\,e^{-x}\cos x$$

Example 8.22 : $\int_0^\infty \frac{2\lambda\sin\lambda x}{\lambda^4+4}\,d\lambda = \frac{\pi}{2}e^{-x}\sin x$

Solution : Step 1 : As $\sin\lambda x$ is present in the expression we find Fourier sine transform

$$\therefore \quad F(\lambda) \;=\; \sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int_0^\infty \frac{\pi}{2}\,e^{-u}\,2\sin u\sin\lambda u\,du$$

Step 2 : Use $2\sin\lambda u\sin u = \cos(\lambda-1)\,u-\cos(\lambda+1)\,u$

$$\therefore \quad F(\lambda) \;=\; \sqrt{\frac{\pi}{2}}\cdot\frac{1}{2}\int_0^\infty e^{-u}\left[\cos(\lambda-1)\,u-\cos(\lambda+1)\,u\right]du$$

Step 3 : Integrate

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \left\{ \left[\frac{e^{-u}}{1+(\lambda-1)^2} (-\cos(\lambda-1)+(\lambda-1)\sin(\lambda-1)) \right]_0^{\infty} - \left[\frac{e^{-u}}{1+(\lambda+1)^2} (-\cos(\lambda+1)+(\lambda+1)\sin(\lambda+1)) \right]_0^{\infty} \right\}$$

Step 4 : Substituting the limits

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}} \left[\frac{1}{1+(\lambda-1)^2} - \frac{1}{1+(\lambda+1)^2} \right]$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}} \left[\frac{\lambda^2+2\lambda+2-\lambda^2+2\lambda-2}{\lambda^4+4} \right]$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}} \left[\frac{4\lambda}{\lambda^4+4} \right]$$

Step 5 : Consider inverse Fourier sine transform

$$\therefore \quad f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{2}\sqrt{\frac{\pi}{2}} \frac{4\lambda}{\lambda^4+4} \sin\lambda x \, d\lambda$$

$$\therefore \quad \frac{\pi}{2} e^{-x} \sin x = \int_0^{\infty} \frac{2\lambda}{\lambda^4+4} \sin\lambda x \, d\lambda$$

Type II

Example 8.23 : *Solve the integral equation.*

$$\int_0^{\infty} f(x)\cos\lambda x \, dx = \begin{cases} 1-\lambda & 0 \le \lambda \le 1 \\ 0 & \lambda > 1 \end{cases}$$ *Hence show that*

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$$

Solution : Step 1 : Multiply the given equation by $\sqrt{2/\pi}$

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cdot \cos\lambda x \, dx = \begin{cases} \sqrt{\frac{2}{\pi}}(1-\lambda) & 0 \le \lambda \le 1 \\ 0 & \lambda > 1 \end{cases}$$

Step 2 : L.H.S. is the formula for $F_c(\lambda)$

$$\therefore \quad F_c(\lambda) = \begin{cases} \sqrt{\frac{2}{\pi}}(1-\lambda) & 0 \le \lambda \le 1 \\ 0 & \lambda > 1 \end{cases}$$

Step 3 : To find f(x) consider inverse Fourier cosine transforms

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(\lambda) \cos \lambda x \, d\lambda$$

Step 4 : Split the integral and put the values of F(λ)

$$f(x) = \sqrt{\frac{2}{\pi}} \left[\int_0^1 \sqrt{\frac{2}{\pi}} (1-\lambda) \cos \lambda x \, d\lambda + \int_1^{\infty} 0 \, d\lambda \right]$$

Step 5 : Integrate using general rule of integration by parts

$$f(x) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \left[(1-\lambda)\left(\frac{\sin \lambda x}{x}\right) - (-1)\left(\frac{-\cos \lambda x}{x^2}\right) \right]_0^1$$

Step 6 : Put the limits

$$f(x) = \frac{2}{\pi} \left[\left(0 - \frac{\cos x}{x^2}\right) - \left(0 - \frac{1}{x^2}\right) \right]$$

Step 7 : Simplify

$$= \frac{2}{\pi} \left(\frac{1-\cos x}{x^2} \right)$$

Step 8 : Use $(1-\cos x) = 2 \sin^2 (x/2)$

$$= \frac{2}{\pi} \frac{2 \sin^2 \frac{x}{2}}{x^2}$$

Step 9 : Simplify

$$f(x) = \frac{1}{\pi} \frac{\sin^2 \frac{x}{2}}{\frac{x^2}{4}} = \frac{1}{\pi} \frac{\sin^2 (x/2)}{(x/2)^2}$$

Step 10 : Substituting in the given equation, we get

$$\int_0^{\infty} \frac{1}{\pi} \frac{\sin^2 (x/2)}{(x/2)^2} \cos \lambda x \, dx = \begin{cases} 1-\lambda & 0 \le \lambda \le 1 \\ 0 & \lambda > 1 \end{cases}$$

Step 11 : Put $\lambda = 0$

$$\frac{1}{\pi} \int_0^{\infty} \frac{\sin^2 (x/2)}{(x/2)^2} 1 \, dx = 1$$

Step 12 : Put $x/2 = u, \ x = 2u, \ dx = 2 \, du$

$$\frac{1}{\pi} \int_0^{\infty} \frac{\sin^2 u}{u^2} 2 \, du = 1$$

Step 13 : Simplify

$$\int_0^\infty \frac{\sin^2 u}{u^2}\, du = \frac{\pi}{2}$$

Example 8.24 : *Solve the integral equation* $\int_0^\infty f(x) \sin \lambda x \, dx = \begin{cases} 1-\lambda & 0 \le \lambda \le 1 \\ 0 & \lambda > 1 \end{cases}$

Solution : Step 1 :

Multiply the given equation by $\sqrt{\frac{2}{\pi}}$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \lambda x \, dx = \begin{cases} \sqrt{\frac{2}{\pi}}\,(1-\lambda) & 0 \le \lambda \le 1 \\ 0 & \lambda > 1 \end{cases}$$

$$\therefore \quad F_s(\lambda) = \begin{cases} \sqrt{\frac{2}{\pi}}\,(1-\lambda) & 0 \le \lambda \le 1 \\ 0 & \lambda > 1 \end{cases}$$

Step 2 : To find f(x) consider inverse Fourier sine transform

$$\therefore \quad f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda$$

Step 3 : Substituting $F_s(\lambda)$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 \sqrt{\frac{2}{\pi}}\,(1-\lambda) \sin \lambda x \, d\lambda + \int_1^\infty 0 \right]$$

Step 4 : Integrate w.r.t λ

$$= \frac{2}{\pi} \left[(1-\lambda)\left(-\frac{\cos \lambda x}{x}\right) - (-1)\left(\frac{-\sin \lambda x}{x^2}\right) \right]_0^1$$

Step 5 : Put the limits for λ

$$= \frac{2}{\pi} \left[\frac{1}{x} - \frac{\sin x}{x^2} \right]$$

$$f(x) = \frac{2}{\pi} \left[\frac{x - \sin x}{x^2} \right]$$

Step 3 : Put the value of $F(\lambda)$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sqrt{\frac{2}{\pi}}\, e^{-\lambda} \sin \lambda x \, d\lambda$$

Step 4 : Integrate w.r.t λ

$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}(-\sin \lambda x - x\cos \lambda x)\right]_0^{\infty}$$

Step 5 : Substitute the limits for λ

$$f(x) = \frac{2}{\pi}\frac{x}{1+x^2}$$

Example 8.28 : *Solve the integral equation* $\int_0^{\infty} f(x) \sin \lambda x \, dx = \begin{cases} 1 & 0<\lambda<1 \\ 2 & 1<\lambda<2 \\ 0 & \lambda>2 \end{cases}$

Solution : **Step 1 :** Given

$$F_s(\lambda) = \begin{cases} 1, & 0<\lambda<1 \\ 2, & 1<\lambda<2 \\ 0, & \lambda>2 \end{cases}$$

Step 2 : Consider inverse sine transform

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\lambda) \sin \lambda x \, d\lambda$$

Step 3 : Split the integral and substitute the value of $F(\lambda)$

$$f(x) = \frac{2}{\pi}\left[\int_0^1 1 \sin \lambda x \, d\lambda + \int_1^2 2 \sin \lambda x \, d\lambda\right]$$

Step 4 : Integrate w.r.t λ.

$$f(x) = \frac{2}{\pi}\left[\left(\frac{-\cos \lambda x}{x}\right)_0^1 - 2\left(\frac{\cos \lambda x}{x}\right)_1^2\right]$$

Step 5 : Substitute the limits for λ

$$f(x) = \frac{-\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}$$

$$f(x) = \frac{2}{\pi}\left[\frac{1+\cos x - 2\cos 2x}{x}\right]$$

Example 8.29 : $\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases} 1 & 0<\lambda<1 \\ 2 & 1<\lambda<2 \\ 0 & \lambda>2 \end{cases}$

Solution : Step 1 : Given

$$\therefore \quad F_c(\lambda) = \begin{cases} 1 & 0<\lambda<1 \\ 2 & 1<\lambda<2 \\ 0 & \lambda>2 \end{cases}$$

Consider the inverse cosine transform

Step 2 :
$$f(x) = \frac{2}{\pi}\int_0^\infty f(\lambda)\cos\lambda x\,d\lambda$$

Step 3 : Split the integral and substitute the value of $F(\lambda)$

$$f(x) = \frac{2}{\pi}\left[\int_0^1 1\cos\lambda x\,d\lambda + \int_1^2 2\cos\lambda x\,d\lambda\right]$$

Step 4 : Integrate w.r.t λ

$$f(x) = \frac{2}{\pi}\left[\left(\frac{\sin\lambda x}{x}\right)_0^1 + 2\left(\frac{\sin\lambda x}{x}\right)_1^2\right]$$

Step 5 : Substitute the limits of λ

$$f(x) = \frac{2}{\pi}\left[\frac{\sin x}{x} + \frac{2\sin 2x}{x} - \frac{2\sin x}{x}\right]$$

$$\therefore \quad f(x) = \frac{2}{\pi}\left[\frac{2\sin 2x - \sin x}{x}\right]$$

Type III : Typical Problems

Example 8.30 : *Find the Fourier integral representation for the function*

$$f(x) = \begin{cases} 0 & x<0 \\ e^{-x} & x>0 \\ 1/2 & x=0 \end{cases}$$

Solution : Step 1 : Note that the given function f(x) is neither an even function nor an odd function.

Step 12 : Use $\quad e^{i\lambda x} = \cos\lambda x + i\sin\lambda x$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1-i\lambda}{1+\lambda^2}\right](\cos\lambda x + i\sin\lambda x)\, d\lambda$$

Step 13 : Use $(a+ib)(c+id) = (ac-bd) + i(ad+bc)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2} + i\frac{-\lambda\cos\lambda x + \sin\lambda x}{1+\lambda^2}\right] d\lambda$$

Step 14 : Separate the two integrals

$$= \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\left[\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\right] d\lambda + i\int_{-\infty}^{\infty}\left[\frac{-\lambda\cos\lambda x + \sin\lambda x}{1+\lambda^2}\right] d\lambda\right\}$$

($\uparrow$ is an even function of λ) $\qquad$ ($\uparrow$ is an odd function of λ)

Step 15 : Use $\int_{-\infty}^{\infty} F(\lambda)\, d\lambda = 2\int_{0}^{\infty} F(\lambda)\, d\lambda$. If $F(\lambda)$ is an even function of λ and

$$\int_{-\infty}^{\infty} F(\lambda)\, d\lambda = 0 \quad \text{If } F(\lambda) \text{ is an odd function of } \lambda.$$

Thus

$$f(x) = \frac{2}{2\pi}\int_{0}^{\infty}\left[\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\right] d\lambda + 0$$

Step 16 : Simplify

$$\therefore \qquad f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\right] d\lambda$$

which is the Fourier integral representation of f(x).

Example 8.31 : *If* $f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & x < 0,\ x > \pi \end{cases}$

then prove that

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} \frac{\cos\lambda x + \cos[\lambda(\pi - x)]}{1-\lambda^2}$$

Hence deduce that

$$\int_{0}^{\infty} \frac{\cos\lambda\pi/2}{1-\lambda^2}\, d\lambda = \frac{\pi}{2}$$

Solution : Step 1 : Here f(x) is defined over the interval $-\infty < x < \infty$, and f(x) is neither an even function nor an odd function.

Step 2 : Consider F.T. formula

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$$

Step 3 : Substitute the value of f(u)

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\pi} \sin u \;\; e^{-i\lambda u}\, du$$

Step 4 : Integrate w.r.t. u

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\lambda u}}{(-i\lambda)^2+1} (-i\lambda \sin u - \cos u) \right]_0^{\pi}$$

Step 5 : Put the limits for u

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\lambda\pi}}{-\lambda^2+1}(-\cos\pi) - \frac{1}{-\lambda^2+1}(-\cos 0) \right]$$

Step 6 : Simplify

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\lambda\pi}+1}{1-\lambda^2} \right]$$

Step 7 : Consider inverse transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$$

Step 8 : Put the value of F(λ)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1+e^{-i\lambda\pi}}{1-\lambda^2} \right) e^{i\lambda x}\, d\lambda$$

Step 9 : Simplify

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{e^{i\lambda x}+e^{-i\lambda(\pi-x)}}{1-\lambda^2} \right) d\lambda$$

Step 10 : Use $e^{ix} = \cos x + i \sin x$

$$= \int_{-\infty}^{\infty} \frac{\cos\lambda x + i\sin\lambda x + \cos\lambda(\pi-x) + i\sin\lambda(\pi-x)}{2\pi(1-\lambda^2)}\, d\lambda$$

Step 11 : Separate the real and imaginary integrals

$$= \int_{-\infty}^{\infty} \frac{\cos\lambda x + \cos\lambda(\pi - x)}{2\pi(1-\lambda^2)} d\lambda + \int_{-\infty}^{\infty} \frac{i\sin\lambda x + i\sin\lambda(\pi - x)}{2\pi(1-\lambda^2)} d\lambda$$

[↑ even function of λ] [↑ odd function of λ]

Step 12 : Use $\int_{-\infty}^{\infty} F(\lambda)\, d\lambda = 2\int_{0}^{\infty} F(\lambda)\, d\lambda$ If $F(\lambda)$ is an even function of λ and

$\int_{-\infty}^{\infty} F(\lambda)\, d\lambda = 0$ If $F(\lambda)$ is an odd function of λ.

Thus
$$f(x) = \frac{1}{2\pi} 2\int_{0}^{\infty} \left(\frac{\cos\lambda x + \cos\lambda(\pi - x)}{1-\lambda^2}\right) d\lambda + 0$$

Step 13 : Simplify

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} \frac{\cos\lambda x + \cos[\lambda(\pi - x)]}{1-\lambda^2} d\lambda$$

Step 14 : Put $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi}\int_{0}^{\infty} \frac{\cos\lambda\pi/2 + \cos[\lambda(\pi - \pi/2)]}{1-\lambda^2} d\lambda$$

$\because \left[f\left(\frac{\pi}{2}\right) = \sin\pi/2 = 1\right]$

Step 15 : Simplify

$$\int_{0}^{\infty} \frac{\cos\lambda\pi/2}{1-\lambda^2} d\lambda = \frac{\pi}{2}$$

This is the required deduction.

Example 8.32 : *Show that the Fourier transform of* $e^{-x^2/2}$ *is* $e^{-\lambda^2/2}$

Solution : Step 1 : $f(x) = e^{-x^2/2}$ is an even function of x defined in the interval $-\infty < x < \infty$.

∴ Use Fourier cosine transform.

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(u)\cos\lambda u\, du$$

Step 10 : Write $e^c = A$.

$$\therefore \quad F_c(\lambda) = A\,e^{-\lambda^2/2}$$

Step 11 : To find constant A, put $\lambda = 0$, then

$$F_c(0) = A\,e^0 = A$$

Step 12 : To find F(0) put $\lambda = 0$ in step 2.

$$F_c(0) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-u^2/2}\,du$$

Step 13 : Put $\quad u^2 = 2t \quad$ or $\quad u = \sqrt{2t}$

$$du = \sqrt{2}\,\frac{1}{2}\,t^{-1/2}\,dt$$

Step 14 : Substituting we get

$$A = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t}\,\frac{1}{\sqrt{2}}\,t^{-1/2}\,dt = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{2}} \int_0^{\infty} e^{-t}\,t^{-1/2}\,dt$$

Step 15 : We know $\Gamma n = \int_0^{\infty} e^{-t}\,t^{n-1}\,dt$ and $\Gamma 1/2 = \sqrt{\pi}$

$$A = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{2}}\,\Gamma 1/2 = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{2}}\,\sqrt{\pi} = 1$$

Step 16 : Substituting A in step 10 we get,

$$\therefore \quad F_c(\lambda) = e^{-\lambda^2/2}$$

Example 8.33 : *Show that the F.T. of* e^{-x^2} *is* $\frac{1}{\sqrt{2}}\,e^{-\lambda^2/4}$

Solution : Step 1 : $f(x) = e^{-x^2}$ is an even function of x defined in the interval $-\infty < x < \infty$.

$\therefore$ Use Fourier cosine transform.

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \cos \lambda u\,du$$

Step 2 : Put the value of f(u)

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-u^2} \cos \lambda u\,du$$

Step 3 : [Use rule of D.U.I.S.]

$$\left\{\frac{d}{d\lambda}\int_a^b f(x,\lambda)\,dx = \int_a^b \frac{\partial}{\partial\lambda} f(x,\lambda)\,dx\right\}$$

$$\therefore \quad F'(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial}{\partial\lambda} e^{-u^2}\cos\lambda u\,du$$

Differentiate w.r.t. λ

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \left(e^{-u^2}(-u)\right)\sin\lambda u\,du$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \left(e^{-u^2}(-2u)\right)\sin\lambda u\,du$$

Step 4 : Integrating by parts

$$= \frac{1}{\sqrt{2\pi}}\left[e^{-u^2}\sin\lambda u\right]_0^\infty - \int_0^\infty e^{-u^2}\lambda\cos\lambda u\,du$$

$$= \frac{1}{\sqrt{2\pi}}\left[0-\lambda\int_0^\infty e^{-u^2}\cos\lambda\,u\,du\right]$$

Step 5 :

$$\therefore \quad F'(\lambda) = -\frac{\lambda}{2}F(\lambda)$$

$$\frac{F'(\lambda)}{F(\lambda)} = -\frac{\lambda}{2}$$

Step 6 : Integrating w.r.t. λ

$$\log F(\lambda) = -\frac{\lambda^2}{4}+c \quad \Rightarrow \quad F(\lambda) = e^{-\lambda^2/4}\; e^c$$

Step 7 : Write $e^c = A$.

$$\therefore \quad F(\lambda) = A\,e^{-\lambda^2/4}$$

Step 8 : To find F(0) put λ = 0, then

$$F(0) = A\,e^0 = A$$

Step 9 : To find A put λ = 0 in step 1

$$F(0) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-u^2}du$$

Step 10 : Put $u^2 = t$, $\therefore$ $2u\, du = dt$, $\therefore$ $du = \frac{1}{2} t^{-1/2}\, dt$

$$\therefore \quad F(0) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{2} e^{-t}\, t^{-1/2}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \overline{|1/2} = \frac{1}{\sqrt{2}} \quad \text{As} \quad \overline{|1/2} = \sqrt{\pi}$$

$$\therefore \quad A = \frac{1}{\sqrt{2}}$$

Step 11 : Substituting we get

$$\therefore \quad F(\lambda) = \frac{1}{\sqrt{2}} e^{-\lambda^2/4}$$

Example 8.34 : *Find Fourier sine transform of* $\frac{1}{x}$.

Solution : Step 1 : Consider F S T formula

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \sin \lambda u\, du$$

Step 2 : Put the value of f (u)

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{u} \sin \lambda u\, du$$

Step 3 : Put $\lambda u = t$, $\therefore$ $du = \frac{dt}{\lambda}$

Step 4 : Substituting we get

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin t}{t/\lambda} \frac{dt}{\lambda}$$

Step 5 : Use standard integral $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin t}{t} dt = \sqrt{\frac{2}{\pi}} \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

Example 8.35 : *Find Fourier sine transform of* $\frac{e^{-ax}}{x}$ *and hence evaluate* $\int_0^{\infty} \tan^{-1} \frac{\lambda}{a} \sin \lambda x\, d\lambda$

Solution : Step 1 : Consider F.S.T. formula

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \sin \lambda u\, du$$

Example 8.36 : *Find the Fourier sine and cosine transforms of the function* $f(x) = x^{m-1}$.

Solution : Step 1 : F.C.T. formula

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty u^{m-1} \cos\lambda u \, du \qquad \ldots (1)$$

Step 2 : F.S.T. formula

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty u^{m-1} \sin\lambda u \, du \qquad \ldots (2)$$

Step 3 : By definition of Gamma function

We know $$\Gamma m = \int_0^\infty e^{-x} x^{m-1} \, dx$$

Step 4 : Put $x = i\lambda u$, thus $dx = i\lambda du$,

$$\Gamma m = \int_0^\infty e^{-i\lambda u} (i\lambda u)^{m-1} \, i\lambda \, du$$

Step 5 : Simplify

$$\Gamma m = i^m \lambda^m \int_0^\infty e^{-i\lambda u} u^{m-1} du$$

Step 6 : $(\because i = e^{i\pi/2})$ substituting we get

$$\Gamma m = \left(e^{i\pi/2}\right)^m \lambda^m \int_0^\infty e^{-i\lambda u} u^{m-1} \, du$$

Step 7 : Simplify

$$\frac{\Gamma m}{\lambda^m} e^{-im\pi/2} = \int_0^\infty e^{-i\lambda u} u^{m-1} du$$

Step 8 : Use $e^{-ix} = \cos x - i \sin x$

$$\therefore \frac{\Gamma m}{\lambda^m}\left(\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right)$$

$$= \int_0^\infty (\cos\lambda u - i\sin\lambda u) \, u^{m-1} \, du$$

Step 15 : Put $\lambda = 0$ in (i) and (ii)

$$F(0) = \left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos\lambda u}{1+u^2}\,du\right]_{\lambda=0} = A + B$$

$$F'(0) = \left[\sqrt{\frac{2}{\pi}}\left\{\int_0^\infty \frac{\sin\lambda u}{u\,(1+u^2)}\,du - \frac{\pi}{2}\right\}\right]_{\lambda=0} = A - B$$

Step 16 : Simplify

$$\therefore \quad \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+u^2}\,du = A + B, \; \sqrt{\frac{2}{\pi}}\left\{-\frac{\pi}{2}\right\} = A - B$$

Step 17 : Integrate first

$$\therefore \quad \sqrt{\frac{2}{\pi}}\left[\tan^{-1} u\right]_0^\infty = A + B$$

i.e. $$\sqrt{\frac{2}{\pi}}\left\{\frac{\pi}{2}\right\} = A + B \text{ and } \sqrt{\frac{2}{\pi}}\left\{-\frac{\pi}{2}\right\} = A - B$$

i.e. $$\sqrt{\frac{\pi}{2}} = A + B \text{ and } -\sqrt{\frac{\pi}{2}} = A - B$$

Step 18 : Solving we get, $A = 0$, $B = \sqrt{\frac{\pi}{2}}$

Step 19 : Substituting A and B in equation (i) and (ii), we get

$$F_c(\lambda) = \sqrt{\frac{\pi}{2}}\, e^{-\lambda}$$

and $$F_c'(\lambda) = -\sqrt{\frac{\pi}{2}}\, e^{-\lambda}$$

Step 20 : Substituting f (λ) and $f'(\lambda)$ in step 2 and step 4 we get

$$\therefore \quad \sqrt{\frac{\pi}{2}}\, e^{-\lambda} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+u^2}\cos\lambda u\,du \text{ and}$$

$$-\sqrt{\frac{\pi}{2}}\, e^{-\lambda} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{-u}{1+u^2}\sin\lambda u\,du$$

Step 21 : Thus we can say that the Fourier cosine transform of $\frac{1}{1+x^2}$ is $\sqrt{\frac{\pi}{2}}\, e^{-\lambda}$

Step 22 : Also we can sat that the Fourier sine transform of $\frac{x}{1+x^2}$ is $\sqrt{\frac{\pi}{2}}\, e^{-\lambda}$

Example 8.38 : *Using inverse sine transform, find $f(x)$ if $F_s(\lambda) = \frac{1}{\lambda} e^{-a\lambda}$.*

Solution : Step 1 : Inverse sine transform of $F_s(\lambda)$ is given by (Note the formula for inverse)

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda$$

Step 2 : Put the value of $F(\lambda)$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \frac{e^{-a\lambda}}{\lambda} \sin\lambda x \, d\lambda$$

Step 3 : [Use the rule of D.U.I.S.]

$$\therefore \quad f'(x) = \frac{2}{\pi}\int_0^\infty \frac{\partial}{\partial x} \frac{e^{-a\lambda}}{\lambda} \sin \lambda x \, d\lambda$$

Step 4 : Differentiate w.r.t. x

$$= \frac{2}{\pi}\int_0^\infty e^{-a\lambda} \cos x\lambda \, d\lambda$$

Step 5 : Integrate w.r.t. λ

$$= \frac{2}{\pi}\left[\frac{e^{-a\lambda}}{a^2+x^2}(-a \cos \lambda x + x \sin \lambda x)\right]_0^\infty$$

Step 6 : Put the limits for λ

$$f'(x) = \frac{2}{\pi} \frac{a}{a^2+x^2} \qquad \text{As } e^{-\infty} = 0, \; e^0 = 1$$

Step 7 : Integrating, w.r.t x, we get

$$f(x) = \frac{2}{\pi}\tan^{-1}\frac{x}{a} + A$$

Step 8 : Put $x = 0$,

$$f(0) = 0 + A$$

Step 9 : To find $f(0)$, put $x = 0$ in step 2.

$$\left[\frac{2}{\pi}\int_0^\infty \frac{e^{-a\lambda}}{\lambda} \sin\lambda x \, d\lambda\right]_{x=0} = A \qquad \therefore A = 0$$

Step 10 : Put $A = 0$ in step 7

$$\therefore \quad f(x) = \frac{2}{\pi}\tan^{-1}\frac{x}{a}$$

ii) $f(x) = \begin{cases} 0 & x < -a \\ 1 & -a \le x \le a \\ 0 & x > a \end{cases}$ [Ans. : $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin a\lambda \cos\lambda x}{\lambda} d\lambda$]

iii) $f(x) = \begin{cases} \frac{\pi}{2}\sin x & |x| < \pi \\ 0 & |x| > \pi \end{cases}$ [Ans. : $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\pi \sin\lambda x}{1-\lambda^2} d\lambda$]

iv) $f(x) = \begin{cases} \frac{\pi}{2}\cos x & |x| < \pi \\ 0 & |x| > \pi \end{cases}$ [Ans. : $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda \sin\lambda\pi \cos\lambda x}{1-\lambda^2} d\lambda$]

v) $f(u) = e^{-|u|} \quad -\infty < u < \infty$ [Ans. : $f(u) = \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda u}{1+\lambda^2} d\lambda$]

vi) $f(u) = e^{-u^2/2} \quad -\infty < u < \infty$ [Ans. : $f(u) = \frac{2}{\pi}\int_0^\infty e^{-\lambda^2/2} \cos\lambda u \, d\lambda$]

vii) $f(x) = 5 \quad |x| < a$
$\quad\quad 0 \quad |x| > a$ [Ans. : $f(x) = \frac{10}{\pi}\int_0^\infty \frac{\sin a\lambda}{\lambda} \cos\lambda x \, d\lambda$]

2. *Find Fourier Sine Transform of the following*

i) $f(x) = \begin{cases} 0 & 0 \le x \le 1 \\ x & 1 < x \le 2 \\ 0 & x > 2 \end{cases}$ [Ans. : $\sqrt{\frac{2}{\pi}}\left\{\frac{\cos\lambda - \cos 2\lambda}{\lambda} + \frac{\sin 2\lambda - \sin\lambda}{\lambda^2}\right\}$]

ii) $f(x) = \begin{cases} x^2 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$ [Ans. : $F_s(\lambda) = \sqrt{\frac{2}{\pi}}\left\{\frac{2\sin\lambda}{\lambda^2} - \frac{\cos\lambda}{\lambda} + \frac{2\cos\lambda}{\lambda^3} - \frac{2}{\lambda^3}\right\}$]

iii) $f(x) = \begin{cases} \pi & 0 \le x \le \pi \\ 0 & x > \pi \end{cases}$ [Ans. : $F_s(\lambda) = \sqrt{\frac{2}{\pi}}\left\{\frac{\pi(1-\cos\lambda)}{\lambda}\right\}$]

iv) $f(x) = e^{-2x} + e^{-3x} \quad x > 0$ [Ans. : $F_s(\lambda) = \sqrt{\frac{2}{\pi}}\left\{\frac{\lambda}{\lambda^2+4} + \frac{\lambda}{\lambda^2+9}\right\}$]

v) $f(x) = \begin{cases} \sin x & 0 < x < m \\ 0 & x > m \end{cases}$ [Ans. : $F_s(\lambda) = \sqrt{\frac{2}{\pi}}\left\{\frac{\sin(\lambda-1)m}{\lambda-1} + \frac{\sin(\lambda+1)m}{\lambda+1}\right\}$]

vi) $f(x) = \begin{cases} 0 & 0 < x < a \\ x & a \le x \le b \\ 0 & x > b \end{cases}$ [Ans. : $\sqrt{\frac{2}{\pi}}\left\{\left(\frac{a\cos a\lambda - b\cos b\lambda}{\lambda}\right) + \left(\frac{\sin b\lambda - \sin a\lambda}{\lambda^2}\right)\right\}$]

3. *Find Fourier Cosine Transforms of the following*

i) $f(x) = \begin{cases} \pi & 0 \le x \le \pi \\ 0 & x > \pi \end{cases}$ [Ans. : $F_c(\lambda) = \sqrt{\frac{2}{\pi}}\left(\frac{\pi\sin\lambda}{\lambda}\right)$]

ii) $f(x) = \begin{cases} x^2 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$ [Ans. : $F_c(\lambda) = \sqrt{\frac{2}{\pi}}\left\{\frac{2\cos\lambda}{\lambda^2} + \frac{\sin\lambda}{\lambda} - \frac{2\sin\lambda}{\lambda^3}\right\}$]

iii) $f(x) = \begin{cases} x & 0 \le x \le a \\ 0 & x > a \end{cases}$ [Ans. : $F_c(\lambda) = \sqrt{\frac{2}{\pi}}\left\{\frac{\sin a\lambda}{\lambda} + \frac{\cos a\lambda - 1}{\lambda^2}\right\}$]

viii) $\int_0^{\infty} \left(\frac{1-\cos\lambda\pi}{\lambda}\right) \sin\lambda x \, d\lambda = \begin{cases} \frac{\pi}{2} & 0 < x < \pi \\ 0 & x > \pi \end{cases}$

ix) $\frac{2}{\pi}\int_0^{\infty} \left(\frac{1-\cos k\lambda}{\lambda}\right) \sin\lambda x \, d\lambda = \begin{cases} 1 & 0 < x < k \\ \frac{1}{2} & x = k \\ 0 & x > k \end{cases}$

6. *Find the Fourier cosine integral representation for the following functions*

i) $f(x) = \begin{cases} x & 0 \le x \le a \\ 0 & x > a \end{cases}$ **[Ans. :** $f(x) = \frac{2}{\pi}\int_0^{\infty} \left(\frac{\lambda a \sin a\lambda + \cos a\lambda - 1}{\lambda^2}\right) \cos\lambda x \, d\lambda$**]**

ii) $f(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$ **[Ans. :** $f(x) = \frac{2}{\pi}\int_0^{\infty} \left(\frac{(\lambda^2-2)\sin\lambda + 2\lambda\cos\lambda}{\lambda^3}\right) \cos\lambda x \, d\lambda$**]**

iii) $f(x) = e^{-x} + e^{-2x}, \quad x \ge 0$ **[Ans. :** $f(x) = \frac{6}{\pi}\int_0^{\infty} \left(\frac{\lambda^2+2}{\lambda^4+5\lambda^2+4}\right) \cos\lambda x \, d\lambda$**]**

iv) $f(x) = \frac{1}{1+x^2}, \quad x \ge 0$ **[Ans. :** $f(x) = \int_0^{\infty} e^{-\lambda} \cos\lambda x \, d\lambda$**]**

7. *Find Fourier transform of* $f(x) = x e^{-x}, \ 0 \le x \le \infty$

Hint : As $0 \le x \le \infty$ *we can find F.C.T. or F.S.T. Let us find F.S.T. consider F.C.T of* e^{-x} *find it and then use D.U.I.S.* **[Ans. :** $\sqrt{\frac{2}{\pi}}\left[\frac{2\lambda}{(1+\lambda^2)^2}\right]$**]**

8. *Find Fourier Sine Transform of* $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$ *hence evaluate* $\int_0^{\infty} \frac{\sin^3 x}{x^3} dx$

9. *Find Fourier Cosine Transform of* $f(x) = \begin{cases} \cos x & 0 < x < a \\ 0 & x > a \end{cases}$ *and write integral Fourier representation for f (x)* $f(x) = \frac{1}{\pi}\int_0^{\infty} \left[\frac{\sin(\lambda+1)a}{\lambda+1} + \frac{\sin(\lambda-1)a}{\lambda-1}\right] \cos\lambda x \, d\lambda.$

10. *If* $f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & x > \pi, \ x < 0 \end{cases}$ *then prove that* $f(x) = \frac{1}{\pi}\int_0^{\infty} \left[\frac{\cos\lambda x + \cos\lambda(\pi - x)}{1-\lambda^2}\right] d\lambda.$ *Hence deduce* $\int_0^{\infty} \frac{\cos\lambda\frac{\pi}{2}}{1-\lambda^2} d\lambda = \frac{\pi}{2}$

11. *If* $f(x) = \begin{cases} \cos x & 0 < x < \pi \\ 0 & x < 0, \ x > \pi \end{cases}$ *then prove that* $f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{\lambda[\sin\lambda x + \sin\lambda(\pi - x)]}{1-\lambda^2} d\lambda.$ *Hence deduce* $\int_0^{\infty} \frac{\lambda \sin\lambda\pi}{1-\lambda^2} d\lambda = \frac{-\pi}{2}.$

12. *Find Fourier Cosine Transform of* $f(x) = e^{-x}$ *and hence deduce* $\int_0^{\infty} \frac{\cos 2x}{1+x^2} dx = \frac{\pi}{2e^2}$

13. *Find complex Fourier integral of* e^{-x^2} **[Ans. :** $f(x) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-\lambda^2/4} \cos \lambda x \, d\lambda$**]**

14. *Solve the following integral equations*

   i) $\int_0^{\infty} f(x) \cos \lambda x \, dx = \frac{\pi}{2} e^{-\lambda^2/2} \quad \lambda > 0$ **[Ans. :** $f(x) = \sqrt{\frac{\pi}{2}} e^{-x^2/2}$**]**

   ii) $\int_0^{\infty} f(x) \sin \lambda x \, dx = \begin{cases} 1-\lambda & 0 \le \lambda \le 1 \\ 0 & \lambda > 1 \end{cases}$ **[Ans. :** $f(x) = \frac{2}{\pi x^2}(x - \sin x)$**]**

   iii) $\int_0^{\infty} f(x) \sin \lambda x \, dx = \frac{\lambda}{\lambda^2 + k^2}$ **[Ans. :** $f(x) = e^{-kx}$**]**

   iv) $\int_0^{\infty} f(x) \cos \lambda x \, dx = \frac{k}{\lambda^2 + k^2}$ **[Ans. :** $f(x) = e^{-kx}$**]**

15. *Use inverse Fourier Sine Transform to find f (x) if* $F_s(\lambda) = \frac{\lambda}{a^2 + \lambda^2}$ **[Ans. :** e^{-ax}**]**

16. *Using inverse Fourier Cosine Transform find f (x) if* $F_c(\lambda) = \begin{cases} \sqrt{\frac{2}{\pi}}\left(a - \frac{\lambda}{2}\right) & \lambda \le 2a \\ 0 & \lambda > 2a \end{cases}$

   [Ans. : $\frac{2 \sin^2 ax}{\pi x^2}$**]**

8.11 Properties and Theorems of Fourier Transforms

If F (λ) = F[f(x)] and G (λ) = F[g(x)] are the complex Fourier transforms of f (x) and g (x) then :

1) Linearity Property

$$\begin{aligned} F[k_1 f(x) + k_2 g(x)] &= k_1 F[f(x)] + k_2 F[g(x)] \\ &= k_1 F(\lambda) + k_2 G(\lambda) \end{aligned}$$

2) Change of Scale Property

$$F[f(ax)] = \frac{1}{a} F\left(\frac{\lambda}{a}\right), \; a \ne 0$$

3) Shifting Property

$$F[f(x-a)] = e^{-i\lambda a} F(\lambda)$$

4) Modulation Theorem

$$F\,[f\,(x)\cos ax] = \frac{1}{2}[F\,(\lambda+a)+F\,(\lambda-a)]$$

Also,

a) $$F_c\,[f\,(x)\cos ax] = \frac{1}{2}[F_c\,(\lambda+a)+F_c\,(\lambda-a)]$$

b) $$F_c\,[f\,(x)\sin ax] = \frac{1}{2}[F_s\,(\lambda+a)-F_s\,(\lambda-a)]$$

c) $$F_c\,[f\,(x)\cos ax] = \frac{1}{2}[F_s\,(\lambda+a)+F_s\,(\lambda-a)]$$

d) $$F_c\,[f\,(x)\sin ax] = \frac{1}{2}[F_c\,(\lambda-a)-F_c\,(\lambda+a)]$$

This theorem is of great importance in the theory of communication.

5) Convolution Theorem

The convolution of the functions f (x) and g (x) over the interval $-\infty < x < \infty$ is defined as

$$f\,(x) * g\,(x) = \int_{-\infty}^{\infty} f\,(u)\,g\,(x-u)\,du \text{ and then}$$

$$F\,[f\,(x) * g\,(x)] = F\,[f\,(x)]\,F\,[g\,(x)] = F\,(\lambda)\,G\,(\lambda)$$

8.12 Finite Fourier Transforms

For a function f (x) is defined in the finite interval (0 < x < L), and satisfying the Dirichlet's conditions, we can obtain its half range cosine or sine series. Using this representation, we define the Finite Cosine or Sine Transforms of f (x) as follows :

1) Finite Fourier Cosine Transform

For a function f (x) defined in 0< x< L, then half range cosine series of f (x) is given by :

$$f\,(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \qquad \text{(where n is an integer)} \qquad \dots (1)$$

where $$a_0 = \frac{2}{L}\int_0^L f\,(x)\,dx$$

and $$a_n = \frac{2}{L}\int_0^L f\,(x)\cos\frac{n\pi x}{L}\,dx$$

Now, the **Finite Fourier Cosine Transform** of f (x) is defined as :

$$\boxed{F_c\,[f(n)] \text{ or } c(n) = \int_0^L f(x) \cos\frac{n\pi x}{L}\,dx}$$ where n is an integer

So that $$a_0 = \frac{2}{L}\int_0^L f(x)(1)\,dx = \frac{2}{L} F_c\,(0)$$

and $$a_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L}\,dx = \frac{2}{L} F_c\,[f(n)]$$

and hence, using (1), the **Inverse Finite Fourier Cosine Transform** of F_c [f(n)] is given by :

$$\boxed{f(x) = \frac{1}{L} F_c\,(0) + \frac{2}{L}\sum_{n=1}^{\infty} F_c[f(n)] \cos\frac{n\pi x}{L}}$$

2) Finite Fourier Sine Transform

For a function f (x) is defined in 0 < x < L, the half range sine series of f (x) is given by :

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{(where n in an integer)} \quad \ldots (2)$$

where, $$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx \quad \ldots (2)$$

Now, the **Finite Fourier Sine Transform** of f (x) is defined as

$$\boxed{F_s\,[f(n)] \text{ or } S\,(n) = \int_0^L f(x) \sin\frac{n\pi x}{L}\,dx}$$ where n is an integer

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx = \frac{2}{L} F_s\,[f(n)]$$

and hence, using (2), the **Inverse Finite Fourier Sine Transform** of F_s [f(n)] is given by

$$\boxed{f(x) = \frac{2}{L}\sum_{n=1}^{\infty} F_s\,[f(n)] \sin\frac{n\pi x}{L}}$$

8.13 Illustrations

➡ **Example 8.41 :** *Find Finite Fourier sine transform of* $f(x) = \begin{cases} kx & 0 \le x \le \pi/2 \\ k(\pi - x) & \pi/2 \le x \le \pi \end{cases}$

Solution : Here the interval (0, π) is finite. Hence, we get the Finite Fourier Transform.

Step 1 : By the definition the formula for Finite Fourier sine transform in (0, L) is

$$F_s\,[f(n)] = \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Put L = π

$$= \int_0^{\pi} f(x) \sin n\pi x \; dx$$

Step 2 : Split the integral and substitute f(x).

$$= \int_0^{\pi/2} kx \sin nx \, dx + \int_{\pi/2}^{\pi} k(\pi - x) \sin nx \, dx$$

Step 3 : Integrate by parts and substitute limits of x

$$= k\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2}$$

$$+ k\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= k\left[\left(-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right) - (0+0)\right.$$

$$\left. + \left[(0-0) - \left(-\frac{\pi}{2n}\cos\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{n\pi}{2}\right)\right]\right]$$

$$= k\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$\therefore \quad F_s\,[f(n)] = k\frac{2}{n^2}\sin\frac{n\pi}{2}$$

➡ **Example 8.42 :** *Find Fourier cosine transform of* $f(x) = lx - x^2$ *in* $0 \le x \le l$.

Solution : Step 1 : Here, the intervals of x are [0, l] is finite. Hence, we get the Finite Fourier Transforms.

By definition the Finite Fourier Cosine Transform of

University Questions

Dec. - 98

1. *Find Fourier sine transform of* $\left(\frac{1}{x}\right)$. **[5 Marks]**

2. *Find Fourier cosine transform of* $f(x) = e^{-x}$ *and hence prove that :*

$$\int_0^\infty \frac{\cos mx}{1+x^2}\,dx = \frac{\pi e^{-m}}{2}$$ **[5 Marks]**

3. *Use Fourier integral representation to show that*

$$\int_0^\infty \frac{\lambda^2 \sin \lambda x}{\lambda^4 + 4}\,d\lambda = \frac{\pi}{2} e^{-x} \cos x, \quad x > 0$$ **[6 Marks]**

May - 99

1. *By considering Fourier sine integral for* e^{-mx} *prove that :*

$$\int_0^\infty \frac{\lambda \sin \lambda x}{\lambda^2 + m^2}\,d\lambda = \frac{\pi}{2} e^{-mx} \quad m > 0, \; x > 0.$$ **[5 Marks]**

2. *Using Fourier transform solve the following integral equation :*

$$\int_0^\infty f(x) \sin \lambda x\,dx = \begin{cases} 1-\lambda & 0 \le \lambda \le 1 \\ 0 & \lambda \ge 1 \end{cases}$$ **[5 Marks]**

3. *Find the Fourier transform of :*

$$f(x) = \begin{cases} 1-x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

and hence evaluate

$$\int_0^\infty \left(\frac{x \cos x - \sin x}{x^3}\right) \cos \frac{x}{2}\,dx$$ **[6 Marks]**

Dec. - 99

1. *Find Fourier sine transform of* $\frac{e^{-ax}}{x}$ **[6 Marks]**

2. *Solve the integral equation*

$$\int_0^\infty f(x) \cos \lambda x\,dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0\;, & \lambda \ge 1 \end{cases}$$

and hence show that

$$\int_0^\infty \frac{\sin^2 z}{z^2}\,dz = \frac{\pi}{2}$$ **[5 Marks]**

3. *Find Fourier transform of*

$$f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0\;, & |x| > 1 \end{cases}$$ **[4 Marks]**

May - 2000

1. *Using Fourier inverse sine transform ; find $f(x)$; if*

$$F_s(\lambda) = \frac{e^{-a\lambda}}{\lambda}$$ **[5 Marks]**

2. *Find Fourier transform on :* $f(x) = \begin{cases} 1 - x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ **[5 Marks]**

Dec. - 2000

1. *Obtain Fourier sine and cosine transform of the function* $f(x) = x^{m-1}$. **[6 Marks]**

2. *If* $f(x) = \begin{cases} \sin x & when \quad 0 < x < \pi \\ 0 & when \quad x < 0 \, and \, when \, x > \pi \end{cases}$

then show that :

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos \lambda x + \cos[\lambda(\pi - x)]}{1 - \lambda^2} \, d\lambda$$ **[6 Marks]**

May - 2001

1. *Using Fourier integral representation show that :*

$$\int_0^\infty \frac{1 - \cos \pi\lambda}{\lambda} \sin \lambda x \, d\lambda = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$$ **[5 Marks]**

2. *Find Fourier cosine transform of* $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 1 \\ 0, & x > 2 \end{cases}$ **[5 Marks]**

Dec. - 2001

1. *Using fourier integral representation show that :*

$$\int_0^\infty \frac{\lambda^3 \sin \lambda x}{\lambda^4 + 4} \, d\lambda = \frac{\pi}{2} e^{-x} \cos x \, ; \; x \ge 0$$ **[6 Marks]**

2. *Find the fourier sine transform of* $\frac{e^{-ax}}{x}$. **[5 Marks]**

May - 2002

1. *Show that the fourier transform of*

$f(x) = e^{-|x|}$ *is* $\frac{2}{1 + \lambda^2}$ **[5 Marks]**

2. *By considering the fourier sine integral for* $f(x) = e^{-mx}$, $m > 0$,

Show that :

$$\int_0^\infty \frac{\lambda \sin \lambda x}{\lambda^2 + m^2} \, d\lambda = \frac{\pi}{2} e^{-mx}, \quad m > 0, \; x > 0.$$ **[5 Marks]**

3. *Use fourier integral representation to show that :*

$$\int_0^{\infty} \frac{\lambda^3 \sin \lambda x}{\lambda^4 + 4} d\lambda = \frac{\pi}{2} e^{-x} \cos x \ , \quad x > 0.$$

[5 Marks]

Dec. - 2002

1. *Find the Fourier cosine transform of* $f(x) = e^{-x^2}$ **[5 Marks]**

2. *Using inverse sine transform find* $f(x)$, *if* $F_s(\lambda) = \frac{1}{\lambda} e^{-a\lambda}$. **[5 Marks]**

3. *If* $f(x) = \sin x, \quad 0 < x < \pi$

$$= 0 \quad , \quad x < 0 \text{ and } x > \pi$$

then show that

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos \lambda x + \cos \lambda(\pi - x)}{1 - \lambda^2} d\lambda$$

Hence deduce that

$$f(x) = \int_0^{\infty} \frac{\cos \lambda \frac{\pi}{2}}{1 - \lambda^2} d\lambda = \frac{\pi}{2}$$

[6 Marks]

May - 2003

1. *Solve the integral equation :*

$$\int_0^{\infty} f(x) \sin \lambda x \, dx = e^{-\lambda} (\lambda > 0)$$

[5 Marks]

2. *Find cosine (Fourier) transform of*

$$f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0 & , \ x > a \end{cases}$$

[5 Marks]

3. *Using Fourier integral representation show that :*

$$\int_0^{\infty} \frac{\lambda^3 \sin \lambda x}{\lambda^4 + 4} d\lambda = \frac{\pi}{2} e^{-x} \cos x, \ (x > 0)$$

[8 Marks]

Dec. - 2003

1. *What is the function f (x) whose Fourier cosine transform is :*

$$\frac{\sin \sigma\lambda}{\lambda}$$

[5 Marks]

2. *Using Fourier integral representation show that :*

$$\int_0^{\infty} \frac{1 - \cos \pi\lambda}{\lambda} \sin \lambda x \, d\lambda = \frac{\pi}{2} \qquad 0 < x < \pi$$

$$0 \qquad x > \pi$$

[5 Marks]

3. *Find Fourier transform of* $f(x) = e^{-x^2/2}; \ x > 0.$ **[8 Marks]**

May - 2004

1. *Find the Fourier sine transform of* $e^{-|x|}$, *hence evaluate :*

$$\int_0^{\infty} \frac{x \sin mn}{1+x^2} dx$$

[5 Marks]

2. *Solve :*

$$\int_0^{\infty} f(x) \cos \lambda x \, dx = e^{-\lambda}$$

[5 Marks]

Dec. - 2004

1. *Find Fourier sine transform of :*

$$f(x) = \begin{matrix} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & x > 2 \end{matrix}$$

[5 Marks]

2. *Using Fourier integral representation, show that*

$$\int_0^{\infty} \frac{\cos \frac{\pi\lambda}{2} \cos \lambda x}{1-\lambda^2} d\lambda = \begin{matrix} \frac{\pi}{2} \cos x & |x| \le \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{matrix}$$

[6 Marks]

3. *Find $f(x)$ whose Fourier sine transform is*

$$F_s(\lambda) = \frac{1}{\lambda} e^{-a\lambda}$$

[5 Marks]

May - 2005

1. *Find the Fourier transform of :*

$$f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0 \quad , & |x| > 1 \end{cases}$$

and hence evaluate :

$$\int_0^{\infty} \left(\frac{x \cos x - \sin x}{x^3} \right) \cos \frac{x}{2} dx$$

[6 Marks]

2. *Obtain Fourier sine and cosine transform of the function :*

$f(x) = x^{m-1}$ **[6 Marks]**

3. *Solve the integral equation :*

$$\int_0^{\infty} f(x) \cos \lambda x \, dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0 \quad , & \lambda > 1 \end{cases}$$

and hence show that :

$$\int_0^{\infty} \frac{\sin^2 z}{z^2} dz = \frac{\pi}{2}$$

[6 Marks]

Dec. - 2005

1. *Show that Fourier transform of $e^{-x^2/2}$ is itself.* **[6 Marks]**
2. *Using Fourier integral representation, show that :*

$$\int_0^{\infty} \frac{\lambda^3 \sin \lambda x}{\lambda^4 + 4} d\lambda = \frac{\pi}{2} e^{-x} \cos x, \text{ where } x > 0.$$

[5 Marks]

3. *Find inverse sine transform of* $\frac{1}{\lambda}e^{-a\lambda}$. **[5 Marks]**

May - 2006

1. *Find Fourier sine transform of :*

$$f(x) = \frac{e^{-ax}}{x},\ 0 < x < \infty$$ **[5 Marks]**

2. *Find the Fourier cosine transform of*

$$f(x) = \begin{cases} x^2 + 2x + 1, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$ **[4 Marks]**

3. *Solve :*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ -\infty < x < \infty$$

subject to the condition :

$u(x,0) = e^{-|x|}$ *for all* x. **[7 Marks]**

❑❑❑

9

Applications of Fourier Transform

9.1 To obtain the Solutions of Partial Differential Equations

a) $\dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2}$ (Heat equation)

b) $\dfrac{\partial^2 u}{\partial t^2} = k^2 \dfrac{\partial^2 u}{\partial x^2}$ (Wave equation)

c) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ (Laplace's equation)

Depending on the boundary conditions and initial conditions, we take infinite Fourier or finite Fourier transform of both sides of above equations, simplifying and using boundary and initial conditions, and then using corresponding inverse transform gives the solution.

9.2 Fourier Transforms of Derivatives of a Function

In solving the problems on applications we need to find Fourier transform of $\dfrac{\partial u}{\partial x}, \dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial u}{\partial t}, \dfrac{\partial^2 u}{\partial t^2}$ etc.

Note :

1) For the interval $-\infty < x < \infty$ we always assume u and $\dfrac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$ even if it is not given in the problem.

2) For the interval $0 < x < \infty$ we always assume u and $\dfrac{\partial u}{\partial x} \to 0$ as $x \to \infty$ even if it is not given in the problem.

A) Fourier transform of $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial^2 u}{\partial x^2}$

Let $\bar{u}(\lambda, t)$ be Fourier transform of u (x, t)

$$\therefore \qquad \bar{u}(\lambda, t) = \int_{-\infty}^{\infty} u(x, t)\, e^{-i\lambda x}\, dx$$

$\therefore$ Fourier transform of $\frac{\partial u}{\partial x}$ is given by

$$F\left[\frac{\partial u}{\partial x}\right] = \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{-i\lambda x} \, dx$$

Integrating by parts

$$= \left[e^{-i\lambda x} u\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-i\lambda) \, e^{-i\lambda x} \cdot u \, dx$$

$$= (0-0) + i\lambda \int_{-\infty}^{\infty} u \, e^{-i\lambda x} \, dx$$

$$F\left[\frac{\partial u}{\partial x}\right] = (i\lambda) \, \bar{u} \, (\lambda, t)$$

$$\therefore \quad \boxed{F\left[\frac{\partial u}{\partial x}\right] = i\lambda \, \bar{u} \, (\lambda, t)}$$

Similarly we can show that

$$\therefore \quad \boxed{\begin{aligned} F\left[\frac{\partial^2 u}{\partial x^2}\right] &= (i\lambda)^2 \, \bar{u} \, (\lambda, t) \\ &= -\lambda^2 \, \bar{u} \, (\lambda, t) \end{aligned}}$$

Similarly we can prove that

$$F\left[\frac{\partial^3 u}{\partial x^3}\right] = (i\lambda)^3 \, \bar{u} \, (\lambda, t) \qquad \text{and so on.}$$

B) Fourier transform of $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial t^2}$

We know that

$$\bar{u} \, (\lambda, t) = \int_{-\infty}^{\infty} u(x, t) \, e^{-i\lambda x} \, dx \qquad \dots (1)$$

$\therefore$ Fourier transform of $\frac{\partial u}{\partial t}$ is

$$F\left[\frac{\partial u}{\partial t}\right] = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\lambda x} \, dx$$

Using D.U.I.S i.e. partial derivative inside the integral becomes total derivative outside the integral.

$$F\left[\frac{\partial u}{\partial t}\right] = \frac{d}{dt}\int_{-\infty}^{\infty} u\, e^{-i\lambda x}\, dx$$

$$F\left[\frac{\partial u}{\partial t}\right] = \frac{d}{dt}\, \bar{u}\,(\lambda, t) \qquad \text{from (1)}$$

$$\therefore \quad \boxed{F\left[\frac{\partial u}{\partial t}\right] = \frac{d}{dt}\, \bar{u}\,(\lambda, t)}$$

Similarly we can show

$$\boxed{F\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{d^2}{dt^2}\, \bar{u}\,(\lambda, t)} \qquad \text{and so on.}$$

C) Fourier sine transform of $\frac{\partial^2 u}{\partial x^2}$

We know that

$$\bar{u}_s\,(\lambda, t) = \int_{-\infty}^{\infty} u(x,t) \sin\lambda x\, dx \qquad \dots (1)$$

$\therefore$ Fourier sine transform of $\frac{\partial^2 u}{\partial x^2}$ is

$$F_S\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{0}^{\infty} \frac{\partial^2 u}{\partial x^2} \sin\lambda x\, dx$$

Use integration by parts

$$= \left[\sin\lambda x \cdot \frac{\partial u}{\partial x}\right]_0^{\infty} - \int_0^{\infty} \lambda \cos\lambda x \frac{\partial u}{\partial x}\, dx$$

$$= (0-0) - \lambda\int_0^{\infty} \cos\lambda x\, \frac{\partial u}{\partial x}\, dx$$

Again use integration by parts

$$= -\lambda[\cos\lambda x \cdot u]_0^{\infty} - \int_0^{\infty} (-\lambda \sin\lambda x)\, u\, dx$$

$$= -\lambda\left\{[0 - 1\cdot u\,(0,t)] + \lambda\int_0^{\infty} u \sin\lambda x\, dx\right\}$$

$$= +\,\lambda\, u\,(0,t) - \lambda^2\, \bar{u}_s\,(\lambda, t)$$

$$\boxed{F_S\left[\frac{\partial^2 u}{\partial x^2}\right] = \lambda\;\; u\,(0,t) - \lambda^2\, \bar{u}_s\,(\lambda, t)}$$

Note : For the interval $0 < x < \infty$, we always assume u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$, even if it is not given in the problem.

9.4 Illustrations : (A) Heat Flow in Semi Infinite Bar

Type I Problems on sine transform

Example 9.1 : *Solve the equation* $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$ $(0 < x < \infty)$ *(Heat equation) subject to the boundary condition u (0, t) = 0 and initial condition u (x, 0) = f (x).*

Solution : Step 1 : As u (0, t) = 0

∴ We use Fourier sine transform.

Let F.S.T of u (x, t) be $\bar{u}_s(\lambda, t)$

i.e.
$$\bar{u}_s(\lambda, t) = \int_0^\infty u(x, t)\sin\lambda x \, dx \quad \text{... (1)}$$

Step 2 : Consider the given heat equation

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$$

Step 3 : Take F.S.T on both sides.

$$\therefore \int_0^\infty \frac{\partial u}{\partial t}\sin\lambda x \, dx = K\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin\lambda x \, dx$$

Step 4 : Use D.U.I.S on L.H.S (i.e. partial derivative inside the integral becomes total derivative outside the integral).

i.e.
$$\frac{d}{dt}\int_0^\infty u\sin\lambda x \, dx = K\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin\lambda x \, dx$$

Step 5 : Use integration by parts formula twice on R.H.S.

i.e.
$$\int uv \, dx = u\left(\int v dx\right) - \int\left(\frac{du}{dx}\int v \, dx\right)dx$$

$$\therefore \frac{d}{dt}\bar{u}_s(\lambda, t) = K\left\{\left(\sin\lambda x\frac{\partial u}{\partial x}\right)_0^\infty - \int_0^\infty \lambda\cos\lambda x\frac{\partial u}{\partial x}dx\right\}$$

$$= K\left\{(0-0) - \lambda\left[(\cos\lambda x \cdot u)_0^\infty - \int_0^\infty -\lambda\sin\lambda x \cdot u \, dx\right]\right\}$$

$$= 0 - \lambda K\left[(0-0) + \lambda\bar{u}_s(\lambda, t)\right] \quad \text{... from (1)}$$

$$\therefore \quad \frac{d}{dt}\bar{u}_s(\lambda, t) = -K\lambda^2 \, \bar{u}_s(\lambda, t) \qquad \dots (2)$$

Step 6 :

$$\therefore \quad \frac{\frac{d}{dt}\bar{u}_s(\lambda, t)}{\bar{u}_s(\lambda, t)} = -K\lambda^2$$

Integrating w.r.t. t

$$\log \bar{u}_s(\lambda, t) = -K\lambda^2 t + \log A$$

$$\therefore \quad \bar{u}_s(\lambda, t) = A\, e^{-k\lambda^2 t} \qquad \dots (3)$$

Step 7 : To find A put t = 0

$$\bar{u}_s(\lambda, 0) = A$$

Step 8 : We know from (1)

$$\bar{u}_s(\lambda, t) = \int_0^\infty u(x, t) \sin \lambda x \, dx$$

put t = 0

$$\therefore \quad \bar{u}_s(\lambda, 0) = \int_0^\infty u(x, 0) \sin \lambda x \, dx$$

Given u (x, 0) = f (x)

$$\therefore \quad \bar{u}_s(\lambda, 0) = \int_0^\infty f(x) \sin \lambda x \, dx$$

Step 9 : As the actual value of f(x) is not given

$\therefore$ We call the value of this integral as $F(\lambda)$.

i.e. $$\bar{u}_s(\lambda, 0) = \int_0^\infty f(x) \sin \lambda x \, dx = F(\lambda)$$

$$\therefore \quad A = F(\lambda)$$

Step 10 : Substituting in (3) we get

$$\bar{u}_s(\lambda, t) = F(\lambda)\, e^{-k\lambda^2 t}$$

Step 11 : Consider inverse sine transform

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \bar{u}_s(\lambda, t) \sin \lambda x \, d\lambda$$

$$\therefore \qquad u(x, t) = \frac{2}{\pi}\int_0^\infty F(\lambda)\, e^{-k\lambda^2 t} \sin\lambda x \, d\lambda$$

Example 9.2 : *Solve* $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \quad t > 0, \quad x > 0$

Subject to (i) $u(0, t) = 0$ *ii)* $u(x, 0) = \begin{cases} 1 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$

Solution : Step 1 : As the condition is u (0, t) = 0

∴ We take Fourier sine transform

Let F.S.T of u (x, t) = $\bar{u}_s(\lambda \cdot t)$

$$\therefore \qquad \bar{u}_s(\lambda, t) = \int_0^\infty u(x, t) \sin\lambda x \, dx \qquad \text{... (1)}$$

Step 2 : Consider the given equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Step 3 : Taking Fourier sine transform of both sides

$$\int_0^\infty \frac{\partial u}{\partial t} \cdot \sin\lambda x \, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cdot \sin\lambda x \, dx$$

Step 4 : Use D.U.I.S on L.H.S and integration by parts on R.H.S.

$$\frac{d}{dt}\int_0^\infty u \sin\lambda x \, dx = \left\{\left(\sin\lambda x \cdot \frac{\partial u}{\partial x}\right)_0^\infty - \int_0^\infty \lambda \cos\lambda x \frac{\partial u}{\partial x} dx\right\}$$

$$\frac{d}{dt}\left[\int_0^\infty u \sin\lambda x \, dx\right] = \left[0 - \lambda\int_0^\infty \cos\lambda x \frac{\partial u}{\partial x} dx\right]$$

Step 5 : Again by parts on R.H.S.

$$\frac{d}{dt}\bar{u}_s(\lambda, t) = (-\lambda)\left[(\cos\lambda x\, u)_0^\infty - \int_0^\infty -\lambda(\sin\lambda x)\, u \cdot dx\right]$$

$$\frac{d}{dt}\bar{u}_s(\lambda, t) = \sqrt{\frac{2}{\pi}}(-\lambda)\left[0 + \lambda\int_0^\infty u \sin\lambda x \, dx\right]$$

$$= -\lambda^2 \cdot \left[\sqrt{\frac{2}{\pi}}\int_0^\infty u \sin\lambda x \, dx\right]$$

Step 11 : Consider inverse sine transform

$$u(x, t) = -\int_0^\infty \bar{u}_s(\lambda, t) \cdot \sin \lambda x \, d\lambda$$

$$\therefore \quad u(x, t) = \frac{2}{\pi} \int_0^\infty \left(\frac{1-\cos\lambda}{\lambda}\right) \cdot e^{-\lambda^2 t} \sin\lambda x \, d\lambda$$

Example 9.3 : *Use Fourier sine transform to solve* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < \infty$, $t > 0$ *subject to*

i) $u(0, t) = 0$

ii) $u(x, 0) = e^{-x} \quad x > 0$

iii) u *and* $\frac{\partial u}{\partial x} \to 0$ *as* $x \to \infty$

Solution : We assume the 3rd condition even if it is not given in the problem.

As u (0, t) = 0 we take Fourier sine transform.

Step 1 : As the condition is u (0, t) = 0

∴ We take Fourier sine transform

Let F.S.T of u (x, t) = $\bar{u}_s(\lambda \cdot t)$

$$\therefore \quad \bar{u}_s(\lambda, t) = \int_0^\infty u(x, t) \sin\lambda x \, dx \quad \ldots (1)$$

Step 2 : Consider the given equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Step 3 : Taking Fourier sine transform of both sides

$$\int_0^\infty \frac{\partial u}{\partial t} \cdot \sin\lambda x \, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cdot \sin\lambda x \, dx$$

Step 4 : Use D.U.I.S on L.H.S and integration by parts on R.H.S.

$$\frac{d}{dt}\int_0^\infty u \sin\lambda x \, dx = \left\{\left(\sin\lambda x \cdot \frac{\partial u}{\partial x}\right)_0^\infty - \int_0^\infty \lambda \cos\lambda x \frac{\partial u}{\partial x} dx\right\}$$

$$\frac{d}{dt}\left[\int_0^\infty u \sin\lambda x \, dx\right] = \left[0 - \lambda \int_0^\infty \cos\lambda x \frac{\partial u}{\partial x} dx\right]$$

Step 5 : Again by parts on R.H.S.

$$\frac{d}{dt} \bar{u}_s(\lambda, t) = (-\lambda)\left[(\cos\lambda x \, u)_0^\infty - \int_0^\infty -\lambda(\sin\lambda x) \, u \cdot dx\right]$$

Step 11 : Consider inverse sine transform

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \bar{u}_s(\lambda, t)\sin\lambda x \, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\lambda}{1+\lambda^2} e^{-\lambda^2 t}\sin\lambda x \, d\lambda$$

Example 9.4 : *Solve the following equation* $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$, $0 < x < \infty$, $t > 0$ *subject to*

i) $u(0, t) = 0$

ii) $u(x, 0) = f(x)$ *where* $f(x) = \begin{cases} 1 & a < x < b \\ 0 & \text{otherwise} \end{cases}$ $b > a > 0$

Solution : Step 1 : As $u(0, t) = 0$

∴ We use Fourier sine transform.

Let F.S.T of $u(x, t)$ be $\bar{u}_s(\lambda, t)$

i.e. $$\bar{u}_s(\lambda, t) = \int_0^\infty u(x, t)\sin\lambda x \, dx \quad \text{... (1)}$$

Step 2 : Consider the given heat equation

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$$

Step 3 : Take F.S.T on both sides.

$$\therefore \int_0^\infty \frac{\partial u}{\partial t}\sin\lambda x \, dx = K\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin\lambda x \, dx$$

Step 4 : Use D.U.I.S on L.H.S (i.e. partial derivative inside the integral becomes total derivative outside the integral).

i.e. $$\frac{d}{dt}\int_0^\infty u\sin\lambda x \, dx = K\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin\lambda x \, dx$$

Step 5 : Use integration by parts formula twice on R.H.S.

i.e. $$\int uv \, dx = u\left(\int v dx\right) - \int\left(\frac{du}{dx}\int v \, dx\right)dx$$

$$\therefore \frac{d}{dt}\bar{u}_s(\lambda, t) = K\left\{\left(\sin\lambda x\frac{\partial u}{\partial x}\right)_0^\infty - \int_0^\infty \lambda\cos\lambda x\frac{\partial u}{\partial x}dx\right\}$$

$$= K\left\{(0-0) - \lambda\left[(\cos\lambda x\cdot u)_0^\infty - \int_0^\infty -\lambda\sin\lambda x\cdot u \, dx\right]\right\}$$

Step 5 : Which is linear differential equation of the form $\frac{dy}{dt} + Py = Q$ whose general solution is $y\, e^{\int P\,dt} = \int Q\, e^{\int P\,dt}\, dt + C$

i.e. $$\bar{u}\; e^{\int k\lambda^2 dt} = \int_0^\infty K\lambda\, f(t) \cdot e^{\int k\lambda^2 dt} dt + C$$

As $t > 0$ we integrate for all $t > 0$

$$\therefore \quad \bar{u} \cdot e^{k\lambda^2 t} = \int_0^\infty K\lambda\, f(t)\, e^{k\lambda^2 t}\, dt + C$$

$$\therefore \quad \bar{u}_s\,(\lambda, t) = e^{-k\lambda^2 t} \int_0^\infty K\lambda\, f(t)\, e^{k\lambda^2 t}\, dt + C\, e^{-K\lambda^2 t} \qquad \ldots (3)$$

Step 6 : As the variable is not important replace t by z inside the integral

$$\bar{u}_s\,(\lambda, t) = e^{-k\lambda^2 t} \int_0^\infty K\lambda\, f(z)\, e^{k\lambda^2 z}\, dz + C\, e^{-k\lambda^2 t}$$

$$\bar{u}_s\,(\lambda, t) = K\lambda \int_0^\infty f(z)\, e^{k\lambda^2 (z-t)} dz + C\, e^{-k\lambda^2 t} \qquad \ldots (4)$$

Step 7 : Put $t = 0$

$$\therefore \quad \bar{u}_s\,(\lambda, 0) = 0 + C$$

Step 8 : Put $t = 0$ in (1)

$$\bar{u}_s\,(\lambda, 0) = \int_0^\infty u\,(x, 0) \sin \lambda x\, dx$$

$$= 0 \quad \text{As } u\,(x, 0) = 0 \text{ (given)}$$

Thus $C = 0$

Step 9 : $\therefore$ From (4)

$$\bar{u}_s\,(\lambda, t) = K\lambda \int_0^\infty f(z)\, e^{k\lambda^2 (z-t)}\, dz \qquad \ldots (5)$$

Step 10 : Taking inverse Fourier sine transform

$$u\,(x, t) = \frac{2}{\pi} \int_0^\infty \bar{u}_s\,(\lambda, t) \sin \lambda x\, d\lambda$$

Integrating w.r.t t

$$\log \bar{u}_c(\lambda, t) = -\lambda^2 t + \log A$$

$$\bar{u}_c(\lambda, t) = A\, e^{-\lambda^2 t} \qquad \ldots (2)$$

Step 5 : To find A put t = 0

$$\therefore \quad \bar{u}_c(\lambda, 0) = A$$

Step 6 : Put t = 0 in (1)

$$\bar{u}_c(\lambda, 0) = \int_0^\infty u(x, 0) \cos \lambda x \, dx$$

Substituting the value of u (x, 0)

$$\therefore \quad A = \left\{ \int_0^1 (1 - x^2) \cos \lambda x \, dx + \int_1^\infty 0 \cos \lambda x \, dx \right\}$$

Use generalized rule of integrating by parts i.e. $\int uv dx = uv_1 - u'v_2 + u''v_3 \ldots$

$$= \left[(1 - x^2) \cdot \left(\frac{\sin \lambda x}{\lambda} \right) - (-2x) \left(\frac{-\cos \lambda x}{\lambda^2} \right) + (-2) \left(\frac{\sin \lambda x}{\lambda^3} \right) \right]_0^1$$

Substituting the limits of x

$$= \left\{ \left(0 - \frac{2 \cos \lambda}{\lambda^2} + \frac{2 \sin \lambda}{\lambda^3} \right) - (0 - 0 - 0) \right\}$$

$$A = \frac{2}{\lambda^3} [\sin \lambda - \lambda \cos \lambda]$$

Step 7 : Subsituting A in (2) we get

$$\bar{u}_c(\lambda, t) = \frac{2}{\lambda^3} (\sin\lambda - \lambda \cos\lambda)\, e^{-\lambda^2 t} \qquad \ldots (3)$$

Step 8 : Consider inverse Fourier cosine transform

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \bar{u}_c(\lambda, t) \cos \lambda x \, d\lambda$$

$$= \frac{2}{\pi} \cdot 2 \cdot \int_0^\infty \left(\frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) e^{-\lambda^2 t}, \cos \lambda x \, d\lambda$$

$$u(x, t) = \frac{4}{\pi} \int_0^\infty \left(\frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) e^{-\lambda^2 t} \cos \lambda x \, d\lambda$$

is the required temperature.

Step 6 : Put t = 0

$$\bar{\theta}_c(\lambda, 0) = -K^2 \mu + C$$

Step 7 : Put t = 0 in (1)

$$\bar{\theta}_c(\lambda, 0) = \int_0^{\infty} \theta(x, 0) \cos \lambda x \, dx$$

$$= 0 \quad \text{As } \theta(x, 0) = 0 \text{ (given)}$$

Thus $\quad C = K^2 \mu$

Step 8 : Subsituting in (3)

$$\bar{\theta}_c(\lambda, t) = -K^2\mu + K^2\mu \, e^{-K^2\lambda^2 t}$$

$$= K^2\mu \left(e^{-K^2\lambda^2 t} - 1\right)$$

Step 9 : Taking inverse Fourier cosine transform

$$\theta(x, t) = \frac{2}{\pi}\int_0^{\infty} \bar{\theta}_c(\lambda, t) \cos \lambda x \, d\lambda$$

Subsituting $\bar{\theta}_c(\lambda, t)$ we get

$$\theta(x, t) = \frac{2}{\pi} K^2\mu \int_0^{\infty} \left(e^{-K^2\lambda^2 t} - 1\right) \cos \lambda x \, d\lambda$$

Exercise 9.2

1) *Solve* $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$ *subject to i)* $\frac{\partial u}{\partial x} = 0$ *at* $x = 0$ *for all t ii)* $u(x, 0) = \begin{cases} x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$

[Ans. : $\frac{2}{\pi}\int_0^{\infty} \frac{\lambda \sin\lambda + \cos\lambda - 1}{\lambda^2} e^{-K\lambda^2 t} \cos\lambda x \, d\lambda$**]**

2) *The semi-infinite plate* $0 \le x \le \infty$, $-\infty \le y \le \infty$ *is insulated along the edge* $x = 0$ *and the temperature satisfies the equation* $\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{K}\frac{\partial \theta}{\partial t}$ *If at* $t = 0$, $\theta = ce^{-x}$, *prove by Fourier transform method that* $\theta(x, t) = \frac{2c}{\pi}\int_0^{\infty} \frac{1}{1+\lambda^2} e^{-K\lambda^2 t} \cos \lambda x \, d\lambda$.

Hint *: Since the plate is insulated at* $x = 0$, *take* $\frac{\partial \theta}{\partial x} = 0$ *at* $x = 0$. *Assume that* θ *and* $\frac{\partial \theta}{\partial t} \to 0$ *as* $x \to \infty$. *Hence use Fourier cosine tranform to obtain required solution.*

$$\log \bar{u}(\lambda, t) = -K\lambda^2 t + \log A$$

$$\therefore \quad \bar{u}(\lambda, t) = A\, e^{-K\lambda^2 t} \qquad \text{... (2)}$$

Step 5 : Substituting $t = 0$ we get $\bar{u}(\lambda, 0) = A$

To find A put $t = 0$ in (1)

$$\bar{u}(\lambda, 0) = \int_{-\infty}^{\infty} u(x, 0)\, e^{-i\lambda x}\, dx$$

$$= \int_{-a}^{a} u_0\, e^{-i\lambda x}\, dx$$

$$= u_0 \left[\frac{e^{-i\lambda x}}{-i\lambda}\right]_{-a}^{a}$$

$$\bar{u}(\lambda, 0) = \left[\frac{e^{i\lambda a} - e^{-i\lambda a}}{i\lambda}\right]$$

$$A = \frac{u_0}{\lambda} \cdot 2\sin a\lambda$$

Step 6 : Substituting in (2) we get

$$\bar{u}(\lambda, t) = 2u_0 \frac{\sin a\lambda}{\lambda} e^{-K\lambda^2 t}$$

Step 7 : Consider inverse Fourier transform

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{u}(\lambda, t) \cdot e^{i\lambda x}\, dx$$

$$= \frac{1}{2\pi} \cdot 2u_0 \int_{-\infty}^{\infty} \frac{\sin a\lambda}{\lambda} e^{-K\lambda^2 t} \cdot e^{i\lambda x}\, dx$$

$$= \frac{2u_0}{2\pi} \int_{-\infty}^{\infty} \frac{\sin a\lambda}{\lambda} \cdot e^{-K\lambda^2 t} \left(\cos \lambda x + i \sin \lambda x\right) d\lambda$$

$$= \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin a\lambda}{\lambda} \cdot e^{-K\lambda^2 t} \cos \lambda x\, d\lambda + \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin a\lambda}{\lambda} \cdot e^{-K\lambda^2 t}\, i \sin \lambda x\, d\lambda$$

↑ is an even function of λ ↑ is an odd function of λ

$$\therefore \qquad u(x, t) = \frac{2u_0}{\pi}\int_0^{\infty} \frac{\sin \lambda a}{\lambda} e^{-K\lambda^2 t} \cos \lambda x \, d\lambda$$

$$= \frac{u_0}{\pi}\int_0^{\infty} \frac{e^{-k\lambda^2 t}}{\lambda} [2 \sin a\lambda \cos \lambda x] \, d\lambda$$

$$= \frac{u_0}{\pi}\int_0^{\infty} \frac{e^{-k\lambda^2 t}}{\lambda} [\sin (a+x)\lambda + \sin (a-x)\lambda]$$

Example 9.10 : *Use Fourier transform to solve the boundary value problem* $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$ $-\infty < x < \infty$, $t > 0$ *subject to the conditions*

a) $u, \frac{\partial u}{\partial x} \to 0$ *as* $x \to \pm\infty$

b) $u(x, 0) = f(x)$

Solution : Step 1 : As the interval is $-\infty < x < \infty$ we use infinite transform.

Let $$\bar{u}(\lambda, t) = \int_{-\infty}^{\infty} u(x, t) e^{-i\lambda x} \, dx \qquad \dots (1)$$

Step 2 : Taking Fourier transform of both sides of $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$ we get

$$F\left[\frac{\partial u}{\partial t}\right] = K F\left[\frac{\partial^2 u}{\partial x^2}\right] \qquad \dots (2)$$

Step 3 : Using the results of Fourier transforms of derivatives

$$F\left[\frac{\partial u}{\partial t}\right] = \frac{d}{dt}\bar{u}(\lambda, t)$$

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = -\lambda^2 \bar{u}(l, t)$$

Thus from (2) we get

$$\frac{d}{dt}\bar{u}(\lambda, t) = K[-\lambda^2 \bar{u}(\lambda, t)]$$

Step 4 :

$$\therefore \qquad \frac{\frac{d}{dt}\bar{u}(\lambda, t)}{\bar{u}(\lambda, t)} = -K\lambda^2$$

Integrating w.r.t t

$$\log \bar{u}(\lambda, t) = -K\lambda^2 t + \log A$$

$$\therefore \quad \bar{u}(\lambda, t) = A\,e^{-K\lambda^2 t} \qquad \ldots (3)$$

Step 5 : Subsituting t = 0 we get $\bar{u}(\lambda, 0) = A$

To find A put t = 0 in (1)

$$\bar{u}(\lambda, 0) = \int_{-\infty}^{\infty} u(x, 0)\,e^{-i\lambda x}\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\,e^{-i\lambda x}\,dx$$

$$= F(\lambda) \quad \text{(say)} \qquad \ldots (4)$$

Step 6 : Substituting in (3) we get

$$\bar{u}(\lambda, t) = F(\lambda)\,e^{-K\lambda^2 t}$$

Step 7 : Consider inverse Fourier transform

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{u}(\lambda, t)\,e^{i\lambda x}\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[F(\lambda)\,e^{-K\lambda^2 t}\right] e^{i\lambda x}\,dx$$

Step 8 : Substituting $F(\lambda) = \int_{-\infty}^{\infty} f(u)\,e^{-i\lambda u}\,du$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\,e^{-i\lambda u}\,du\right] e^{-K\lambda^2 t}\cdot e^{i\lambda x}\,d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\,e^{-K\lambda^2 t}\,e^{i\lambda(x-u)}\,du\,d\lambda$$

which is the required solution.

Exercise 9.3

1. *Initial temperature along the length of an infinite bar is given as* $f(x) = \begin{cases} 2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$. *If temperature satisfies the equation* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.

[Ans. : $\frac{2}{\pi}\int_{0}^{\infty} e^{-\lambda^2 t}\left[\frac{\sin(1+x)\lambda + \sin(1-x)\lambda}{\lambda}\right] d\lambda$**]**

2. *Use Fourier transforms to solve the boundary value problem,* $\frac{\partial u}{\partial t} = \frac{K\partial^2 u}{\partial x^2}$ $-\infty < x < \infty,\ t > 0$

Subject to i) $u, \frac{\partial u}{\partial x} \to 0$ *as* $x \to \pm\infty$ *ii)* $u(x, 0) = f(x)$ **[Ans. :** $u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{-k^2 t} e^{i\lambda x} d\lambda$**]**

3. *Solve the equation* $\frac{\partial\theta}{\partial t} = c^2 \frac{\partial^2\theta}{\partial x^2}$ $-\infty < x < \infty$ $t > 0$ *subject to the condition* $\theta(x,0) = f(x)$ *where*

[Ans. : $\theta(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-c^2\lambda^2 t} e^{i\lambda(x-u)} du\, d\lambda$**]**

4. *If the initial temperature of an infinite bar is given by* $\theta(x,0) = \begin{cases} \theta_0 & |x| < a \\ 0 & |x| > a \end{cases}$ *determine the temperature out any point x and at any time t.* **[Ans. :** $\theta(x,t) = \frac{2\theta_0}{\pi}\int_0^{\infty} \frac{\sin a\lambda}{\lambda} e^{-c^2\lambda^2 t}\cos\lambda x\, d\lambda$**]**

5. *Solve the equation* $\frac{\partial\theta}{\partial t} = c^2\frac{\partial^2\theta}{\partial x^2}$ $-\infty < x < \infty,$ $t > 0$ *subject to* $\theta(x,0) = f(x)$ *where*

$f(x) = \begin{cases} u_0 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$ **[Ans. :** $\theta(x,t) = \frac{u_0}{\pi}\int_0^{\infty} e^{-c^2\lambda^2 t}\left[\frac{\sin(1+x)\lambda + \sin(1-x)\lambda}{\lambda}\right] d\lambda$**]**

9.7 Problems on Finite Fourier Transforms

If the interval is finite interval $0 < x < L$ then depending on the boundary conditions we use finite Fourier sine or cosine transforms.

a) Finite fourier sine transform

$$\bar{u}_s(n,t) = F_s[u] = \int_0^L u(x,t)\sin\left(\frac{n\pi x}{L}\right)dx$$

and the inverse sine transform is

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} \bar{u}_s(n,t)\sin\frac{n\pi x}{L}$$

b) Finite fourier cosine transform

$$\bar{u}_c(n,t) = F_c[u] = \int_0^L u(x,t)\cos\frac{n\pi x}{L}dx$$

and the inverse cosine transform is

$$u(x,t) = \frac{1}{L}\bar{u}_c(0,t) + \frac{2}{L}\sum_{n=1}^{\infty}\bar{u}_c(n,t)\cos\frac{n\pi x}{L}$$

9.8 Choice of Finite Fourier Sine or Cosine Transform

If the interval is $0 < x < L$ and

i) If boundary conditions are $u(0, t) = u(L, t) = 0$ for all t, we use finite Fourier sine transform.

ii) If boundary conditions are $u_x(0, t) = u_x(L, t) = 0$ for all t, we use finite Fourier cosine transform.

Interval	Boundary conditions	Type of F.T.
$0<x<L$	$u(0, t) = u(L, t) = 0$	Finite Fourier sine transform
$0<x<L$	$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0$	Finite Fourier cosine transform

9.9 Type IV : Problems on Finite Fourier Sine Transform

Example 9.11 : *Use finite Fourier transform to solve* $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$ *for* $0 < x < L,\ t > 0$ *subject to*

i) $u(0, t) = u(L, t) = 0 \qquad 0 < x < L, \quad t > 0$

ii) $u(x, 0) = \frac{u_0\, x}{L}$ *for* $0 < x < L$

Solution : Step 1 : As $u(0, t) = u(L, t) = 0$

$\therefore$ We use finite Fourier sine transform

Let $$\bar{u}_s(n, t) = F_S[u] = \int_0^L u(x, t)\sin\left(\frac{n\pi x}{L}\right)dx \qquad \text{... (i)}$$

Step 2 : Taking finite fourier sine transform of

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \text{ we get}$$

$$\int_0^L \frac{\partial u}{\partial t}\sin\left(\frac{n\pi x}{L}\right)dx = K\int_0^L \frac{\partial^2 u}{\partial x^2}\sin\left(\frac{npx}{L}\right)dx$$

Step 3 : Use D.U.I.S on L.H.S i.e. the partial derivative inside the integral becomes total derivative outside the integral. Also apply integration by parts rule on R.H.S.

$$\frac{d}{dt}\int_0^L u\sin\left(\frac{n\pi x}{L}\right)dx = K\left\{\left(\frac{\partial u}{\partial x}\sin\frac{n\pi x}{L}\right)_0^L - \int_0^L \frac{\partial u}{\partial x}\cdot\frac{n\pi}{L}\cdot\cos\frac{n\pi x}{L}\,dx\right\}$$

$$\frac{d}{dt}\bar{u}_s(n, t) = K\left\{(0-0) - \frac{n\pi}{L}\int_0^L \frac{\partial u}{\partial x}\cos\frac{n\pi x}{L}\,dx\right\}$$

$$= (-K)\cdot\frac{n\pi}{L}\int_0^L \frac{\partial u}{\partial x}\cos\frac{n\pi x}{L}\,dx$$

Apply integration by parts again

$$= (-K)\frac{n\pi}{L}\left\{\left(u\cdot\cos\frac{n\pi x}{L}\right)_0^L - \int_0^L u\cdot\left(-\frac{n\pi}{L}\right)\sin\frac{n\pi x}{L}\,dx\right\}$$

$$= (-K)\cdot\frac{n\pi}{L}\left\{(0-0)+\frac{n\pi}{L}\cdot\int_0^L u\sin\frac{n\pi x}{L}\,dx\right\}$$

$$= -K\left(\frac{n\pi}{L}\right)^2\cdot\bar{u}_s(n,t) \qquad \text{... using (i)}$$

Step 4 : $$\frac{\frac{d}{dt}\bar{u}_s(n,t)}{\bar{u}_s(n,t)} = -K\left(\frac{n\pi}{L}\right)^2$$

Integrate w.r.t. t

$$\log\bar{u}_s(n,t) = -K\left(\frac{n\pi}{L}\right)^2\cdot t+\log A$$

$$\therefore \quad \bar{u}_s(n,t) = A\,e^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad \text{... (ii)}$$

Step 5 : Put $t = 0$

$$\bar{u}_s(n,0) = A$$

Step 6 : put $t = 0$ in (i)

$$\bar{u}_s(n,0) = \int_0^L u(x,0)\sin\frac{n\pi x}{L}\,dx$$

$$= \int_0^L \frac{u_0\,x}{L}\sin\frac{n\pi x}{L}\,dx \qquad \left\{\text{As given } u(x,0)=\frac{u_0\,x}{L}\right.$$

Integrating by parts we get

$$A = \frac{u_0}{L}\left\{x\left(\frac{-\cos\frac{n\pi x}{L}}{\left(\frac{n\pi}{L}\right)}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{L}}{\left(\frac{n\pi}{L}\right)^2}\right)\right\}_0^L$$

$$A = \frac{u_0}{L}\left\{\left(-\frac{L^2}{n\pi}\cos n\pi-0\right)+(0-0)\right\}$$

$$A = -\frac{u_0\,L}{n\pi}(-1)^n$$

$$A = \frac{u_0 L}{n\pi}(-1)^{n+1}$$

Step 7 : Substituting in (ii) we get

$$\bar{u}_s(n, t) = \frac{u_0 L}{n\pi}(-1)^{n+1} \cdot e^{-K\left(\frac{n\pi}{L}\right)^2 t} \quad \text{... (iii)}$$

Step 8 : Take inverse finite sine transform

$$u(x, t) = \frac{2}{L}\sum_{n=1}^{\infty} \bar{u}_s(n, t) \sin\frac{n\pi x}{L}$$

$$u(x, t) = \frac{2}{L} \cdot \frac{u_0 L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-K\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L}$$

is the required solution.

Example 9.12 : *Solve the equation* $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}$ $0 < x < \pi$, $t > 0$ *subject to the conditions*

i) $u(x, 0) = 1 \quad 0 < x < \pi$

ii) $u(0, t) = u(\pi, t) = 0, \quad 0 < x < \pi, \quad t > 0$

Solution : Step 1 : As $u(0, t) = u(\pi, t) = 0$ $\therefore$ We use finite Fourier sine transform.

Let $$\bar{u}_s(n, t) = F_S(u) = \int_0^L u(x, t) \sin\frac{n\pi x}{L} dx$$

As $$L = \pi$$

$\therefore$ $$\bar{u}_s(n, t) = F_S(u) = \int_0^{\pi} u(x, t) \sin nx \, dx \quad \text{... (i)}$$

Step 2 : Taking finite Fourier sine transform of $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ in $0 < x < \pi$ we get

$$\int_0^{\pi} \frac{\partial u}{\partial t} \sin nx \, dx = \int_0^{\pi} \frac{\partial^2 u}{\partial x^2} \sin nx \, dx$$

Step 3 : Use D.U.I.S on L.H.S and integration by parts on R.H.S.

$$\frac{d}{dt}\int_0^{\pi} u \sin nx \, dx = \left\{\left(\frac{\partial u}{\partial x} \sin nx\right)_0^{\pi} - \int_0^{\pi} \frac{\partial u}{\partial x} \cdot n \cos nx \, dx\right\}$$

$$\frac{d}{dt}\bar{u}_s(n, t) = \left\{(0-0) - n\int_0^{\pi} \frac{\partial u}{\partial x} \cos nx \, dx\right\}$$

Apply integration by parts again

$$\frac{d}{dt}\,\overline{u}_s(n,t) = (-n)\left\{(u\cdot\cos nx)_0^{\pi} - \int_0^{\pi} u\,(-n)\sin nx\,dx\right\}$$

$$= (-n)\{(0-0)+n\,\overline{u}_s(n,t)\}$$

$$= -n^2\cdot\overline{u}_s(n,\ t) \qquad \text{... using (i)}$$

Step 4 : $$\frac{\frac{d}{dt}\overline{u}_s(n,t)}{\overline{u}_s(n,t)} = -n^2$$

Integrate w.r.t. t

$$\log\ \overline{u}_s(n,t) = -n^2\ t+\log A$$

$$\therefore \qquad \overline{u}_s(n,t) = A\,e^{-n^2 t} \qquad \text{... (ii)}$$

Step 5 : Put $t = 0$

$$\overline{u}_s(n,\ 0) = A$$

Step 6 : Put $t = 0$ in (i)

$$\overline{u}_s(n,0) = \int_0^{\pi} u(x,0)\sin nx\,dx$$

$$A = \int_0^{\pi} (1)\sin nx\,dx$$

$$= \left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= -\frac{1}{n}\,[\cos n\pi-1]$$

$$A = \frac{1}{n}\left[1-(-1)^n\right]$$

Step 7 : Substituting in (ii) we get

$$\overline{u}_s(n,t) = \frac{\left[1-(-1)^n\right]}{n}\,e^{-n^2 t} \qquad \text{... (iii)}$$

Step 8 : Take inverse finite Fourier sine transform

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\overline{u}_s(n,t)\sin\frac{n\pi x}{L}$$

$$L = \pi$$

$$\therefore \qquad u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n}\, e^{-n^2 t}\cdot \sin nx$$

is the required solution.

Exercise 9.4 : Problems on Finite Fourier Sine Transform

1) $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$ $0 < x < 4,\ t > 0$ *subject to the conditions :*

   a) $u(x, 0) = 2x, \quad 0 < x < 4$

   b) $u(0, t) = u(4, t) = 0, \quad 0 < x < 4, \quad t > 0$

   [Ans. : $\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{kn^2\pi^2 t}{16}}\sin\frac{n\pi x}{4}$]

2) *Solve* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $0 < x < \pi, \quad t > 0$ *subject to the conditions*

   a) $u(x, 0) = 1 \qquad 0 < x < \pi$

   b) $u(0, t) = u(\pi, t) = 0 \qquad 0 < x < \pi, \quad t > 0$ *using the appropriate transform.*

   [Ans. : $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n}e^{-n^2 t}\sin nx$]

3) *Solve by using finite Fourier transform,* $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$, $0 \le x \le 6,\ t > 0$ *subject to the conditions*

   $V(0, t) = V(6, t) = 0,\ V(x, 0) = \begin{cases} 1 & 0 \le x \le 3 \\ 3 & 3 \le x \le 6 \end{cases}$

   [Ans. : $V(x, t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(1-\cos\frac{n\pi}{2}\right)e^{-\frac{n^2\pi^2 t}{36}}\sin\frac{n\pi x}{6}$]

4) *Using finite Fourier transform, find the solution of the wave equation* $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$ *subject to*

   i) $u(0,t) = u(\pi,t) = 0$

   ii) $u_t(0,t) = 0$ *for* $0 \le x \le \pi,\ t > 0$

   iii) $u(x,0) = 3\sin x + 4\sin 4x$

   [Ans. : $u(x, t) = 3\cos at \sin x + 4\cos 4at \sin 4x$]

9.10 Type V : Problems on Finite Fourier Cosine Transform

➡ Example 9.13 : *Solve the equation* $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$ *subject to the conditions.*

i) u *is not infinite when* $t \to \infty$

ii) $\frac{\partial u}{\partial x} = 0$ *for* $x = 0$ *and* $x = l$

iii) $u = lx - x^2$ *for* $t = 0$ *between* $x = 0$ *and* $x = l$.

Solution : Step 1 : As $\frac{\partial u}{\partial x} = 0$ we use finite Fourier cosine transform.

$$\text{Let} \qquad \bar{u}_c(n, t) = \int_0^l u(x, t)\cos\frac{n\pi x}{l}\,dx \qquad \ldots (1)$$

Step 6 : To find $\bar{u}_c(n, 0)$ Put t = 0 in (1)

$$\bar{u}_c(n, 0) = \int_0^l u(x, 0) \cos \frac{n\pi x}{l} dx$$

Substituting the value of $u(x, 0)$

$$\therefore \quad A = \int_0^l (lx - x^2) \cos \frac{n\pi x}{l} dx$$

Use generalised rule of integration by parts

$$= \left[(lx - x^2) \left(\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} \right) - (l - 2x) \left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} \right) + (-2) \left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3} \right) \right]_0^l$$

$$= \left[\left(0 - \frac{l(-1)^n}{\left(\frac{n\pi}{l}\right)^2} - 0 \right) - \left(0 + \frac{l}{\left(\frac{n\pi}{l}\right)^2} \right) + 0 \right]$$

$$\therefore \quad \bar{u}_c(n, 0) = \frac{l^2}{n^2\pi^2} \cdot l \, [-1 - (-1)^n]$$

$$A = \frac{-l^3}{n^2\pi^2} [1 + (-1)^n]$$

Step 7 : For $\quad n = 0$

$$\bar{u}(0, 0) = \int_0^l u(x, 0) \, dx$$

$$= \int_0^l u(lx - x^2) dx = \frac{l^3}{6}$$

Step 8 : Substituting in (3)

$$\bar{u}(n, t) = \frac{-l^3}{n^2\pi^2} \left[1 + (-1)^n \right] e^{-k \frac{n^2\pi^2 t}{l^2}}$$

and $\quad \bar{u}_c(0, t) = \frac{l^3}{6}$

Let $$\bar{y}(n,t) = \int_0^l y(x,t) \sin \frac{n\pi x}{l}\, dx \qquad \text{... (i)}$$

Step 2 :

Consider $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{... (ii)}$$

Taking finite Fourier sine transform of both sides

$$\int_0^l \frac{\partial^2 y}{\partial t^2} \sin \frac{n\pi x}{l}\, dx = c^2 \int_0^l \frac{\partial^2 y}{\partial x^2} \sin \frac{n\pi x}{l}\, dx$$

Step 3 : Using D.U.I.S on L.H.S and integration by parts twice on R.H.S.

$$\frac{d^2}{d t^2} \bar{y}(n,t) = c^2 \left\{ \left(\sin \frac{n\pi x}{l} \cdot \frac{\partial y}{\partial x} \right)_0^l - \int_0^l \frac{n\pi}{l} \cos \frac{n\pi x}{l} \cdot \frac{\partial y}{\partial x}\, dx \right\}$$

$$= c^2 \left\{ 0 - \frac{n\pi}{l} \int_0^l \cos \frac{n\pi x}{l} \cdot \frac{\partial y}{\partial x}\, dx \right\}$$

$$= c^2 \left(\frac{-n\pi}{l} \right) \cdot \left\{ \left(\cos \frac{n\pi x}{l}\, y \right)_0^l - \int_0^l \frac{-n\pi}{l} \sin \frac{n\pi x}{l} \cdot y\, dx \right\}$$

$$= c^2 \left(\frac{-n\pi}{l} \right) \left[0 + \left(\frac{n\pi}{l} \right) \cdot \bar{y}(n,t) \right]$$

$$\frac{d^2}{dt^2} \bar{y}(n,t) = -c^2 \left(\frac{n\pi}{l} \right)^2 \cdot \bar{y}(n,t)$$

$$\frac{d^2}{dt^2} \bar{y}(n,t) = -c^2 \left(\frac{n\pi}{l} \right)^2 \cdot \bar{y}(n,t)$$

If $(D^2 + p^2)\, y = 0 \;\Rightarrow\; y = c_1 \cos p\,t + c_2 \sin p\,t$

Step 4 :

$\therefore$ The solution of the above differential equation is

$$\bar{y}(n,t) = c_1 \cos \frac{c\,n\pi\,t}{l} + c_2 \sin \frac{c\,n\pi\,t}{l} \qquad \text{... (iii)}$$

Step 5 :

Now given $y(x,0) = lx - x^2$

Taking finite Fourier sine transform of both sides

$$= F_s(f(n)) \quad \text{(say)} \qquad \dots \text{(iv)}$$

Step 6 : Also given $\frac{\partial y}{\partial t}(x, 0) = g(x)$

Taking finite Fourier sine transforms we get

$$\frac{\partial \bar{y}}{\partial t}(n, 0) = \int_0^{\pi} g(x) \sin nx \, dx$$

$$= F_s(g(n)) \quad \text{(say)} \qquad \dots \text{(v)}$$

Step 7 : Differentiating (iii) w.r.t. t we get

$$\frac{\partial \bar{y}}{\partial t}(n, t) = cn[-c_1 \sin cnt + c_2 \cos cnt] \qquad \dots \text{(vi)}$$

Step 8 : Substituting t = 0 in (iii)

$$\bar{y}(n, 0) = c_1$$

$\therefore$ from (iv)

$$c_1 = F_s[f(n)]$$

Step 9 : put t = 0 in (vi)

$$\frac{\partial \bar{y}}{\partial t}(n, 0) = cn[0 + c_2]$$

$\therefore$ from (v)

$$c_2 \cdot cn = F_s(g(n))$$

$$c_2 = \frac{1}{cn} F_s(g(n))$$

Step 10 : Substituting c_1, c_2 in (iii)

$$\bar{y}(n, t) = F_s(f(n)) \cos cnt + \frac{1}{cn} \cdot F_s(g(n)) \sin cnt$$

Step 11 : Taking inverse finite Fourier sine transform we get

$$y(x, t) = \sum_{n=1}^{\infty} \bar{y}(n, t) \sin nx$$

$$= \sum_{n=1}^{\infty} \left[F_s(f(n)) \cos cnt + \frac{1}{cn} F_s(g(n)) \sin cnt \right] \sin nx$$

which is the required solution.

$$V(x,y) = \int_0^\infty A(\lambda)\cos\lambda x \ \cosh\lambda y \ d\lambda \qquad \dots (5)$$

Step 6 : To find $A(\lambda)$ put $y = 1$

$$\therefore \quad V(x,1) = \int_0^\infty A(\lambda)\cos\lambda x \cdot \cosh\lambda \, d\lambda$$

$$f(x) = \int_0^\infty [A(\lambda)\cdot\cosh\lambda]\cos\lambda x \ d\lambda$$

Step 7 : Here $A(\lambda)\cosh\lambda$ is the cosine transform of f(x).

$$\therefore \quad A(\lambda)\cosh\lambda = \frac{2}{\pi}\int_0^\infty f(x)\cos\lambda x \cdot dx.$$

Step 8 : As $f(x) = e^{-x} \quad x > 0$

$$\therefore \quad A(\lambda)\cosh\lambda = \frac{2}{\pi}\int_0^\infty e^{-x}\cos\lambda x \ dx$$

$$= \frac{2}{\pi}\left[\frac{e^{-x}}{1+\lambda^2}(-\cos\lambda x + \lambda\sin\lambda x)\right]_0^\infty$$

$$= \frac{2}{\pi}\left[0 - \frac{-1}{1+\lambda^2}\right]$$

$$= \frac{2}{\pi}\frac{1}{1+\lambda^2}$$

$$\therefore \quad A(\lambda) = \frac{2}{\pi}\frac{1}{1+\lambda^2}\frac{1}{\cosh\lambda}$$

Step 9 : Substituting $A(\lambda)$ in (5) we get

$$V(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{(1+\lambda^2)\cosh\lambda}\cosh\lambda y \ \cos\lambda x \, d\lambda$$

which is the required solution.

Example 9.19 : *Show that solution of Laplace's equation $\nabla^2 v = 0$ for the semi infinite strip $x > 0$ and $0 < y < b$ such that $v = f(x)$ when $y = 0$, $0 < x < \infty$ and $V = 0$ when $y = b$, $0 < x < \infty$ and $v = 0$ when $x = 0$, $0 < y < b$ is given by*

$$V(x,y) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(u)\frac{\sinh(b-y)\lambda}{\sinh b\lambda}\sin\lambda x \ \sin\lambda u \, du \, d\lambda$$

Solution : **Step 1 :** To solve

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \text{subject to} \qquad \dots (1)$$

i) $\quad V(0,y) = 0$

$\therefore$
$$f(x) = \int_0^\infty A(\lambda)\ \sin\lambda x\, d\lambda$$

which indicates that $A(\lambda)$ is Fourier sine transform of f(x)

$\therefore$
$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(u)\sin\lambda u\, du$$

Step 7 : Substituting $A(\lambda)$ in (4) we get

$$V(x,y) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(u)\,\frac{\sin h(b-y)\lambda}{\sin h\, b\lambda}\cdot \sin\lambda y\ \sin\lambda x\ du\, d\lambda$$

which is the required solution.

Exercise 9.7 : Problems on Laplace Equation

1) *Show that solution of Laplace's equation* $\nabla^2 V = 0$ *for V inside the semi-infinite strip* $x > 0$, $0 < y < b$ *such that* $V = f(x)$ *when* $y = 0$, $0 < x < \infty$; $V = 0$ *when* $y = b$, $0 < x < \infty$ *and* $V = 0$ *when* $x = 0$, $0 < y < b$ *is given by* $V = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(u)\frac{\sin h(b-y)\lambda}{\sin h b\lambda}\sin\lambda x \sin\lambda u\, du\, d\lambda$

2) *Find the bounded harmonic function V (x, y) in the semi-infinite strip* $x > 0$, $0 < y < 1$ *that satisfies the boundary conditions* $V_x(0,y) = 0$, $V_y = (x,0) = 0$, $V(x,1) = f(x)$ *where*

i) $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$ *ii)* $f(x) = e^{-x}\ x > 0$

[Ans. : $\frac{2}{\pi}\int_0^\infty \frac{1}{(1+\lambda^2)\cosh\lambda}\cosh\lambda y\ \cos\lambda x\ d\lambda$**]**

3) *Find the bounded harmonic function V (x, y) in the semi-infintie strip* $y > 0$, $0 > x > 1$, *which satisfies the conditions.* $V_y(x,0) = 0$, $V(0,y) = e^{-y}$ *and* $V(1,y) = 0$

[Ans. : $V(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda^2}\ \frac{\sinh\lambda(1-x)}{\sinh\lambda}\cos\lambda y\ d\lambda$**]**

4) *Find the bounded harmonic function V (x, y) in the semi-infinite strip* $y > 0$, $0 < x < 1$ *which satisfies the conditions* $V_y(x,0) = 0$, $V(0,y) = 0$ *and* $V_x(1,y) = f(y)$.

[Ans. : $V(x,y) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{f(u)\cos\lambda u}{\lambda\cosh\lambda}\cdot\sinh\lambda x\ \cos\lambda y\ du\ d\lambda$**]**

University Questions

Dec. - 98

1. *Use Fourier transform to solve*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\quad 0 < x < \infty,\quad t > 0$$

where $u(x,t)$ *satisfied the conditions :*

124.5 lies before 127

Thus 50-60 in the median class

$\therefore l = 50$, h = 10, f = 33, c = 94

$$\text{Median} = l + \frac{h\left(\frac{N}{2} - c\right)}{f}$$

$$= 50 + \frac{10(124.5 - 94)}{33}$$

$$= 59.24$$

c) Mode

It is the value of the variate which occurs most frequently in a set of observations, or is the value of variate corresponding to maximum frequency.

$$\text{Mode} = l + \frac{h\,(f_1 - f_0)}{2\,f_1 - f_0 - f_2}$$

l = lower limit of modal class.

h = width of the modal class.

f_1 = frequency of the modal class.

f_0 = frequency of the class preceding the modal class.

f_2 = frequency of the class succeeding the modal class.

Modal class is the class with highest frequency.

Example 10.3 : *Find Mean, mode, median for the following distribution.*

CI	*0 - 10*	*10 - 20*	*20 - 30*	*30 - 40*	*40 - 50*	*50 - 60*	*60 - 70*
f	*4*	*7*	*8*	*12*	*25*	*18*	*10*

Solution :

CI	x_i Midvalues	f_i	$u_i = \frac{x_i - 35}{10}$	CF	$f_i\,u_i$
0 - 10	5	4	– 3	4	– 12
10 - 20	15	7	– 2	11	– 14
20 - 30	25	8	– 1	19	– 8

30 - 40	35	12	0	31	0
40 - 50	45	25	1	56	25
50 - 60	55	18	2	74	36
60 - 70	65	10	3	84	30
Total		**84**			**57**

a) $$\text{Mean} = A + h\left(\frac{\sum f_i u_i}{\sum f_i}\right) = 35 + 10\left(\frac{57}{84}\right) = 41.785$$

b) Median class is the class in which

$$N/2 = 84/2 = 42$$

$\therefore$ lies 40 - 50 is median class

$$l = 40 \quad h = 10 \quad f = 25 \quad c = 31 \quad N = 84$$

$$\text{Median} = l + \frac{h\left(\frac{N}{2} - c\right)}{f}$$

$$= 40 + \frac{10\left(\frac{84}{2} - 31\right)}{25}$$

$$= 44.44$$

c) Modal class is the class with highest frequency i.e. 25

$\therefore$ Modal class = 40 – 50

$l = 40 \quad h = 10 \quad f_1 = 25 \quad f_0 = 12 \quad f_2 = 18$

$$\text{Mode} = l + \frac{h(f_1 - f_0)}{2 f_1 - f_0 - f_2}$$

$$= 40 + \frac{10\,(25-12)}{50-12-18}$$

$$= 46.5$$

d) Geometric Mean

i) For ungrouped data : Geometric mean or G.M. of n observations $x_1, x_2 \ldots x_n$ $(x_i \neq 0)$ is the n^{th} root of their product.

i.e. $$\text{G.M.} = (x_1.x_2 \ldots x_n)^{1/n}$$

Take log on both sides

Simplifying we get

$$\sigma = \sqrt{\frac{\sum f_i x_i^2}{N} - \left(\frac{\sum f_i x_i}{N}\right)^2} \qquad \text{where } \sum f_i = N$$

i.e. $$\sigma = \sqrt{\frac{\sum f_i x_i^2}{\sum f_i} - (\bar{x})^2}$$

If we use method of step deviation for simplification of our calculations

i.e. if $u_i = \dfrac{x_i - A}{h}$ then

i) For ungrouped data

$$\sigma_u = \sqrt{\frac{\sum u_i^2}{n} - \left(\frac{\sum u_i}{n}\right)^2}$$

$$\sigma_x = h\,\sigma_u$$

ii) For grouped data

$$\sigma_u = \sqrt{\frac{\sum f_i u_i^2}{\sum f_i} - \left(\frac{\sum f_i \cdot u_i}{\sum f_i}\right)^2}$$

$$\sigma_x = h\,\sigma_u$$

$$\bar{x} = A + h\,\bar{u}$$

Note :

1) The square of standard deviation is called variance given by σ^2.

2) The coefficient of variation is given by

$$\text{C.V.} = \frac{\sigma}{\text{A.M.}} \times 100$$

For comparing the variability of two series, we calculate the coefficient of variations for each series. The series having lesser C.V. is said to be more consistent.

Example 10.4 : *Goals scored by two teams A and B in a football season were as follows. Determine which team is more consistent.*

Number of goals scored	*Number of Matches*	
	Team A	*Team B*
0	27	17
1	9	9

2	8	6
3	5	5
4	4	3

Solution : Frequency distribution table for team A.

No. of goals (x_i)	Matches f_i	$d_i = x_i - 2$	$f_i\ d_i$	$f_i\ d_i^2$
0	27	− 2	− 54	108
1	9	− 1	− 9	9
2	8	0	0	0
3	5	1	5	5
4	4	2	6	12
	53		− 50	138

Thus for team A

$$\bar{x} = A + \left(\frac{\sum f_i\ d_i}{\sum f_i}\right)$$

$$= 2 + \frac{-50}{53} = 1.06$$

$$\sigma_A = \sqrt{\frac{\sum f_i\ d_i^2}{\sum f_i} - \left(\frac{\sum f_i\ d_i}{\sum f_i}\right)^2}$$

$$= \sqrt{\frac{138}{53} - \left(\frac{-50}{53}\right)^2}$$

$$= 1.31$$

$$\text{C.V.} = \frac{\sigma_A}{\bar{x}} \times 100$$

$$= \frac{1.31}{1.06} \times 100 = 123.6$$

Frequency distribution table for team B

No. of goals (x_i)	Matches f_i	$d_i = x_i - 2$	$f_i\ d_i$	$f_i\ d_i^2$
0	17	− 2	− 34	68
1	9	− 1	− 9	9

840	2	10	20	200
842	2	12	24	288
844	1	14	14	196
	$\sum f = 15$		$\sum f_i d_i = 18$	$\sum f_i d_i^2 = 1412$

$$\text{A.M.} = 830 + \frac{18}{15} = 831.2$$

$$\sigma = \sqrt{\frac{1412}{15} - \left(\frac{18}{15}\right)^2}$$

$$= \sqrt{94.133 - 1.44}$$

$$= 9.628$$

$$\text{coefficient of variation} = \frac{9.628}{831.2} \times 100 = 1.158$$

Coefficient of variation of group A is greater than that of group B.

$\therefore$ group A has greater variability or group B is more consistent.

10.5 Moments

i) **Moments about mean** : The arithmetic mean of various powers of the deviation $(x_i - \bar{x})$ is called moment of the distribution and is denoted by u_i.

Thus for ungrouped data

$$\mu_1 = \frac{\sum (x_i - \bar{x})}{n}$$

$$\mu_2 = \frac{\sum (x_i - \bar{x})^2}{n}$$

$$\mu_3 = \frac{\sum (x_i - \bar{x})^3}{n}$$

$$\vdots$$

$$\mu_r = \frac{\sum (x_i - \bar{x})^r}{n}$$

Also for grouped data

$$\mu_1 = \frac{\sum f_i (x_i - \bar{x})}{N} \qquad \text{where } N = \sum f_i$$

$$\mu_2 = \frac{\sum f_i (x_i - \bar{x})^2}{N}$$

$$\vdots$$

$$\mu_r = \frac{\sum f_i (x_i - \bar{x})^r}{N}$$

μ_r is the r^{th} moment about the mean of a distribution.

Note : Subsituting r = 0, 1, 2, we get

i) $r = 0 \Rightarrow \quad \mu_0 = \dfrac{\sum f_i}{N} = \dfrac{N}{N} = 1$

ii) $r = 1 \Rightarrow \quad \mu_1 = \dfrac{\sum f_i (x_i - \bar{x})}{N}$

$$= \frac{\sum f_i x_i}{N} - \left(\frac{\sum f_i}{N}\right)\bar{x}$$

$$= \bar{x} - \frac{N}{N} \cdot \bar{x}$$

$$= \bar{x} - \bar{x}$$

$$= 0$$

iii) $r = 2 \Rightarrow \quad \mu_2 = \dfrac{\sum f_i (x_i - \bar{x})^2}{N}$

which gives the variance σ^2 of the distribution.

Thus

$$\boxed{\begin{aligned} \mu_1 &= 0 \\ \mu_2 &= \frac{\sum f_i (x_i - \bar{x})^2}{N} \\ \mu_3 &= \frac{\sum f_i (x_i - \bar{x})^3}{N} \\ \mu_4 &= \frac{\sum f_i (x_i - \bar{x})^4}{N} \end{aligned}}$$

are the first four moments of the distribution about mean.

ii) Moments about any number A : The r^{th} moment about any number A is denoted by μ'_r and is given by

$$\mu'_r = \frac{\sum f_i (x_i - A)^r}{N}$$

Substituting r = 0, 1, 2, ... we get

$$\mu'_0 = 1$$

$$\mu'_1 = \frac{\sum f_i (x_i - A)}{N}$$

$$= \frac{\sum f_i x_i}{N} - \left(\frac{\sum f_i}{N}\right) A$$

$$= \bar{x} - A$$

$$\mu'_2 = \frac{\sum f_i (x_i - A)^2}{N}$$

$$= s^2 = \text{mean square deviation}$$

$$\mu'_3 = \frac{\sum f_i (x_i - A)^3}{N}$$

and $$\mu'_4 = \frac{\sum f_i (x_i - A)^4}{N}$$

Note : Proper choice of A can reduce that calculations of calculating μ'_r than that of μ_r.

Relations between μ'_r and μ_r :

By definition

$$\mu'_r = \frac{\sum f_i (x_i - A)^r}{N}$$

Let $$d_i = x_i - A$$

$\therefore$ $$\mu'_r = \frac{\sum f_i (d_i)^r}{N}$$

Also $$\mu_r = \frac{\sum f_i (x_i - \bar{x})^r}{N}$$

$$= \frac{1}{N} \sum f_i (x_i - A + A - \bar{x})^r$$

$$= \frac{1}{N} \sum f_i (d_i - \mu'_1)^r \qquad \{\text{As } \mu'_1 = \bar{x} - A\}$$

using binomial expansion

$$= \frac{1}{N}\sum f_i\left\{(d_i)^r - {}^rC_1(d_i)^{r-1}(\mu_1') + {}^rC_2(d_i)^{r-2}(\mu_1')^2 \ldots + (-1)^r(\mu_1')^r\right\}$$

As $\mu_r' = \frac{\sum f_i (d_i)^r}{N}$ we get

$$\mu_r = \mu_r' - {}^rC_1\, \mu_{r-1}'(\mu_1') + {}^rC_2\, \mu_{r-2}'(\mu_1')^2 \ldots + (-1)^r(\mu_1')^r$$

Substituting r = 2, 3, 4 we get

$$\mu_2 = \mu_2' - 2C_1\, \mu_1' \mu_1' + \mu_0'(\mu_1')^2$$

$$= \mu_2' - 2(\mu_1')^2 + (\mu_1')^2$$

$$= \mu_2' - (\mu_1')^2$$

$$\mu_3 = \mu_3' - 3C_1\, \mu_2' \mu_1' + 3C_2\, \mu_1'(\mu_1')^2 - (\mu_1')^3$$

$$= \mu_3' - 3\mu_2' \mu_1' + 2(\mu_1')^3$$

$$\mu_4 = \mu_4' - 4C_1\, \mu_3' \mu_1' + 4C_2\, \mu_2'(\mu_1')^2 - 4C_3\, \mu_1'(\mu_1')^3 + 4C_4\,(\mu_1')^4$$

$$= \mu_4' - 4\mu_3' \mu_1' + 6\mu_2'(\mu_1')^2 - 4(\mu_1')^4 + (\mu_1')^4$$

$$= \mu_4' - 4\mu_3' \mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

Thus

$$\boxed{\begin{aligned} \mu_0 &= 1 \\ \mu_1 &= 0 \\ \mu_2 &= \mu_2' - (\mu_1')^2 \\ \mu_3 &= \mu_3' - 3\mu_2' \mu_1' + 2(\mu_1')^3 \\ \mu_4 &= \mu_4' - 4\mu_3' \mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 \end{aligned}}$$

10.6 Sheppard's Correction for Moments

If we use $u_i = \frac{x_i - A}{h}$ then also we get the same relations. This involves some error in calculations of moments.

$\therefore$ By W.F. Sheppard the corrected formulae are

μ_2 (corrected) $= \mu_2 - \frac{h^2}{12}$ (where h = width of the interval)

The different measures of skewness are

i) skewness $= \dfrac{3\,(\text{Mean} - \text{Median})}{\text{Standard deviation}}$

ii) Coefficient of skewness $= \beta_1 - \dfrac{\mu_3^2}{\mu_2^3}$

$$= \gamma_1 = +\sqrt{\beta_1}$$

10.8 Kurtosis

If we know the measures of central tendency, dispersion and skewness, we still cannot have a complete idea about the distribution. Observe the Fig. 10.3 there are three curves $C_1 C_2 C_3$ which are symmetrical about mean and have the same range. Therefore we should know about the flatness or peakedness of the curve.

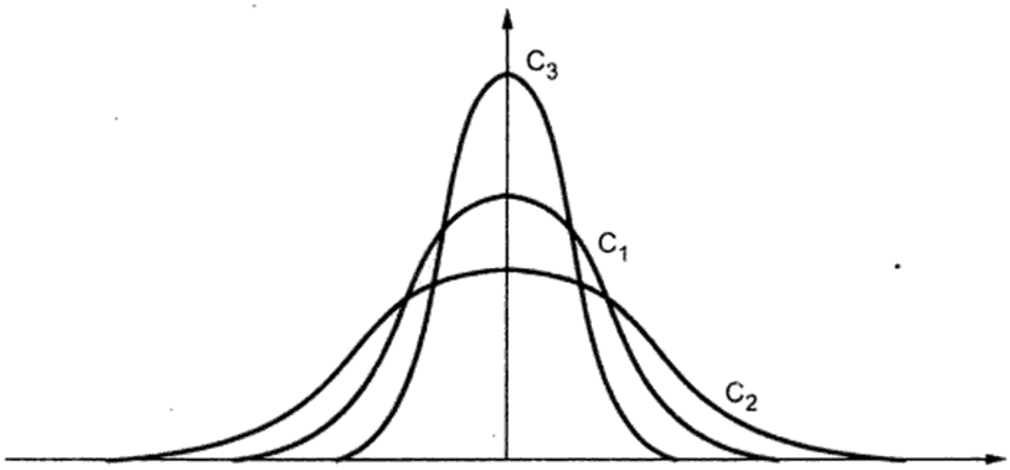


Fig. 10.3

Kurtosis (convexity of curve) is a measure which gives an idea about the flatness or peakedness of the curve. It is measured by the coefficient β_2.

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2} \quad \text{or} \quad \gamma_2 = \beta_2 - 3$$

a) Mesokurtic curve : (Normal curve)

The curve C_1 which is neither flat nor peaked is called the normal curve or mesokurtic curve, for which $\beta_2 = 3$ or $\gamma_2 = 0$.

b) Platykurtic curve :

The curve (C_2) which is flatter than C_1 is platykurtic curve, for which $\beta_2 < 3$ or $r_2 < 0$.

c) Leptokurtic curve :

The curve (C_3) which is more peaked than C_1 is Leptokurtic curve, for which $\beta_2 > 3$ or $r_2 > 0$.

Solution : We first calculate moments about x = 4 (Assumed mean)

x_i	f_i	$d_i = x_i - 4$	$f_i\ d_i$	$f_i\ d_i^2$	$f_i\ d_i^3$	$f_i\ d_i^4$
0	1	– 4	– 4	16	– 64	256
1	8	– 3	24	72	– 216	648
2	28	– 2	– 56	112	– 224	448
3	56	– 1	– 56	56	– 56	56
4	70	0	0	0	0	0
5	56	1	56	56	56	56
6	28	2	56	112	224	448
7	8	3	24	72	216	648
8	1	4	4	16	64	256
	$\sum f_i = 256$		$\sum f_i\ d_i = 0$	$\sum f_i\ d_i^2 = 512$	$\sum f_i\ d_i^3 = 0$	$\sum f_i\ d_i^4 = 2816$

We know that

$$\mu'_r = \frac{1}{N}\sum f_i\,(x_i - 4)^r = \frac{1}{N}\sum f_i\ d_i^r$$

$$\mu'_1 = \frac{1}{N}\sum f_i\ d_i = 0$$

$$\mu'_2 = \frac{1}{N}\sum f_i\ d_i^2 = \frac{512}{256} = 2$$

$$\mu'_3 = \frac{1}{N}\sum f_i\ d_i^3 = 0$$

$$\mu'_4 = \frac{1}{N}\sum f_i\ d_i^4 = \frac{2816}{256} = 11$$

using the relations between μ_r and μ'_r

∴ Moments about mean are

$$\mu_1 = 0 \text{ always}$$

$$\mu_2 = \mu'_2 - \mu_1'^2 = 2 - 0$$

$$\mu_3 = \mu'_3 - 3\mu'_2\,\mu'_1 + 2\mu_1'^3 = 0 - 3 \times 2 \times 0 + 2 \times 0 = 0$$

$$\mu_4 = \mu'_4 - 4\mu'_3\,\mu'_1 + 6\mu'_2\,\mu_1'^2 - 3\mu_1'^4$$

$$= 11 - 4(0)(0) + 6(2)(0) - 3 \times 0 = 11$$

$$\therefore \quad \text{Skewness } \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{0}{2^3} = 0$$

$$\text{Kurtosis } \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{11}{4} = 2.75$$

Example 10.10 : *The first four moments about the working mean 30.2 of a distribution are 0.255, 6.222, 30.211 and 400.25. Calculate the moments about the mean. Also evaluate β_1, β_2 and comment upon the skewness and kurtosis of the distribution.*

Solution : Given the first four moments about the arbitrary origin 30.2 are

$$\mu_1' = 0.255, \ \mu_2' = 6.222, \mu_3' = 30.211, \mu_4' = 400.25$$

We know that $\mu_1' = \bar{x} - A$

$$0.255 = \bar{x} - 30.2$$

$$\therefore \quad \bar{x} = 30.455$$

Now using the relations between μ_r and μ_r'

$$\mu_2 = \mu_2' - \mu_1'^2 = 6.222 - (0.255)^2 = 6.15698$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 30.211 - 3(6.222)(0.255) + 2(0.255)^3$$

$$= 30.211 - 4.75983 + 0.03316275$$

$$\mu_3 = 25.48433$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= 440.25 - 4(30.211)(0.255) + 6(6.222)(0.255) - 3(0.255)^4$$

$$\mu_4 = 378.9418$$

$$\therefore \quad \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(25.48433)^2}{(6.15698)^3} = 2.78255$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{378.9418}{(6.15698)^2}$$

$$\beta_2 = 9.99625$$

$$\therefore \quad \gamma_1 = \sqrt{\beta_1} = \sqrt{2.78255} = 1.6681$$

which indicates considerable skewness of the distribution.

$$\gamma_2 = \beta_2 - 3 = 9.99625 = 6.99625$$

which shows that the distribution is leptokurtic (As $\beta_2 > 3$ or $\gamma_2 > 0$)

Example 10.11 : *Calculate the first four moments about the mean of the given distribution. Also find skewness and kurtosis.*

Class intervals	1.75-2.25	2.25-2.75	2.75-3.25	3.25-3.75	3.75-4.25	4.25-4.75	4.75-5.25
Frequency	4	36	60	90	70	40	10

Solution : here h = 0.5 (class width)

Let $$u_i = \frac{x_i - A}{h} = \frac{x_i - 3.5}{0.5}$$

frequency distribution table

Class intervals	x_i Mid values	f_i	$u_i = \frac{x_i - A}{h}$	$f_i u_i$	$f_i u_i^2$	$f_i u_i^3$	$f_i u_i^4$
1.75–2.25	2.0	4	– 3	– 12	36	– 108	324
2.25–2.75	2.5	36	– 2	– 72	144	– 288	576
2.75–3.25	3.0	60	– 1	– 60	60	– 60	60
3.25–3.75	3.5	90	0	0	0	0	0
3.75–4.25	4.0	70	1	70	70	70	70
4.25–4.75	4.5	40	2	80	160	320	640
4.75–5.25	5.0	10	3	30	90	270	810
Total		$\sum f_i = 310$		$\sum f_i u_i = 36$	$\sum f_i u_i^2 = 560$	$\sum f_i u_i^3 = 204$	$\sum f_i u_i^4 = 2480$

$\therefore$ Moments about A = 3.5 are

$$\mu'_1 = \frac{\sum f_i u_i}{N} = \frac{36}{310} = 0.116$$

$$\mu'_2 = \frac{\sum f_i u_i^2}{N} = \frac{560}{310} = 1.806$$

$$\mu'_3 = \frac{\sum f_i u_i^3}{N} = \frac{204}{310} = 0.658$$

$$\mu'_4 = \frac{\sum f_i u_i^4}{N} = \frac{2480}{310} = 8.0$$

To find moments about mean use the relations between μ_r and μ'_r.

$$\mu_0 = 1, \ \mu_1 = 0 \text{ (always)}$$

$$\mu_2 = \mu'_2 - \mu_1'^2 = 1.806 = 0.013456$$

for $\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu_1'^2 - 3\mu_1'^4$

$\therefore \quad \mu'_4 = \mu_4 + 4\mu'_3\mu'_1 - 6\mu'_2\mu_1'^2 + 3\mu_1'^4$

$\therefore \quad \mu'_4 = 1024 + 4\times1544\times10 - 6\times116\times100 + 3\times10000$

$\therefore \quad \mu'_4 = 23184$

As $\gamma_1 = +1$, the distribution is positively skewed. i.e. for the given distribution, tail will be longer towards right.

Also $\beta_2 = 4 > 3$, Distribution is leptokurtic

Exercise 10.1

1. *Find the arithmetic mean, median, standard deviation first four moments about mean of the following distribution. Find the coefficient of skewness and kurtosis.*

x	1	2	3	4	5	6	7	8	9	10
f	6	15	23	42	62	60	40	24	13	5

[Ans. : $\mu_1 = 0, \mu_2 = 3.3778, \mu_3 = -0.0824, \mu_4 = 37.7721, \beta_1 = 0.002, \beta_2 = 3.3106$]

2. *The first four moments of a distribution about $x = 2$ are 1, 2.5, 5.5 and 16. Calculate first four moments about mean and about zero.* **[Ans. : $\mu_1 = 0, \mu_2 = 1.5, \mu_3 = 0, \mu_4 = 6$**
about zero : 3 10.5, 40.5, 168]

3. *Compute coefficient of skewness and kurtosis for the data*

x	4.5	14.5	24.5	34.5	44.5	54.5	64.5	74.5	84.5	94.5
f	1	5	12	22	17	9	4	3	1	1

[Ans. : $\mu_1 = 0, \mu_2 = 2.83, \mu_3 = 3.38, \mu_4 = 30.295 \therefore \beta_1 = 0.504, \beta_2 = 3.782$]

4. *Calculate first four moments about the mean for the distribution.*

x	1	2	3	4	5	6	7	8	9
f	1	6	13	25	30	22	9	5	2

[Ans. : $\mu_1 = 0, \mu_2 = 2.49, \mu_3 = 0.68, \mu_4 = 18.26$]

5. *First four moments of a distribution about the value 5 of the variable are 2, 20, 40 and 50. Find the moments about the mean.* **[Ans. : 0, 16, – 64, 162]**

6. *The scores of two golfers for 10 rounds each are*

A	58	59	60	54	65	66	52	75	69	52
B	84	56	92	65	86	78	44	54	78	68

Which may be regarded as more consistent player. **[Ans. : 'A']**

7. *A Collar manufacturer is considering production of a new type of Collar to attract youngmen. The following statistics of neck circumferences are available based upon the measurements of typical groups of college students.*

Mid value of students	12.5	13.0	13.5	14.0	14.5	15.0	15.5	16.0	16.5
No. of Students	4	19	30	63	66	29	18	1	1

Compute Mean, Standard deviation and variance. **[Ans. : 14.24, 0.72, 0.52]**

8. *Calculate Mean and Standard deviation for the data*

x	56	63	70	77	84	91	98
f	3	6	14	16	13	6	2

[Ans. : 76.53, 9.87]

9. *Compute Mean deviation from median for the data :*

Class Interval	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
No. of Students	5	10	20	5	10

[Ans. : 9]

10. *Find the arithmetic Mean, Median and Standard deviation for the following frequency distribution.*

x_i	5	9	12	15	20	24	30	35	42	49
f_i	3	6	8	8	9	10	8	7	6	2

10.9 Correlation and Regression

Bivariate distribution : The distribution for one variate x is known as univariate distribution. The distribution which involves more than one variable is known as bivariate distribution. If a change in one variable x affects the change in other variable y then the variables are said to be correlated.

If increase in x $\Rightarrow$ increase in y $\Rightarrow$ direct or
(decrease) (decrease) positive correlation

If increase in x $\Rightarrow$ decrease in y $\Rightarrow$ negative correlation
(decrease) (increase) (inverse correlation)

For variables x and y if the ratio $\frac{y}{x}$ = constant then the correlation is linear, otherwise non-linear.

Karl Person's Coefficient of Correlation

To measure the intensity or degree of linear relationship between two variables, Karl Pearson developed a formula called correlation coefficient.

a) Correlation coefficient between two variables x and y is denoted by $r(x, y)$ and is defined as

$$\boxed{r\,(x, y) = \frac{\text{cov}\,(x, y)}{\sigma_x\ \sigma_y}}$$

where

$$\text{cov}\,(x, y) = \text{co-variance of } (x, y)$$

$$= \frac{1}{n}\sum (x_i - \bar{x})(y_i - \bar{y})$$

where $\bar{x} = \dfrac{\sum x_i}{n}$ $\quad \bar{y} = \dfrac{\sum y_i}{n}$

$$\sigma_x = \sqrt{\frac{1}{n}\sum (x_i - \bar{x})^2}$$

$$\sigma_y = \sqrt{\frac{1}{n}\sum (y_i - \bar{y})^2}$$

Now

$$\text{cov}\,(x, y) = \frac{1}{n}\sum (x_i - \bar{x})(y_i - \bar{y})$$

$$= \frac{1}{n}\sum (x_i\, y_i - \bar{y}\cdot x_i - \bar{x}\cdot y_i + \bar{x}\,\bar{y})$$

$$= \frac{1}{n}\left(\sum x_i\, y_i - \bar{y}\sum x_i - \bar{x}\sum y_i + \sum \bar{x}\,\bar{y}\right)$$

$$= \frac{1}{n}\sum x_i\, y_i - \bar{y}\left(\frac{\sum x_i}{n}\right) - \bar{x}\left(\frac{\sum y_i}{n}\right) + \frac{1}{n}\,\bar{x}\,\bar{y}\,n$$

$$= \frac{1}{n}\sum x_i\, y_i - \bar{y}\,\bar{x} - \bar{x}\cdot\bar{y} + \bar{x}\cdot\bar{y}$$

$$= \frac{1}{n}\sum x_i\, y_i - \bar{x}\cdot\bar{y}$$

i.e.

$$\boxed{\begin{aligned} \text{cov}(x, y) &= \frac{1}{n}\sum x_i\, y_i - (\bar{x})(\bar{y}) \\ \text{cov}(x, y) &= \frac{1}{n}\sum x_i\, y_i - \left(\frac{\sum x_i}{n}\right)\left(\frac{\sum y_i}{n}\right) \end{aligned}}$$

Also

$$\sigma_x^2 = \frac{1}{n}\sum (x_i - \bar{x})^2$$

$$= \frac{1}{n}\sum \left(x_i^2 - 2\,\bar{x}\,x_i + (\bar{x})^2\right)$$

$$= \frac{1}{n}\sum x_i^2 - 2\,\bar{x}\left(\frac{\sum x_i}{n}\right) + (\bar{x})^2\,\frac{1}{n}\;n$$

$$= \frac{1}{n}\sum x_i^2 - 2\,\bar{x}(\bar{x}) + (\bar{x})^2$$

$$= \frac{1}{n}\sum x_i^2 - (\bar{x})^2$$

i.e.

$$\boxed{\begin{aligned} \sigma_x^2 &= \frac{1}{n}\sum x_i^2 - (\bar{x})^2 \\ \sigma_x^2 &= \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2 \end{aligned}}$$

Similarly

$$\boxed{\sigma_y^2 = \frac{\sum y_i^2}{n} - \left(\frac{\sum y_i}{n}\right)^2}$$

b) Method of step deviation

If $\quad u_i = \dfrac{x_i - A}{n}$ and $v_i = \dfrac{y_i - B}{k}$

$$\boxed{\begin{aligned} \text{cov}(u, v) &= \frac{\sum f_i u_i v_i}{\sum f_i} - \bar{u}\cdot\bar{v} \\ \sigma_u^2 &= \frac{\sum f_i u_i^2}{\sum f_i} - (\bar{u})^2 \\ \sigma_v^2 &= \frac{\sum f_i v_i^2}{\sum f_i} - (\bar{v})^2 \end{aligned}}$$

where $\bar{u} = \dfrac{\sum f_i u_i}{\sum f_i}$ $\quad \bar{v} = \dfrac{\sum f_i v_i}{\sum f_i}$

and $r(x, y) = r(u, v) = \dfrac{\text{cov}(u, v)}{\sigma_u \cdot \sigma_v}$

Substituting all the above values we can write

$$r(x, y) = \frac{\dfrac{\sum f_i u_i v_i}{N} - \left(\dfrac{\sum f_i u_i}{N}\right)\left(\dfrac{\sum f_i v_i}{n}\right)}{\sqrt{\dfrac{\sum f_i u_i^2}{N} - \left(\dfrac{\sum f_i u_i}{N}\right)^2}\ \sqrt{\dfrac{\sum f_i v_i^2}{N} - \left(\dfrac{\sum f_i v_i}{N}\right)^2}}$$

i.e.

$$\boxed{r(x, y) = \frac{N\cdot\sum f_i u_i v_i - \left(\sum f_i u_i\right)\left(\sum f_i v_i\right)}{\sqrt{N\sum f_i u_i^2 - \left(\sum f_i u_i\right)^2}\ \sqrt{N\sum f_i v_i^2 - \left(\sum f_i v_i\right)^2}}}$$

Example 10.14 : *Compute the coefficient of correlation for the following data*

x	10	14	18	22	26	30
y	18	12	24	6	30	36

Solution : Let $A = 22$ $\quad \therefore \quad u_i = \dfrac{x_i - A}{h} = \dfrac{x_i - 22}{4}$

and $B = 24$ $\quad \therefore \quad v_i = \dfrac{y_i - B}{k} = \dfrac{x_i - 24}{6}$

Table

x_i	y_i	$u_i = \frac{x_i - 22}{4}$	$v_i = \frac{y_i - 24}{6}$	u_i^2	v_i^2	$u_i\ v_i$
10	18	− 3	− 1	9	1	3
14	22	− 2	− 2	4	4	4
18	24	− 1	0	1	0	0
22	6	0	− 3	0	9	0
26	30	1	1	1	1	1
30	36	2	2	4	4	4
Total		− 3	− 3	19	19	12

$$\bar{u} = \frac{\sum u_i}{n} = \frac{-3}{6} = \frac{-1}{2}$$

$$\bar{v} = \frac{\sum v_i}{n} = \frac{-3}{6} = \frac{-1}{2}$$

$$\text{cov}(u, v) = \frac{\sum u_i\ v_i}{n} - (\bar{u})(\bar{v})$$

$$= \frac{12}{6} - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)$$

$$= 2.0 - 0.25$$

$$= 1.75$$

$$\sigma_u^2 = \frac{\sum u_i^2}{n} - (\bar{u})^2$$

$$= \frac{19}{6} - \left(\frac{-1}{2}\right)^2$$

$$= 3.1666 - 0.25$$

$$= 2.9166$$

$$\sigma_v^2 = \frac{\sum v_i^2}{n} - (\bar{v})^2$$

$$= \frac{19}{6} - \left(-\frac{1}{2}\right)^2$$

$$= 2.9166$$

$$\therefore \qquad r(x, y) = r(u, v)$$

$$= \frac{\text{cov}(u, y)}{\sigma_u \sigma_v}$$

$$= \frac{1.75}{\sqrt{(2.9166)(2.9166)}}$$

$$= \frac{1.75}{2.9166}$$

$$= 0.60$$

Example 10.15 : *Following are the marks of 10 students in Maths III and strength of materials calculate the coefficient of correlation.*

Roll No.	1	2	3	4	5	6	7	8	9	10
SOM	78	36	98	25	75	82	90	62	65	39
M. III	84	51	91	60	68	62	86	58	53	47

Solution : Let x, y represents the marks in two subjects.

Arrange x in increasing order and write corresponding y in front of x.

Let $u_i = x_i - 65$, $\quad v_i = y_i - 66$

x_i	y_i	$u_i = x_i - 65$	$v_i = y_i - 66$	u_i^2	v_i^2	$u_i v_i$
25	60	– 40	– 6	1600	36	240
36	51	– 29	– 15	841	225	435
39	47	– 26	– 19	676	361	494
62	58	– 3	– 8	9	64	24
65	53	0	– 13	0	169	0
75	68	10	2	100	4	20
78	84	13	18	169	324	234
82	62	17	– 4	289	16	– 68
90	86	25	20	625	400	500
98	91	33	25	1089	625	825
		0	0	5398	2224	2734

$$\bar{u} = \frac{\sum u_i}{n} = 0, \qquad \bar{v} = \frac{\sum v_i}{n} = 0$$

$$\text{cov}(u, v) = \frac{\sum u_i v_i}{n} - \bar{u} \cdot \bar{v}$$

$$= \frac{2734}{10}$$

$$= 273.4$$

$$\sigma_u^2 = \frac{\sum u_i^2}{n} - (\bar{u})^2$$

$$= \frac{5398}{10} - 0$$

$$= 539.8$$

$$\sigma_v^2 = \frac{\sum v_i^2}{n} - (\bar{v})^2$$

$$= \frac{2224}{10} - 0$$

$$= 222.4$$

$$r(x, y) = r(u, v) = \frac{\text{cov}(u, y)}{\sigma_u \sigma_v}$$

$$= \frac{273.4}{\sqrt{(529.8)(222.4)}}$$

$$= 0.787$$

Example 10.16 : *The following data are ages of husband and wife for twenty couples calculate coefficient of correlation.*

x	22	24	26	26	27	30	27	28	28	29	39	39	31	32	33	36	34	25	35	37
y	18	20	20	24	22	32	24	27	24	21	25	29	27	27	30	30	27	30	31	32

Solution : Let $u_i = x_i - 30$ $\qquad v_i = y_i - 26$

x_i	y_i	$u_i = x_i - 30$	$v_i = y_i - 26$	u_i^2	v_i^2	$u_i v_i$
22	18	− 8	− 8	64	64	64
24	20	− 6	− 6	36	36	36
26	20	− 4	− 6	16	36	24

26	24	– 4	– 2	16	4	8
27	22	– 3	– 4	9	16	12
30	32	0	6	0	36	0
27	24	– 3	– 2	9	4	6
28	27	– 2	1	4	1	– 2
28	24	– 2	– 2	4	4	4
29	21	– 1	– 5	1	25	5
30	25	0	– 1	0	1	0
30	29	0	3	0	9	0
31	27	1	1	1	1	1
32	27	2	1	4	1	2
33	30	3	4	9	16	12
36	30	6	4	36	16	24
34	27	4	1	16	1	4
25	30	5	4	25	16	20
35	31	5	5	25	25	25
37	32	7	6	49	36	42
		0	0	324	348	287

Now $\bar{u} = \dfrac{\sum u_i}{n} = 0, \quad \bar{v} = \dfrac{\sum v_i}{n} = 0$

$$\text{cov}(u, v) = \frac{\sum u_i v_i}{n} - \bar{u}\,\bar{v}$$

$$= \frac{287}{20} - (0)(0) = 14.35$$

$$\sigma_u = \sqrt{\frac{\sum u_i^2}{n} - (\bar{u})^2} = \sqrt{\frac{324}{20}} = 4.05$$

$$\sigma_y = \sqrt{\frac{\sum v_i^2}{n} - (\bar{v})^2} = \sqrt{\frac{348}{20}} = 4.2$$

$\therefore \quad r(u, v) = \dfrac{\text{cov}(u, v)}{\sigma_u \, \sigma_v}$

$$= 0.85$$

Example 10.17 : *From a group of 10 students marks obtained by each in papers of Mathematics and Applied Mechanics are given as.*

x - marks in maths	*23*	*28*	*42*	*17*	*26*	*35*	*29*	*37*	*16*	*46*
y - marks in ap mech	*25*	*22*	*38*	*21*	*27*	*39*	*24*	*32*	*18*	*44*

Calculate Karl Pearson's coefficient of correlation.

Solution :

x_i	y_i	$u_i = x_i - 35$	$v_i = y_i - 39$	u_i^2	v_i^2	$u_i v_i$
16	18	– 19	– 21	361	441	399
17	21	– 18	– 18	324	324	324
23	25	– 12	– 14	144	196	168
26	27	– 9	– 12	81	144	108
28	22	– 7	– 17	49	289	119
29	24	– 6	– 15	36	225	90
35	39	0	0	0	0	0
37	32	2	– 7	4	49	– 14
42	38	7	– 1	49	1	– 7
46	44	11	5	121	25	55
		$\sum u = -51$	$\sum v = -100$	$\sum y^2 = 1169$	1694	1242

$$\bar{u} = \frac{\sum u_i}{n}$$

$$\therefore \quad \bar{u} = \frac{-51}{10} = -5 \cdot 1 \qquad \bar{u}^2 = 26 \cdot 01$$

$$\therefore \quad \bar{v} = -10 \qquad \bar{v}^2 = 100$$

Now
$$\text{cov}(u, v) = \frac{1}{n} \sum u_i \ v_i - \bar{u}\,\bar{v}$$

$$= \frac{1}{10}(1242) - 51 = 73 \cdot 2$$

$$\sigma_{\bar{u}}^2 = \frac{1}{n} \sum u_i^2 - (\bar{u})^2$$

$$= \frac{1169}{10} - 26 \cdot 01 = 90 \cdot 89$$

$$\sigma_u = \sqrt{90 \cdot 89} = 9 \cdot 5336$$

$$\sigma_v^2 = \frac{1}{n} \sum vi^2 - \overline{v}^2$$

$$= \frac{1694}{10} - 100 = 69 \cdot 4$$

$$\sigma_v = \sqrt{69 \cdot 4} = 8 \cdot 33$$

$$r(x, y) = r(u, v) = \frac{\text{cov}(u, v)}{\sigma_u \cdot \sigma_v}$$

$$= \frac{73 \cdot 2}{9 \cdot 534 \times 8 \cdot 33}$$

$$= 0.9217$$

Example 10.18 : *Calculate the coefficient of correlation.*

x	5	9	15	19	24	28	32
y	7	9	14	21	23	29	30
f	6	9	13	20	16	11	7

Solution :

x_i	y_i	f_i	$u_i = x_i - 19$	$v_i = y_i - 21$	$f_i u_i$	$f_i v_i$	$f_i u_i^2$	$f_i v_i^2$	$f_i u_i v_i$
5	7	6	– 14	– 14	– 84	– 84	1176	1176	1176
9	9	9	– 10	– 12	– 90	– 108	900	1296	1080
15	14	13	– 4	– 7	– 52	– 91	208	637	364
19	21	20	0	0	0	0	0	0	0
24	23	16	5	2	80	32	400	64	160
28	29	11	9	8	99	88	891	704	792
32	30	7	13	9	91	63	1183	567	819
		82			**44**	**– 100**	**4758**	**4444**	**4391**

$$\overline{u} = \frac{\sum f_i u_i}{\sum f_i} = \frac{44}{82} = 0.5366 \qquad \overline{u}^2 = 0.288$$

$$\overline{v} = \frac{\sum f_i v_i}{\sum f_i} = \frac{-100}{82} = -1.2196 \quad \overline{v}^2 = 1.4872$$

Solution : Prepare the frequency distribution table.

Let $$u_i = \frac{x_i - 450}{100} \qquad v_i = \frac{y_i - 17.5}{5}$$

Class intervals			200-300	300-400	400-500	500-600	600-700				
Class intervals		x	250	350	450	550	650				
	y	u / v	u = −2	− 1	0	+ 1	+ 2	f	fv	fv^2	fuv
10-15	12.5	v = − 1	-	-	-	−3 3	−14 7	10	− 10	10	− 17
15-20	17.5	0	-	0 4	0 9	0 4	0 3	20	0	0	0
20-25	22.5	1	− 14 7	− 6 6	0 12	5 5	-	30	30	30	− 15
25-30	27.5	2	− 12 3	− 20 10	0 19	16 8	-	40	80	160	− 16
		f	10	20	40	20	10	N = 100	Σ fv = 100	Σ fv^2 = 200	− 48
		fu	− 20	− 20	0	20	20	Σ fu = 00			
		fu^2	40	20	0	20	40	Σ fu^2 = 120			
		fuv	− 26	− 26	0	18	− 14	Σ fuv = − 98			

Thus $N = 100$, $\sum f_i u_i = 0$, $\sum f_i v_i = 100$, $\sum f_i u_i^2 = 120$, $\sum f_i v_i^2 = 200$,

$\sum f_i u_i v_i = -48$

Substituting in we get

$$r(x, y) = \frac{N\sum f_i u_i v_i - \sum f_i u_i \sum f_i v_i}{\sqrt{N\sum f_i u_i^2 - \left(\sum f_i u_i\right)^2}\ \sqrt{N\sum f_i v_i^2 - \left(\sum f_i v_i\right)^2}}$$

$$r(x, y) = \frac{100(-48) - 0 \times 100}{\sqrt{100(120) - (00)^2}\ \sqrt{100(200) - (100)^2}}$$

$$= \frac{-4800}{\sqrt{12000}\ \sqrt{10000}} = \frac{-4800}{10954} = \mathbf{-0.438}$$

Example 10.21 : *The following table gives, according to age the frequency of marks obtained by 200 students in a test to determine talent in mathematics :*

Marks \ Age in yrs	20	21	22	23	24	Total
0 - 10	10	8	6	10	4	38
10 - 20	8	10	8	-	11	37
20 - 30	-	11	7	8	5	31
30 - 40	20	-	10	12	10	52
40 - 50	2	6	7	15	12	42
Total	40	35	38	45	42	200

Solution :

	y_i		20	21	22	23	24	Total	$f_i u_i$	$f_i u_i^2$	$f_i u_i v_i$
CI	$v_i = y_i - 22$		-2	-1	0	1	2	–	–	–	–
	x_i	$u_i = \frac{x_i - 25}{10}$	40	16	0	-20	-16	38	-76	152	20
0-10	5	-2	10	8	6	10	4				
			16	10	0	0	-22	37	-37	37	4
10-20	15	-1	8	10	8	0	11				
			0	0	0	0	0	31	0	0	0
20-30	25	0	0	11	7	8	5				
			-40	0	0	12	20	52	52	52	-8
30-40	35	1	20	0	10	12	10				
			-8	-12	0	30	48	42	84	168	58
40-50	45	2	2	6	7	15	12				
Total			40	35	38	45	42	200	23	409	74
$f_i v_i$			$40 \times (-2) = -80$	-35	0	45	84	14			
$f_i v_i^2$			$40 \times 4 = 160$	35	0	45	168	408			
$f_i v_i u_i$			8	14	0	22	30	74			

Substituting in

$$r(x, y) = \frac{N\sum f_i u_i v_i - \left(\sum f_i u_i\right)\left(\sum f_i v_i\right)}{\sqrt{\left[N\sum f_i u_i^2 - \left(\sum f_i u_i\right)^2\right]\left[N\sum f_i v_i^2 - \left(\sum f_i v_i\right)^2\right]}}$$

$$= \frac{(200)(74)-(14)(23)}{\sqrt{\left[(200)(408)-(14)^2\right]\left[(200)(409)-(23)^2\right]}}$$

$$= 0.178076$$

10.10 Regression

If x and y are correlated. If the points in scatter diagram lies on some curve then that curve is called curve of regression. If the curve is a straight line then it is called as a regression line in such a case the relation between the two variables is a linear relation.

Lines of regression are used for estimating the value of one variable for a given value of other variable.

The regression line is obtained using the method of least squares.

Consider the set of values of (x_i, y_i) i = 1, 2, ... n

Let the line of regression of y on x be

$$y = mx + c$$

$$\therefore \quad \sum y_i = m\sum x_i + c\sum i$$

i.e. $$\sum y_i = nc + m\sum x_i \quad \text{... (I)}$$

Also $$\sum x_i y_i = m\sum x_i^2 + c\sum x_i \quad \text{... (II)}$$

Dividing (I) by n

$$\frac{1}{n}\sum y_i = c + m\frac{\sum x_i}{n}$$

i.e. $$\bar{y} = c + m\bar{x} \quad \text{... (III)}$$

Which shows that the point $(\bar{x}, \bar{y})$ lies on the line of regression.

We know that

$$\text{cov}(x, y) = \frac{1}{n}\sum x_i y_i - \bar{x}\bar{y}$$

Let $$\text{cov}(x, y) = \mu_{11}$$

Then $$\mu_{11} + \bar{x}\bar{y} = \frac{1}{n}\sum x_i y_i \quad \text{... (IV)}$$

Also $$\sigma_x^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2$$

$$\therefore \quad \frac{1}{n}\sum x_i^2 = \sigma_x^2 + \bar{x}^2 \quad \text{... (V)}$$

Dividing (II) by N

$$\frac{1}{n}\sum x_i\, y_i = m\frac{\sum x_i^2}{N} + c\frac{\sum x_i}{N} \quad \text{... (VI)}$$

$$\therefore \quad \mu_{11} + \bar{x}\,\bar{y} = m\left(\sigma_x^2 + \bar{x}^2\right) + c\,\bar{x} \quad \text{... (VII)}$$

Multiplying (III) by $\bar{x}$ we have

$$\bar{x}\,\bar{y} = c\,\bar{x} + m\,\bar{x}^2$$

Subtract from (VIII)

$$\mu_{11} = m\,\sigma_x^2 \Rightarrow m = \frac{\mu_{11}}{\sigma_x^2}$$

$\therefore$ Equation of regression line which passes through $(\bar{x}, \bar{y})$ is

$$y - \bar{y} = m(x - \bar{x})$$

or $$y - \bar{y} = \frac{\mu_{11}}{\sigma_x^2}(x - \bar{x})$$

This gives regression line of y on x we know that

$$r(x, y) = \frac{\operatorname{cov}(x, y)}{\sigma_x\, \sigma_y}$$

$$= \frac{\mu_{11}}{\sigma_x \cdot \sigma_y}$$

$$\therefore \quad \frac{\mu_{11}}{\sigma_x^2} = r(x, y)\cdot \frac{\sigma_y}{\sigma_x}$$

$$\therefore \quad \boxed{y - \bar{y} = r(x, y)\cdot \frac{\sigma_y}{\sigma_x}(x - \bar{x})}$$

is the line of regression of y on x.

Similarly if we start from x = my + c we can get the line of regression of x and y as

$$\boxed{x - \bar{x} = r(x, y)\cdot \frac{\sigma_x}{\sigma_y}(y - \bar{y})}$$

Thus the equation of line of regression of y on x is given by

$$y-\bar{y} = b_{yx}(x-\bar{x})$$

where $\bar{x}, \bar{y}$ are means of distributions for x and y respectively. The equation of line of regression of x and y is given by

$$x-\bar{x} = b_{xy}(y-\bar{y})$$

where, $r\frac{\sigma_y}{\sigma_x}$ = regression coefficient of y on x

$= b_{yx}$

$r\frac{\sigma_x}{\sigma_y}$ = regression coefficient of x on y

$= b_{xy}$

Thus $b_{yx} \cdot b_{xy} = r^2$

Example 10.22 : *Find the lines of regression for the data.*

x	10	14	19	26	30	34	39
y	12	16	18	26	29	35	38

and estimate y for x = 14.5 and x for y = 29.5

Solution : Let A = 26, B = 26 $u = x - A$, $v = y - B$

$\therefore$ $u = x - 26$

$v = y - 26$

x	y	u = x − a x − 26 a	v = y − b y − 26 b	u^2	v^2	uv
10	12	− 16	− 14	256	196	224
14	16	− 12	− 10	144	100	120
19	18	− 7	− 8	49	64	56
26	26	0	0	0	0	0
30	29	4	3	16	9	12
34	35	8	9	64	81	72
39	38	13	12	169	144	156
		$\sum u = -10$	$\sum v = -8$	$\sum u^2 = 698$	$\sum v^2 = 594$	$\sum uv = 640$

Regression line of Y on X is

$$y-\bar{y} = b_{yx}(x-\bar{x})$$
$$y-8 = -0.65(x-6)$$
$$y = -0.65x+3.9+8$$
$$y = -0.65x+11.9$$

Regression line of X on Y is

$$x-\bar{x} = b_{xy}(y-\bar{y})$$
$$x-6 = -1.3(y-8)$$
$$x-6 = -1.3y+10.4$$
$$x = -1.3y+10.4+6$$
$$x = -1.3y+16.4$$

Example 10.25 : *The following are marks obtained by 10 students in Statistics and Economics.*

No.	1	2	3	4	5	6	7	8	9	10
Marks in economics	25	28	35	32	31	36	29	38	34	32
Marks in statistics	43	46	49	41	36	32	31	30	33	39

Marks are out of 50. Obtain regression equation to estimate marks in statistics if mark in economics are 30.

Solution : n = 10. Let marks in economics be x and marks in statistics be y.

Let $u = x-30$ and $v = y-35$.

x	y	u = x – 30	v = y – 35	u^2	v^2	uv
25	43	– 5	8	25	64	– 40
28	46	– 2	11	4	121	– 22
35	49	5	14	25	196	70
32	41	2	6	4	36	12
31	36	1	1	1	1	1
36	32	6	– 3	36	9	– 18

29	31	– 1	– 4	1	16	4
38	30	8	– 5	64	25	– 40
34	33	4	– 2	16	4	– 8
32	39	2	4	4	16	8
–	–	$\sum u = 2$	$\sum v = 30$	$\sum u^2 = 180$	$\sum v^2 = 488$	$\sum uv = -33$

$$\bar{u} = \frac{\sum u}{n} = \frac{20}{10} = 2 \quad \text{and} \quad \bar{v} = \frac{\sum v}{n} = \frac{30}{10} = 3$$

$u = x - 30 \quad \therefore \quad u = \bar{x} - 30$

$\therefore \quad \bar{x} = \bar{u} - 30 = 2 + 30 = 32$

$v = y - 35 \quad \therefore \quad \bar{v} = \bar{y} - 35$

$\therefore \quad \bar{y} = \bar{v} + 35 = 3 + 35 = 38$

$$\sigma_{\bar{u}}^2 = \frac{\sum u^2}{n} - (\bar{u})^2 = \frac{180}{10} - (2)^2 = 18 - 4 = 14$$

$$\sigma_{\bar{v}}^2 = \frac{\sum v^2}{n} - (\bar{v})^2 = \frac{488}{10} - (3)^2 = 48.8 - 9 = 39.8$$

$\therefore \quad \sigma_u = 3.742 \quad \text{and} \quad \sigma_v = 6.309$

If $\quad u_i = \dfrac{x_i - A}{h} \qquad v_i = \dfrac{y_i - B}{k}$

then $\quad \sigma_x = h\,\sigma_u \qquad \sigma_y = k\,\sigma_v \qquad \text{here } h = k = 1$

$\therefore \quad \sigma_x = \sigma_u \qquad \sigma_y = \sigma_v$

$\therefore \quad \sigma_x = 3.743 \quad \text{and} \quad \sigma_y = 6.309$

$\therefore \quad \sigma_{\bar{x}}^2 = 14 \quad \text{and} \quad \sigma_{\bar{y}}^2 = 39.8$

$$\text{cov}(u, v) = \frac{\sum uv}{n} - \bar{u}\,\bar{v} = \frac{-33}{10} - 2\,(3) = -3.3 - 6$$

$\therefore \quad \text{cov}(u, v) = -9.3$

Now $\quad r(x, y) = r(u, v) = \dfrac{\text{cov}(u, v)}{\sigma_u \sigma_v}$

$$= \frac{-9.3}{(3.742)(6.309)}$$

$$= -0.3943$$

$$\therefore \quad b_{yx} = -r\frac{\sigma_y}{\sigma_x}$$

$$= -0.3943\,\frac{6.309}{3.742}$$

$$= -0.6646$$

$\therefore$ The regression line of y on x is

$$y-\bar{y} = b_{yx}(x-\bar{x})$$

$$y-38 = -0.664\,(x-32)$$

$$\therefore \quad y = -0.664\,x + 59.248$$

Put $x = 30$ and find y

$$y = -0.664\,(30) + 59.248$$

$$= 39.328$$

Thus marks in economics are approximately 39.

Example 10.26 : *Find the coefficient of correlation for distribution in which S.D. of $x = 4$ and S.D. of $y = 1.8$. Coefficient of regression of y on x is 0.32.*

Solution : $\sigma_x = 4, \quad \sigma_y = 1.8 \quad$ and $\quad b_{yx} = 0.32$

We have, $$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

$$0.32 = r\times\frac{1.8}{4}$$

$$\therefore \quad r = \frac{0.32\times 4}{1.8}$$

$$r = 0.711$$

Example 10.27 : *Given $n = 6$, $\sum(x-18.5) = -3$, $\sum(y-50) = 0$, $\sum(x-18.5)^2 = 19$, $\sum(y-50)^2 = 850$, $\sum(x-18.5)(y-50) = -120$.*
Calculate coefficient of correlation.

Solution : Let $u = x - 18.5$ and $v = y - 50$

$$\therefore \quad \bar{u} = \frac{-3}{6} = -0.5$$

and $$\bar{v} = \frac{20}{6} = 3.33$$

From the given data $\sum u = -3$, $\sum v = 20$, $\sum u^2 = 19$, $\sum v^2 = 850$ and $\sum uv = -120$.

Coefficient of correlation is given by

$$r = \frac{n(\sum uv)-(\sum u)(\sum v)}{\sqrt{\left[n\sum u^2-(\sum u)^2\right]\left[n\sum v^2-(\sum v)^2\right]}}$$

$$r = \frac{6(-120)-(-3)(20)}{\sqrt{\left[6(19)-(-3)^2\right]\left[6(850)-(20)^2\right]}}$$

$$r = \frac{-720+60}{\sqrt{[105][4700]}} = -0.9395$$

Example 10.28 : *Given the following information*

	Variable x	Variable y
Arithmetic mean	8.2	12.4
Standard deviation	6.2	20

Coefficient of correlation between x and y is 0.9. Find the linear regression estimate of x given y = 10.

Solution : Given that $\bar{x} = 8.2$, $\bar{y} = 12.4$, $\sigma_x = 6.2$, $\sigma_y = 20$ and $r_{xy} = 0.9$. We want to find x for y = 10.

Line of regression of x on y is

$$x-\bar{x} = b_{xy}(y-\bar{y})$$

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} = 0.9\times\frac{6.2}{20}$$

$$= 0.279$$

Substituting value of $\bar{x}$, $\bar{y}$ and b_{xy} in above equation we get,

$$x - 8.2 = 0.279\,(y - 12.4)$$

$$x = 0.279\,y - 3.4596 + 8.2$$

$$x = 0.279\,y + 4.7404$$

Putting y = 10 in equation we get,

$$x = 0.279 \times 10 + 4.7404$$

$$x = 7.5304$$

Example 10.29 : *Given :*

	x series	y series
Mean	18	100
Standard deviation	14	20

and coefficient of correlation is 0.8. Find most probable values of y if x = 70 and most probable values of x if y = 90.

Solution : We are given $\bar{x} = 18 \quad \sigma_x = 14$

$\bar{y} = 100 \quad \sigma_y = 20$

The equation of line of regression of y on x is

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 100 = \frac{0.8 \times 20}{14}(x - 18)$$

$$y = 1.413\,x + 79.421 \qquad \dots (1)$$

Probable value of y when x = 70 is given by (1)

$$y = 1.413\,(70) + 79.421$$

$$= 178.338$$

Equation of line of regression of x on y is

$$x - \bar{x} = r\,\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$x - 18 = (0.8)\frac{14}{20}(y - 100)$$

i.e. $$x = 0.56\,y - 38 \qquad \dots (2)$$

Probable value of x when y = 90 is given by (2)

$$x = 0.56\,(90) - 38$$

$$= 12.4$$

Example 10.30 : *From record of analysis of correlation data the following results are available variance of x = 9 and lines of regression are given by*

$8x - 10y + 66 = 0$

$40\,x - 18y = 214$

Find out a) Mean values for x and y services. b) Standard deviation of y services. c) Coefficient of correlation between x and y services.

Now $\quad b_{yx} = r\frac{\sigma_y}{\sigma_x} \quad$ As $\quad \sigma_x^2 = 9 \quad \therefore \quad \sigma_x = 3$

$$\therefore \quad \frac{8}{10} = 0.6\,\frac{\sigma_y}{3}$$

$$\therefore \quad \sigma_y = \frac{24}{6} = 4$$

Example 10.31 : *If the two lines of regression are $9x+y-\lambda = 0$ and $4x+y=\mu$ and the means of x and y are 2 and -3 respectively, find the values of λ, μ and the coefficient of correlation between x and y.*

Solution : $\bar{x} = 2$ and $\bar{y} = -3$

The lines of regression are $9x + y = \lambda$ and $4x + y = \mu$.

The point of intersection of two regression lines is (x, y) i.e. $(\bar{x}, \bar{y})$ lies on both the regression lies.

$$9\,\bar{x}+\bar{y} = \lambda \qquad \text{... (1)}$$

$$4\bar{x}+\bar{y} = \mu \qquad \text{... (2)}$$

Substituting values of $\bar{x}$ and $\bar{y}$ we get,

$$9(2)+(-3) = \lambda$$

$$\lambda = 18-3=15$$

and $\quad 4(2)+(-3) = \mu$

$$\therefore \quad \mu = 8-3=5$$

Thus, the regression lies are,

$$9x + y = 15 \quad \text{and} \quad 4x + y = 15$$

Let $9x + y = 15$ be the regression line of x and y, so it can be written as

$$x = \frac{15}{9}-\frac{y}{9}$$

$$\therefore \quad b_{xy} = -\frac{1}{9} = -0.11$$

Let $4x + y = 5$ be the regression line of y on x. So it can be written as $y = 5 - 4x$.

$$\therefore \quad b_{yx} = -4$$

Correlation coefficient between x and y is given as,

$$r = \sqrt{b_{yx}\,b_{xy}}$$

$$= \sqrt{(-4)\times(-0.11)}$$

12. *Determine the reliability of estimates for the data :*

x	10	14	19	26	30	34	39
y	12	16	18	26	29	35	38

[Ans. : $r^2 = 0.988$ high]

13. *The following marks have been obtained by a group of students in Engineering Mathematics.*

Paper I	80	45	55	56	58	60	65	68	70	75	85
Paper II	82	56	50	48	60	62	64	65	70	74	90

Calculate the coefficient of correlation. **[Ans. 9277] (May-95)**

14. *For the following tabulated data, find the coefficient of correlation.*

y \ x	18	18	20	21	Total
10 - 20	4	2	2	–	8
20 - 30	5	4	6	4	19
30 - 40	6	8	10	11	35
40 - 50	4	4	6	8	22
50 - 60	–	2	4	4	10
60 - 70	–	2	3	1	6
Total	19	22	31	28	100

[Ans. : 0.25]

15. *The two regression equations of variables x and y are $x = 4y - 3$ and $9y = x + 13$ find :*

i) mean of x and y and **[Ans. : $\bar{x} = 5, \bar{y} = 2$]**

ii) coefficient of correlation between x and y **[Ans. : $r = \frac{2}{3}$]**

16. *A panel of two judges 'A' and B graded dramatic performances by independently awarding marks as follows :* **(May-2000, civil)**

Performance No.	1	2	3	4	5	6	7	8
Marks by A	36	32	34	31	32	32	35	38
Marks by B	35	33	31	30	34	32	36	?

The eight performance, however, which judge B could not attend, got 38 marks by judge A. If judge B had also been present, how many marks would be expected to have been awarded by him to the eight performance ? **[Ans. : 35.9 ≅ 36 (Approximately)]**

17. *For a group of children, mean age is 10 years with standard deviation of 2.5 years. The average height of the group is 125 cms, with standard deviation of 13 cms, the coefficient of correlation between the age and height is 0.6, write the equations of two regression lines and explain their use.*

[Ans. : Line of regression of y on x is, y = 3.12x + 93.8,

Line of regression of x on y is, x = 0.115y – 4.43]

18. *Find the correlation coefficient and the equations of regression lines from the following data :*

a)

x	1	2	3	4	5
y	2	5	3	8	7

[Ans. : r = 0.81, x = 0.5y + 0.5, y = 1.3x + 1.1]

b)

x	80	45	55	56	58	60	65	68	70	75	85
y	8	56	50	48	60	62	64	65	70	74	90

[Ans. : r = 0.918, y – 65.45 = 0.989 (x – 65.18), x – 65.18 = 0.85 (y – 65.45)]

c)

x	2	4	5	6	8	11
y	18	12	10	8	7	5

[Ans. : r = – 0.92, y – 10 = – 1.34 (x – 6), x – 6 = – 0.632 (y – 10)]

University Questions

Dec. - 98

1. *From the following frequency-distribution compute the standard-deviation of 100 students :* **[5 Marks]**

Mass in kg.	Number of students
60 - 62	5
63 - 65	18
66 - 68	42
69 - 71	27
72 - 74	8

2. *Calculate the coefficient of correlation between the marks obtained by eight students in Mathematics and Statistics from the following data. Find also the lines of regression.* **[8 Marks]**

Student	Maths (x)	Statistics (y)
A	25	08
B	30	10
C	32	15
D	35	17
E	37	20
F	40	22
G	42	24
H	45	25

3. *Define Mean, Median and Mode of a frequency distribution.* **[3 Marks]**

May - 99

1. *A computer while calculating the correlation coefficient between the variables X and Y obtained the following results :*

$N = 30, \sum X = 120, \sum X^2 = 600, \sum Y = 90, \sum Y^2 = 250, \sum XY = 335$

It was, however, later discovered at the time of checking that it has copied down two pairs of observations as :

X	Y
8	10
12	7

while the correct values were :

X	Y
8	12
10	8

Obtain the correct value of the correlation coefficient between X and Y. **[8 Marks]**

Dec. - 99

1. *Find the lines of regression for the following data :*

x	10	14	19	26	30	34	39
y	12	16	18	26	29	35	38

and determine the reliability of estimate of y for x = 14.5. **[9 Marks]**

May - 2000

1. *From the following data obtain the two regression equations :*

x :	6	2	10	4	8
y :	9	11	5	8	7

Find the estimate of y corresponding to x = 5. What is the reliability of this estimate ? **[8 Marks]**

Dec. - 2000

1. *The two regression equations of the variables x and y are* $x = 19.13 - 0.87\,y$ *and* $y = 11.64 - 0.50\,x$. *Find* $\overline{x}$, $\overline{y}$ *and the correlation coefficient between x and y.* **[8 Marks]**

May - 2001

1. *In the following table are recorded data showing that test scores made by salesmen on an intelligence test and their weekly sales :*

Salesmen	Test scores	Sales
1	40	2.5
2	70	6.0
3	50	4.5
4	60	5.0
5	80	4.5
6	50	2.0
7	90	5.5
8	40	3.0
9	60	4.5
10	60	3.0

Find the two regression lines and estimate the weekly sales volume if salesman makes a score of 52. **[8 Marks]**

Dec. - 2001

1. *The following data gives the experience of machine operators and their performance ratings as given by the number of good parts turned at per 100 pieces.*

Operators	Experience in years (X)	Performance rating (Y)
1	16	87
2	12	88

3	18	89
4	4	68
5	3	28
6	10	80
7	5	75
8	12	83

Calculate the regression lines of performance ratings on experience and estimate the probable performance if an operator has 7 years experience. **[8 Marks]**

May - 2002

1. *Find the lines of regression for the data :*

x	10	14	19	26	30	34	39
y	12	16	18	26	29	35	38

and determine the reliability of estimate of y for x = 14.5. **[8 Marks]**

Dec. - 2002

1. *From the following data, obtain the two regression equations :* **[8 Marks]**

x	y
1	4
2	8
3	2
4	12
5	10
6	14
7	16
8	6
9	18

May - 2003

1. *Find the regression line of y on x for the following data :*

x	1	3	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9

Estimate the value of y when x = 10. **[8 Marks]**

Dec. - 2003

1. *Find the lines of regression for the following data :* **[9 Marks]**

x	y
10	12
14	16
19	18
26	26
30	29
34	35
39	38

Dec. - 2004

1. *Find the correlation coefficient between x and y when lies of regression are* $2x - 9y + 6 = 0$ *and* $x - 2y + 1 = 0$. **[8 Marks]**

May - 2005

1. *If* θ *is the acute angle between the two regression lines in case of two variables x and y, then show that :*

$$\tan\theta = \frac{\sigma_x \cdot \sigma_y}{\left(\sigma_x^2 + \sigma_y^2\right)} \cdot \frac{\left|1 - r^2\right|}{r}$$ **[4 Marks]**

2. *Find the first four moment about the mean of the following :* **[7 Marks]**

X	f
61	5
64	18
67	42
70	27
73	8

Also calculate coefficient of skewness, coefficient of kurtosis. State the nature of the distribution.

Dec. - 2005

1. *The first four moments about the working mean 30.2 of a distribution are 0.255, 6.222, 30.211 and 400.25. Calculate the moments about the mean. Also evaluate* β_1, β_2. **[6 Marks]**

2. *The following are marks obtained by 10 students in Statistics and Economics :*

No.	Marks in Economics	Marks in Statistics
1	25	43
2	28	46
3	35	49
4	32	41
5	31	36
6	36	32
7	29	31
8	38	30
9	34	33
10	32	39

Marks are out of 50. Obtain regression equation to estimate marks in Statistics if marks in Economics are 30. **[7 Marks]**

May - 2006

1. *From the record of correlation data the following results are available :*

   Variance of $x = 9$

   Lines of regression are given by $8x - 10y + 66 = 0$, $40x - 18y = 214$.

   Find :

   i) Mean values for x and y series.

   ii) Standard deviation for x-series.

   iii) Coefficient of correlation between x and y series. **[7 Marks]**

2. *Goals scored by two teams A and B in a football season are as follows :*

No. of goals in a match	No. of matches	
	A	B
0	27	17
1	9	9
2	8	6
3	5	5
4	4	3

Find out which team is more consistent scorer ? **[5 Marks]**

11 Probability and Probability Distributions

11.1 Introduction

When an experiment is conducted and each outcome of the experiment has the same chance of appearing as any other then we call the outcomes as equally likely.

$$P(A) = \text{Probability of occurrence of any event A}$$

$$= \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}}$$

When an event succeeds in "S" ways and fails in "F" ways (n = F + S = total number of ways) then

$$P\text{ (success)} = \frac{S}{S+F} \quad \text{and} \quad P\text{ (Failure)} = \frac{F}{S+F}$$

Generally the probability of success is denoted by p

and the probability of failure is denoted by q.

Obviously $p + q = 1$ i.e. $q = 1 - p$.

11.2 Theory of Probability

Some important definitions

1) Trial and event : Let an experiment be repeated under essentially the same conditions and let it result in any one of the several possible outcomes. Then, the experiment is called a trial and the possible outcomes are known as events or cases. For example tossing of a coin is a trial and the turning up of head or tail is an event.

2) Independent events : Two events are said to be independent when the outcome of one does not affect and is not affected by the other.

e.g. coin is tossed twice, outcome of 2^{nd} throw is independent of the outcome of 1^{st}.

3) Dependent events : Dependent events are those in which the occurrence or non-occurrence of one event in any trial affects the probability of other event in other trials.

4) Mutually exclusive : Two events are said to be mutually exclusive or incompatible when both cannot happen simultaneously in a single trial or in other words the occurrence of any one of them precludes the occurrence of the other.

e.g. coin is tossed, either it will be head or tail, both can not be up at the same time.

Similarly person can not be alive and dead simultaneously.

$\therefore$ Mutually exclusive events are either or

5) Simple and compound events : In case of simple events we consider the probability of the happening or not happening of single events.

e.g. we might be interested in probability of Red ball from 2 bag of 10 white and 6 red balls.

On the other hand if a bag contains 10 white and 6 red balls and two successive draws of 3 balls each are made. Then finding probability of 3 white in first draw and 3 red in second is the case of joint occurrence. In this case we are dealing with compound event.

6) Favourable events : The cases which entail the happening of an event are said to be favourable to event. It is the total number of possible outcomes in which the specified event happens. In throwing of two dice the number of cases favourable to getting a sum '6' is five i.e. (1, 5), (5, 1), (2, 4), (4, 2), (3, 3).

7) Equally likely events : Events are said to be equally likely if there is no reason to expect any other. In throwing a die, all the six faces are equally likely to come.

8) Exhaustive events : Events are said to be exhaustive when their totality includes all the possible outcomes of a random experiment.

e.g. while tossing a die, the possible outcome are 1, 2, 3, 4, 5, 6.

$\therefore$ exhaustive number of cases is '6'.

If two dices are thrown once (together) the possible outcomes are $6 \times 6 = 36$.

Similarly for three dices it will be $6 \times 6 \times 6 = 216$ as total number of cases.

Similarly black and red cards are examples of collectively exhaustive events each being 26 in number.

9) Complementary events : Two events are said to be complementary if they are mutually exclusive and exhaustive.

for e.g. a dice is thrown, getting an even 2, 4, 6 and odd no. (1, 3, 5) are complementary events.

i.e. if A occurs B does not (Exclusive) and vice-versa.

11.3 Theorems on Probability

a) Addition theorem

1) If A and B are mutually exclusive events.

then Prob (A or B)

i.e. $$P(A \cup B) = P(A) + P(B)$$

If A, B, C are mutually exclusive then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

2) When the events are not mutually exclusive then probability that atleast one of the two events A and B will occur is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

i.e. $$P(A \text{ or } B) = P(A) + P(B) - P(A \,\&\, B)$$

In case of three events

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$$
$$= -P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

b) Multiplication theorem

If A and B are two independent events, then the probability that both will occur is equal to the product of their individual probabilities.

$$P(A \,\&\, B) = P(A) \times P(B)$$

Similarly $P(A, B \,\&\, C) = P(A) \times P(B) \times P(C)$

c) Conditional probability

Multiplication theorem is not applicable when events are dependent.

e.g. when we are computing prob. of a particular event A. When given information about occurrence of B. Such a prob. is referred to as conditional probability.

$\therefore$ for two dependent events A and B prob. of B, given A has occurred is denoted by

$$P(B \mid A) = \frac{P(A, B)}{P(A)}$$

Similarly, prob. of A given B has occurred is

$$P(A/B) = \frac{P(A, B)}{P(B)}$$

$$\therefore \quad P(A \,\&\, B) = P(A) \times P(B/A)$$

$$P(A \,\&\, B) = P(B) \times P(A/B)$$

for three events A, B and C.

$$P(A, B \,\&\, C) = P(A) \times P(B/A) \times P(C/A, B)$$

11.4 Illustrations

Example 11.1 : *What is the probability that a leap year will contain 53 Mondays ?*

Solution : A leap year has 366 days.

This contains complete 52 weeks and two more days. These two days may take following combinations.

i) Monday - Tuesday (ii) Tuesday - Wednesday (iii) Wednesday - Thursday

(iv) Thursday - Friday (v) Friday - Saturday (vi) Saturday - Sunday and

(vii) Sunday - Monday.

Out of these 7 combinations only 2 contain Monday.

$\therefore$ Required probability $= \frac{2}{7}$

Example 11.2 : *Prof. X and Madam Y appear for an interview for two posts. The prob. of Prof. X's selection is $\frac{1}{7}$ and that of Madam Y's selection is $\frac{1}{5}$. Find the Prob. that only one of them is selected. What is prob. that at least one of them is selected.*

Solution :

$$P(X) = \frac{1}{7} \qquad P(Y) = \frac{1}{5}$$

$$P(\overline{X}) = 1-\frac{1}{7} = \frac{6}{7} \qquad P(\overline{Y}) = 1-\frac{1}{5} = \frac{4}{5}$$

As only one of this is selected

$\Rightarrow$ If X is selected, Y is not selected (Case A)

$\Rightarrow$ If Y is selected, X is not selected (Case B)

$$P(A) = P(X) \times P(\overline{Y}) = \frac{1}{7} \times \frac{4}{5} = \frac{4}{35}$$

$$P(B) = P(\overline{X}) \times P(Y) = \frac{6}{7} \times \frac{1}{5} = \frac{6}{35}$$

$\therefore$ Required prob.

$$P(A \text{ or } B) = P(A) + P(B)$$

$$\frac{4}{35} + \frac{6}{35} = \frac{10}{35} = \frac{2}{7}$$

$$P(\overline{X}) = \frac{6}{7}, \quad P(\overline{Y}) = \frac{4}{5}$$

$\therefore$ Prob. that none is selected $= P(\overline{X}) \times P(\overline{Y})$

$$= \frac{6}{7} \times \frac{4}{5}$$

Example 11.8 : *A problem of statistics is given to 5 students A, B, C, D and E. Their chances of solving the problem are 1/2, 1/3, 1/4, 1/5 and 1/6. What is the probability that problem is solved.*

Solution : Probability that problem is solved.

Atleast one of the student solve it.

$$P(A) = \frac{1}{2} \Rightarrow P(\overline{A}) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(B) = \frac{1}{3} \Rightarrow P(\overline{B}) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$P(C) = \frac{1}{4} \Rightarrow P(\overline{C}) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(D) = \frac{1}{5} \Rightarrow P(\overline{D}) = 1 - \frac{1}{5} = \frac{4}{5}$$

$$P(E) = \frac{1}{6} \Rightarrow P(\overline{E}) = 1 - \frac{1}{6} = \frac{5}{6}$$

$\therefore$ Probability that problem is not solved.

$$= P(\overline{A}) \times P(\overline{B}) \times P(\overline{C}) \times P(\overline{D}) \times P(\overline{E})$$

$\therefore$ Probability that problem is solved.

$$= 1 - P(\overline{A}) \times P(\overline{B}) \times P(\overline{C}) \times P(\overline{D}) \times P(\overline{E})$$

$$= 1 - \frac{1}{6} = \frac{5}{6}$$

Example 11.9 : *An aircraft gun can take a minimum of four shots at an enemy plane moving away from it. The probability of hitting plane at 1^{st}, 2^{nd}, 3^{rd}, 4^{th} shot are 0.4, 0.3, 0.2, and 0.1 respectively. What is the probability that gun hits the plane ?*

Solution : Let probability of hitting at 1^{st}, 2^{nd}, 3^{rd} and 4^{th} shots be $P(S_1)$, $P(S_2)$, $P(S_3)$ and $P(S_4)$.

$$P(S_1) = 0.4 \Rightarrow P(\overline{S}_1) = 1 - 0.4 = 0.6$$

$$P(S_2) = 0.3 \Rightarrow P(\overline{S}_2) = 1 - 0.3 = 0.7$$

$$P(S_3) = 0.2 \Rightarrow P(\overline{S}_3) = 1 - 0.2 = 0.8$$

$$P(S_4) = 0.1 \Rightarrow P(\overline{S}_4) = 1 - 0.1 = 0.9$$

$\therefore$ Probability that no shot hits the plane.

$$P(\overline{S}_1, \overline{S}_2, \overline{S}_3, \overline{S}_4) = P(\overline{S}_1) \times P(\overline{S}_2) \times P(\overline{S}_3) \times P(\overline{S}_4)$$

$\therefore$ Probability that at least one shot hits the plane.

$$= 1 - P(\overline{S}_1, \overline{S}_2, \overline{S}_3, \overline{S}_4)$$

$$= 1 - P(\overline{S}_1) \times P(\overline{S}_2) \times P(\overline{S}_3) \times P(\overline{S}_4)$$

$$= 0 \cdot 6976$$

iii) 2 spades out of 13 can be drawn in ${}^{13}C_2$ ways.

2 hearts out of 13 can be drawn in ${}^{13}C_2$ ways.

$\therefore$ Favourable number of cases = ${}^{13}C_2 \times {}^{13}C_2 = \dfrac{13 \times 12 \times 13 \times 12}{2 \times 1 \times 2 \times 1} = 6084$

$\therefore$ Required probability $P = \dfrac{6084}{270725} = 0.0.02247$

Example 11.12 : *Find the probability of drawing (i) a card of spades (ii) a king, (iii) a king or a queen or a knave from a pack of cards.*

Solution : i) There are 13 cards of spades in a pack of cards and drawing any shall be a success. Obviously the total number of ways is 52.

$$\therefore \qquad P = \frac{13}{52} = 0.25$$

ii) There are 4 kings in a pack of cards and drawing any one of these shall be a success.

$$\therefore \qquad P = \frac{4}{52} = \frac{1}{13}$$

iii) These are 4 kings, 4 queues, 4 jacks i.e. a total of 12 and drawing any one of these is a success.

$$\therefore \qquad P = \frac{12}{52} = \frac{3}{13}$$

Example 11.13 : *From a pack of cards, four are drawn at random. What is the probability that there will be the four honours from the same suit ?*

Solution : Four cards can be selected out of 52 in ${}^{52}C_4$ ways and this gives us the total number of ways.

There are only four favourable ways, as there can be 4 honours from the same suit in spade, heart, diamond or clud.

Thus, $$P = \frac{4 \times 4!}{{}^{52}C_4} = \frac{4 \times 4 \times 3 \times 2 \times 1}{52 \times 51 \times 50 \times 49}$$

$$= \frac{96}{6497400} = 0.0000147$$

Example 11.14 : *A bag contains 7 white, 6 red and 5 black balls. Two balls are drawn at random. Find the probability that they will both be white.*

Solution : Total number of balls = 7 + 6 + 5 = 18.

Out of 18 balls, two can be drawn in ${}^{18}C_2$ ways.

$\therefore$ Exhaustive number of cases = ${}^{18}C_2 = \dfrac{18 \times 17}{2 \times 1} = 153$

out of 7 white balls 2 can be drawn in ${}^7C_2$ ways $= \dfrac{7\times 6}{2\times 1} = 21$

$\therefore$ favourable number of cases = 21

$$\therefore \qquad P = \frac{21}{153} = \frac{7}{51}$$

Example 11.15 : *A bag contains 6 red and green balls. If a draw of three is made, what is chance that the three balls drawn are red ?*

Solution : Total number of balls in bag are $6+4=10$. Three can be drawn in ${}^{10}C_3$ ways.

The event shall be a success if all the balls are red i.e. the balls drawn are from the 6 red balls. Thus the number of favourable ways is ${}^6C_3$.

$$\therefore \qquad P = \frac{{}^6C_3}{{}^{10}C_3} = \frac{6\times 5\times 4}{10\times 9\times 8} = \frac{1}{6}$$

Example 11.16 : *A bag contains 10 white and 15 black balls. Two balls are drawn in succession, what is the probability that*

i) One of the ball is black and other white

ii) Both of them are black.

Solution : Total number of balls in bag are 10 + 15 = 25.

i) It will contain one white and one black in ${}^{10}C_1\times {}^{10}C_1$ ways.

$$\therefore \qquad P = \frac{{}^{10}C_1\times {}^{15}C_1}{{}^{25}C_2} = \frac{10\times 15\times 2}{25\times 24} = \frac{1}{2}$$

ii) If both balls are black then favourable ways = ${}^{15}C_2$

$$\therefore \qquad P = \frac{{}^{15}C_2}{{}^{25}C_2} = \frac{15\times 14}{25\times 24} = \frac{7}{20}$$

Example 11.17 : *A bag contains 5 white and 3 black balls. Two balls are drawn at random one after the other without replacement. Find the prob. that both balls drawn are black.*

Solution : There are 5 + 3 = 8 balls.

Prob. of drawing black P (A) $= \dfrac{3}{8}$

Prob. of drawing 2nd black given that 1st ball drawn is black

$$P(B/A) = \frac{2}{7}$$

$\therefore$ Prob. that both balls drawn are black is given by

$$P(A \,\&\, B) = P(A)\times P(B/A)$$

$$= \frac{3}{8}\times\frac{2}{7} = \frac{3}{28}$$

$$\text{Prob. (THH)} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

There are mutually exclusive events.

$$\therefore \ P\ (A \text{ or } B \text{ or } C) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

Example 11.22 : *A, B play a game of alternate tossing a coin one who gets head first wins the game. Find the probability that B wins the game if A has a start.*

Solution : Following are the cases where B wins the game

1) TH 2) TTTH 3) TTTTTH ....

We know $P\ (T) = \frac{1}{2}$ $P\ (H) = \frac{1}{2}$

$$\therefore \quad P\ ((1)) = P(T) \cdot P(H) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^2}$$

$$P\ ((2)) = P(T)\,P(T)\,P(T)\,P(T) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^4}$$

$$P\ ((3)) = P(T) \cdot P(T) \cdot P(T) \cdot P(T) \cdot P(T) \cdot P(H) = \frac{1}{2^6}$$

$$\therefore \ \text{Required probability} = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} \ldots$$

which is a geometric series

$$a + a_r + a_r^2 \ \ldots\ldots. + \frac{a}{1-r} \quad \text{with} \quad a = \frac{1}{2^2} \quad \text{and} \quad r = \frac{1}{2^2}$$

$$\therefore \ \text{Required probability} = \frac{1/4}{1-1/4} = \frac{1}{3}$$

Example 11.23 : *A six faced die is so biased that it is twice as likely to show an even number as an odd number when thrown. It is thrown twice. What is the probability that the sum of the two numbers thrown is even ?*

Solution : Let probability of an odd number be P, so probability of an even number appearing is 2P.

Now there are 6 outcomes (1, 2, 3, 4, 5, 6)

$$\therefore \quad P\ (1) = P\ (3) = P\ (5) = P$$

$$P\ (2) = P\ (4) = P\ (6) = 2P$$

$$P\ (1) + P\ (2) + P\ (3) + P\ (4) + P\ (5) + P\ (6) = 1$$

$$\therefore \ P = 1 \qquad \therefore \ P = \frac{1}{9}$$

$$P\ (ii) = \frac{1}{3} \times \frac{2}{8} = \frac{1}{12}$$

Probability of A winning

$$= P\ (i) + P\ (ii) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

2) Probability of A's losing $= 1 - \frac{1}{6} = \frac{5}{6}$

Hence odds against A's winning are $\frac{5}{6} ; \frac{1}{6}$, i.e. 5 : 1.

Example 11.25 : *In a single throw of two dice, determine the probability of obtaining a total of 7 or 9.*

Solution : Total of 7 can be obtained as (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)

Total of 9 (4, 5), (3, 6), (6, 3), (5, 4)

When two dices are thrown

$$\text{Exhaustive no.} = 6 \times 6 = 36$$

$$P\ (7) = \frac{6}{36}, \quad P\ (9) = \frac{4}{36}$$

$$\therefore \quad P\ (7 \text{ or } 9) = \frac{6}{36} + \frac{4}{36} = \frac{10}{36}$$

Example 11.26 : *A student takes his examination in four subjects $\alpha, \beta, \gamma, \delta$. He estimates his chances of passing α as $\frac{4}{5}$, in β as $\frac{3}{4}$ in γ as $\frac{5}{6}$ in δ as $\frac{2}{3}$.*

To qualify, he must pass in α and at least two other subjects. What is the probability that he qualifies.

Solution : Here

$$P\ (\alpha) = \frac{4}{5}, \quad P\ (\beta) = \frac{3}{4}, \quad P\ (\gamma) = \frac{5}{6}, \quad \text{and} \quad P\ (\delta) = \frac{2}{3}$$

$$P\ (\bar{\alpha}) = 1 - \frac{4}{5} = \frac{1}{5}, \qquad P\left(\bar{\beta}\right) = 1 - \frac{3}{4} = \frac{1}{4}$$

$$P\ (\gamma) = 1 - \frac{5}{6} = \frac{1}{6}, \qquad P\left(\bar{\delta}\right) = 1 - \frac{2}{3} = \frac{1}{3}$$

There are four possibilities of passing at least two subjects,

I) Passing in β, γ and failing in δ

$$= P(\beta) \times P(\gamma) \times P\left(\bar{\delta}\right)$$

$$= \frac{3}{4} \times \frac{5}{6} \times \frac{1}{3} = \frac{5}{24}$$

II) Passing in γ, δ and failing in β

$$= P(\gamma) \times P(\delta) \times P(\bar{\beta})$$

$$= \frac{5}{6} \times \frac{2}{3} \times \frac{1}{4} = \frac{5}{36}$$

III) Passing in δ, β and failing in γ

$$= P(\delta) \times P(\beta) \times P(\bar{\gamma})$$

$$= \frac{2}{3} \times \frac{3}{4} \times \frac{1}{6} = \frac{1}{12}$$

II) Passing in β, γ, δ

$$= P(\beta) \times P(\gamma) \times P(\delta)$$

$$= \frac{3}{4} \times \frac{5}{6} \times \frac{2}{3} = \frac{5}{12}$$

$\therefore$ Probability of passing in at least two other subjects

$$= \frac{5}{24} + \frac{5}{36} + \frac{1}{12} + \frac{5}{12} = \frac{61}{72}$$

$\therefore$ Probability of passing α and at least two other subjects

$$= \frac{4}{5} \times \frac{61}{72} = \frac{61}{90}$$

Example 11.27 : *Find the probability of throwing '6' in the first only of two successive throws with an ordinary dice.*

Solution : We required a throw of '6' in the first throw and any number other than '6' in the second throw. Thus this is a problem of multiplication of the probabilities.

The probability of the first event (of throwing '6' in a single throw) = $\frac{1}{6}$

The probability of the second event (of throwing a number other than 6, in the second throw) = $\frac{5}{6}$.

Hence the probability of happening of both the events.

$$= \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$$

Example 11.28 : *A box contains 3 black, 4 white and 5 red balls, what is the probability of drawing 2 black balls in succession if the first ball is replaced after drawing ?*

Solution : Obviously one ball is drawn, in each draw.

The probability of drawing a black ball in one draw.

$$= \frac{3}{3+4+5} = \frac{1}{4}$$

When this ball is replaced, and another drawn, the probability of black is again 1/4.

The compound probability, that the ball is black in both the draws is, therefore.

$$= \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

Exercise 11.1

1. *A box contains 6 red balls 4 white balls 5 blue balls. Three balls are drawn successively from the box. Find the probability that they are drawn in the order red, white and blue if each ball is not replaced.* **[Ans. : $\frac{6}{15}, \frac{4}{14}, \frac{5}{13} = \frac{4}{91}$]**

2. *An urn contains 6 white and 8 red balls. Second urn contains 9 white and 10 red balls. One ball is drawn at random from the first urn and put into the second urn without noticing its colour. A ball is then drawn at random from the second urn. What is the probability that it is red.* **[Ans. : $\frac{3}{7} \cdot \frac{1}{2} + \frac{4}{7} \cdot \frac{11}{20} = \frac{259}{490}$]**

3. *Supposing that out of 12 test matches played between India and Shrilanka during last 3 years, 6 are won by India, 4 are won by Shrilanka and 2 are drawn. If they agree to play a test series consisting of three matches. Find the probability that India wins the test series on the basis of past performance.* **[Ans. : $\frac{19}{72}$]**

4. *A throw is made with two dice. Find the probability that (i) the sum is 7 or less, (ii) the sum is a perfect square.* **[Ans. : (i) $\frac{7}{12}$, (ii) $\frac{7}{36}$]**

5. *Three coins are tossed simultaneously. Find the probability of getting at least 2 Heads.* **[Ans. : $\left(\frac{1}{2}\right)$]**

6. *A bag contains 6 white and a 10 black balls. What is the probability that just 3 will be white out of 8 drawn ?* **[Ans. : $\frac{56}{143}$]**

7. *Assuming that the ratio of male children is $\frac{1}{2}$, find the probability that in a family of 6 children :*

   i) all children will be of same sex. **[Ans. : $\frac{1}{32}$]**

   ii) the four eldest children will be boys. **[Ans. : $\frac{1}{64}$]**

   iii) exactly three children will be boys. **[Ans. : $\frac{1}{1280}$]**

8. *What is the chance of throwing '3' in a single die ?* **[Ans. : $p = \frac{1}{6}$]**

9. *What is the probability of throwing a number greater than 3 in a single throw of one dice ?* **[Ans. : $\frac{1}{2}$]**

10. *Find the probability of throwing '9' with two dice.* **[Ans. : $\frac{1}{9}$]**

11. *A problem in mathematics is given to three students A, B and C whose chances solving it are $\frac{1}{2}, \frac{3}{4}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved ?* **[Ans. : $\frac{29}{32}$]**

12. *From a deck of 52 cards, two cards are drawn at random. Find the probability that (i) Both are hearts, (ii) Both the cards are of different suits.* **[Ans. : (i) $\frac{39}{613}$, (ii) $\frac{13}{17}$]**

13. *There are six married couples in a room. If two persons are chosen at random, find the probability that (i) they are of different sex, (ii) they are married to each other.* **[Ans. : (i) $\frac{6}{11}$; (ii) $\frac{1}{11}$]**

14. *A committee consists of 9 students two of which are from first year, three from second year and four from third year. Three students are to be removed at random. What is the chance that :*
    i) The three students belong to different classes. **[Ans. : $\frac{2}{7}$]**
    ii) Two belong to the same class and third to the different class. **[Ans. : $\frac{55}{84}$]**
    iii) The three belong to the same class. **[Ans. : $\frac{5}{84}$]**

15. *If 15 persons take seats at random at a round table, find the probability that two specified, persons are seated next to each other.* **[Ans. : $\frac{1}{7}$]**

16. *Urn I contains 6 white and 4 black balls and urn II contains 4 white and 5 black balls. From urn I, two balls are transferred to urn II without noticing the colour. Sample of size 2 is then drawn without replacement from urn II. What is the probability that the sample contains exactly 1 white ball ?* **[Ans. : $\frac{4}{5}$]**

17. *Four persons are chosen at random from a group containing 3 men, 2 women and 4 children. Show that the probability that exactly two of them will be children is $\frac{10}{21}$.*

18. *A bag contains 5 green and 7 red balls. Two balls are drawn. What is the probability that one is green and the other red ?* **[Ans. : $\frac{35}{66}$]**

19. *One shot is fired from each of the three guns. E_1, E_2, E_3 denote the events that the target is hit by the first, second and third guns respectively. If $P(E_1) = 0.5$, $P(E_2) = 0.6$, $P(E_3) = 0.7$ and E_1, E_2, E_3 are independent events, then find the probability that at least two hits are registered.* **[Ans. : 0.65]**

20. *A problem on computer Mathematics is given to the three students A, B and C whose chances of solving it are $\frac{1}{2}$, $\frac{3}{4}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved ?* **[Ans. : $\frac{29}{32}$]**

21. *What is the probability of getting a total of 5 or 8 in a single throw with two dice ?* **[Ans. : $\frac{1}{4}$]**

11.7 Mean and Variance of Random Variables

For the probability distribution

X	x_1	x_2	x_3	....	x_n
P (X)	p_1	p_2	p_3	....	p_n

Mean is denoted by μ and is given by

$$\mu = \frac{\sum p_i x_i}{\sum p_i}$$

As $\sum p_i = 1$, mean $\mu = \sum p_i x_i$

Mean is also called Mathematical expectation denoted by E (X)

Variance σ^2 is defined as

$$\sigma^2 = \frac{\sum p_i (x_i - \mu)^2}{\sum p_i}$$

As $\sum p_i = 1 \quad \therefore \quad \sigma^2 = \sum p_i (x_i - \mu)^2$

Simplifying we get

$$\sigma^2 = \sum p_i x_i^2 - \mu^2$$

i.e. $$\sigma^2 = \sum p_i x_i^2 - \left(\sum p_i x_i\right)^2$$

standard deviation $= \sqrt{\text{variance}}$

11.8 Illustrations

Example 11.29 : *Find the probability distribution for number of sixes in three tosses of a dice.*

Solution : Let X denote the random variable which is the number of sixes obtained in 3 tosses.

$\therefore$ X can take values 0, 1, 2, 3.

Prob. of getting a six $= \frac{1}{6}$ i.e. $P = \frac{1}{6}$

$\therefore$ Prob. of not getting a six $= 1 - \frac{1}{6} = \frac{5}{6}$ i.e. $q = \frac{5}{6}$

$$\therefore \quad P(X = 0) = (q \times q \times q) = \left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}\right) = \frac{125}{216}$$

$$P(X=1) = (p \times q \times q) + (q \times p \times q) + (q \times q \times p)$$
$$= \left(\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right)$$
$$= 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right) = \frac{75}{216}$$
$$P(x=2) = (p \times p \times q) + (p \times q \times p) + (q \times p \times p)$$
$$= 3\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = \frac{15}{216}$$
$$P(X=3) = (p \times p \times p) = \left(\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{216}$$

∴ Probability distribution

X	0	1	2	3
P (X)	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

Example 11.30 : *A dice is tossed thrice. A success is getting 1 or 6 on a toss. Find the mean and variance of the number of success.*

Solution : Let

X be the random variable = getting 1 or 6

∴ X can take values 0, 1, 2, 3

Prob. of getting (1 or 6) $= \frac{2}{6} = \frac{1}{3} \Rightarrow p = \frac{1}{3}$

∴ Prob. of not getting (1 or 6) $= 1 - \frac{1}{3} = \frac{2}{3}$

∴ $$P(X=0) = q \times q \times q = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$$
$$P(X=1) = 3(p \times q \times q) = 3 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{12}{27}$$
$$P(X=2) = 3(p \times p \times q) = 3 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{6}{27}$$
$$P(X=3) = p \times p \times p = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

∴ Probability distribution

X	0	1	2	3
P (X)	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

$$= n \cdot p \cdot q^{n-1} + 2 \cdot \frac{n(n-1)}{2} \cdot p^2 q^{n-2} + 3 \cdot \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} \ldots$$

$$p^3 \cdot q^{n-3} + \ldots\ldots n \cdot p^n$$

$$= n\,p \left[q^{n-1} + (n-1)\, q^{n-2} \cdot p + \frac{(n-1)(n-2)}{2 \cdot 1} \right.$$

$$\left. q^{n-3} \cdot p^2 + \ldots + (n-1)_{c_{(n-1)}}\, p^{n-1} \right]$$

$$= np\,(q+p)^{n-1}$$

$$\therefore \qquad \mu = np \quad \text{as} \quad p + q = 1$$

$\therefore$ Mean of binomial distribution is np.

2) Variance

$$\sigma^2 = \sum_{r=0}^{n} r^2\, P(r) - \mu^2$$

$$= \sum_{r=0}^{n} \left(r + r^2 - r\right) \cdot p(r) - \mu^2$$

$$= \sum_{r=0}^{n} \{r + r(r-1)\}\, p(r) - \mu^2$$

$$= \sum_{r=0}^{n} r \cdot p(r) + \sum_{r=0}^{n} r(r-1)\, p(r) - \mu^2$$

$$= \mu + \sum_{r=2}^{n} r(r-1)\, {}^{n}C_{r}\, p^r\, q^{n-r} - \mu^2$$

$$= m + [2 \cdot 1 \cdot {}^{n}c_2\, p^2 \cdot q^{n-2} + 3 \cdot 2\, {}^{n}c_3\, p^3 \cdot q^{n-3}$$

$$+ \ldots\ldots + n(n-1) \cdot {}^{n}c_n\, p^n\, q^0] - \mu^2$$

$$= \mu + \left[2.1\, \frac{n(n-1)}{2 \cdot 1}\, p^2 \cdot q^{n-2} + 3 \cdot 2 \cdot \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} \right.$$

$$\left. p^3 \cdot q^{n-3} + \ldots. + n(n-1)\, p^n \right]$$

$$= \mu + n(n-1)\, p^2\, [q^{n-2} + (n-2)\, q^{n-3} \cdot p + \ldots + p^{n-2}] - \mu^2$$

$$= \mu + n(n-1)\, p^2 \left[n - 2_{c_0} \cdot q^{n-2} + (n-2)_{c_1}\, q^{n-3} \cdot p + \ldots \right.$$

$$\left. + (n-2)_{c_{(n-2)}} \cdot p^{n-2} \right] - \mu^2$$

$$= \mu + n(n-1)\, p^2\, [q+p]^{n-2} - \mu^2$$

Thus for Binomial Distribution :

If p = Probability of success.

q = Probability of failure.

Then, $B(n, p, r) = {}^nC_r \; p^r \; q^{n-r}$

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

where $B(n, p, r)$ = Probability r successes in n trials

11.13 Illustrations

Example 11.32 : *An unbiased coin is thrown 10 times. Find the probability of getting exactly 6 heads, at least 6 Heads.*

Solution : Here $p = q = \frac{1}{2}$ $n = 10$

Occurrence of head is treated as successes.

Probability of getting exactly 6 Heads is

$$P(6) = 10_{C_6}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^4 = 10_{C_6}\left(\frac{1}{2}\right)^{10}$$

Event at least six heads occurs when coin shows up Head 6, 7, 8, 9 or 10 times the probability for these events are

$$P(7) = 10_{C_7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3 = 10_{C_7}\left(\frac{1}{2}\right)^{10}$$

$$P(8) = 10_{C_8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 = 10_{C_8}\left(\frac{1}{2}\right)^{10}$$

$$P(9) = 10_{C_9}\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^1 = 10_{C_9}\left(\frac{1}{2}\right)^{10}$$

$$P(10) = 10_{C_{10}}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^0 = \left(\frac{1}{2}\right)^{10}$$

$$P(\text{at least 6 Heads}) = P(6) + P(7) + P(8) + P(9) + P(10)$$

$$= \left(\frac{1}{2}\right)^{10}\left[10_{C_6} + 10_{C_7} + 10_{C_8} + 10_{C_9} + 10_{C_{10}}\right]$$

$$= \left(\frac{1}{2}\right)^{10}\left[\frac{10!}{6!\,4!} + \frac{10!}{7!\,3!} + \frac{10!}{8!\,2!} + \frac{10!}{9!\,1!} + \frac{10!}{10!\,0!}\right]$$

$$= \left(\frac{1}{2}\right)^{10}\left[\frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1}+\frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}+\frac{10\cdot 9}{2\cdot 1}+\frac{10}{1}+1\right]$$

$$= \left(\frac{1}{2}\right)^{10}[210+120+45+10+1]$$

$$= \frac{386}{2^{10}} = \frac{386}{1024} = 0.3769$$

Example 11.33 : *A pair of dice is thrown 10 times. If getting a doublet is considered a success, find the probability of (i) 4 successes, (ii) No success.*

Solution : Here $n = 10$

$$p = \text{Prob. of getting a doublet} = \frac{3}{36} = \frac{1}{6}$$

$$q = \text{Prob. of not getting a doublet} = 1 - \frac{1}{6} = \frac{5}{6}$$

$$\therefore \quad p(r) = {}^{n}C_{r}\, p^{r}\, q^{n-r}$$

$$p(4) = 10_{C_4}\left(\frac{1}{6}\right)^{4}\left(\frac{5}{6}\right)^{10-4}$$

$$= \frac{10!}{6!\,4!}\left(\frac{1}{6}\right)^{4}\left(\frac{5}{6}\right)^{6}$$

$$= \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} = \frac{(5)^{6}}{(6)^{10}}$$

Probability of no success i.e.

$$p(o) = {}^{n}C_{o}\, p^{0}\, q^{n-o}$$

$$= 10_{C_0}\left(\frac{1}{6}\right)^{0}\left(\frac{5}{6}\right)^{10}$$

$$= \left(\frac{5}{6}\right)^{10}$$

Example 11.34 : *Probability of man aged 60 years will live for 70 year is 1/10. Find the probability of 5 men selected at random 2 will live for 70 years.*

Solution : Here $p = \frac{1}{10}$ $q = \frac{9}{10}$ $r = 2$ $n = 5$

$$p\,(\text{2 men live for 70 years}) = 5_{C2}\left(\frac{1}{10}\right)^{2}\left(\frac{9}{10}\right)^{3}$$

$$\text{i.e.}\quad p\,(2) = \frac{5.4}{1.2}\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)\left(\frac{9}{10}\right)\left(\frac{9}{10}\right)$$

$$= 0\cdot 054675$$

Example 11.35 : *Twelve coins are thrown and the number of heads recorded. If the experiment is repeated 4096 times. Find the theoretical frequencies of different number of heads.*

Solution : Here $N = 4096,\ N = 12,$

$$p = \frac{1}{2},\quad q = \frac{1}{2}$$

Thus the theoretical frequencies are given by

$$N(p+q)^n = 4096\left(\frac{1}{2}+\frac{1}{2}\right)^{12}$$

$$= 4096\left[\left(\frac{1}{2}\right)^{12} + {}^{12}C_1\left(\frac{1}{2}\right)^{11}\left(\frac{1}{2}\right) + {}^{12}C_2\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^2 + {}^{12}C_3\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^3\right.$$

$$+ {}^{12}C_4\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^4 + {}^{12}C_5\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^5 + {}^{12}C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^6 + {}^{12}C_7\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^7$$

$$+ {}^{12}C_8\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^8 + {}^{12}C_9\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^9 + {}^{12}C_{10}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{10} + {}^{12}C_{11}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{11}$$

$$\left. + {}^{12}C_{12}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{12}\right]$$

$$= 4096\cdot\left(\frac{1}{2}\right)^{12}[1+12+66+220+495+792+924+792$$

$$+495+220+66+12+1]$$

$$= 4096\left[\frac{1}{4096}+\frac{12}{4096}+\frac{66}{4096}+\frac{220}{4096}+\frac{495}{4096}+\frac{792}{4096}+\frac{924}{4096}+\frac{792}{4096}\right.$$

$$\left.+\frac{495}{4096}+\frac{220}{4096}+\frac{66}{4096}+\frac{12}{4096}+\frac{1}{4096}\right]$$

Thus we have the following theoretical frequencies,

No. of Heads	0	1	2	3	4	5	6	7	8	9	10	11	12
Frequency	1	12	66	220	495	792	924	792	495	220	66	12	1

The sum of frequencies = 4096

Example 11.36 : *Four coins are tossed simultaneously. What is the probability of getting (1) two heads and two tails (ii) at least two heads (iii) at least one head.*

Solution : In single toss

$$P(H) = \frac{1}{2} \quad \text{i.e.} \quad P = \frac{1}{2}$$

$$\therefore \quad P(T) = P(\bar{H}) = 1 - \frac{1}{2} = \frac{1}{2} \quad \text{i.e.} \quad q\,\frac{1}{2}$$

$$p(r) = {}^nC_r\, p^r\, q^{n-r}$$

$$\therefore \quad p(2) = {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2}$$

$$= \frac{4!}{2!\,2!}\left(\frac{1}{2}\right)^4 = 6\left(\frac{1}{2}\right)^4$$

This is prob. of two heads and two tails.

i) We know that

$$\therefore \quad p(x \geq 2) = p(2) + p(3) + p(4)$$

$$= {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2} + {}^4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{4-3} + {}^4C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^6$$

$$= 6\left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 = \frac{11}{16}$$

ii) Prob. of at least one head = 1 – prob. of no head

$$= 1 - {}^4C_0\, p^0\, q^{4-0}$$

$$= 1 - \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4$$

$$= 1 - \frac{1}{16} = \frac{15}{16}$$

Example 11. 37 : *Ten percent of articles from a certain machine are defective. What is the probability that there shall be 6 defectives in a sample of 25 ?*

Solution : Here $n = 25, \quad p = \frac{10}{100} = 0.1, \quad q = 1 - p = 0.9$

$$\therefore \quad p(r) = {}^nC_r\, p^r\, q^{n-r}$$

$$p(r) = {}^{25}C_r\,(0.1)^r\,(0.9)^{25-r}$$

$$\therefore \quad p(6) = {}^{25}C_6\,(0.1)^6\,(0.9)^{19}$$

$$= \frac{25 \times 24 \times 23 \times 22 \times 21 \times 20}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \times \frac{(9)^{19}}{(10)^{25}} = 0.024$$

$\therefore$ p [Zero old + Two new]

$$= \left[{}^3C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{3-0}\right]\left[{}^7C_2 \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^{7-2}\right]$$

Hence $\quad P(B) = \left(\frac{9}{10}\right)^3 {}^7C_2 \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^5$

$$= \frac{7 \cdot 6}{1 \cdot 2} \times \left(\frac{9}{10}\right)^3 \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^5$$

$$= 0.0296$$

Event, two machines needing an adjustment which are of the same type A + B.

$\therefore \quad P(A+B) = P(A) + P(B)$ [A, B are mutually exclusive]

$$= 0.0188 + 0.0296$$

required probability $= 0.0484$

Example 11.39 : *A dice is thrown 5 times. If getting an odd number is a success, what is the probability of (i) 4 successes (ii) at least 4 success.*

Solution : Odd numbers are 1, 3, 5

$\therefore$ Prob. of an odd $= \frac{3}{6} = \frac{1}{2}$ i.e. $p = \frac{1}{2}$

$\therefore$ Prob. of not an odd $= 1 - \frac{1}{2} = \frac{1}{2}$ i.e. $q = \frac{1}{2}$

$\therefore \quad P(r) = {}^nC_r\, p^r\, q^{n-r}$

$$P(4) = {}^5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{5-4} = 5\left(\frac{1}{2}\right)^5 = \frac{5}{32}$$

Prob. of at least four, i.e.

$$P(x \geq 4) = p(4) + p(5)$$

$$= {}^5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{5-4} + {}^5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{5-5}$$

$$= \frac{5}{32} + \frac{1}{32} = \frac{6}{32}$$

Example 11.40 : *An average box containing 10 articles is likely to have 2 defectives. If we consider a consignment of 100 boxes, how many of them are expected to have three or less defectives.*

Solution : Let $\quad p$ = Probability of box containing defective articles

$$= 2/10 = 1/5$$

Example 11.42 : *Assume that on average telephone number out of 15 called between 2 pm to 3 pm on week days is busy. What is the probability that is 6 randomly selected telephone numbers called. i) not more that 3 ii) at least 3 of them is busy.*

Solution : The probability that the telephone no, called between 2 pm and 3 pm, is busy is

$$p = \frac{1}{15} \quad q = 1 - \frac{1}{15} = \frac{14}{15}$$

Hence probability that r nos. called out of 6 called are.

$$P(r) = 6_{C_r}\, p^r\, q^{6-r}$$

$$= 6_{C_r}\left(\frac{1}{15}\right)^r\left(\frac{14}{15}\right)^{6-r} \qquad \text{... (1)}$$

i) For not more than 3 calls are busy

$$P(r \leq 3) = P(0) + P(1) + P(2) + P(3)$$

$$= 6_{C_0}\left(\frac{14}{15}\right)^6 + 6_{C_1}\left(\frac{1}{15}\right)\left(\frac{14}{15}\right)^5 + 6_{C_2}\left(\frac{1}{15}\right)^2\left(\frac{14}{15}\right)^4 + 6_{C_3}\left(\frac{1}{15}\right)^3\left(\frac{14}{15}\right)^3$$

$$= \frac{(14)^3}{(15)^6}\left[(14)^3 + 6(14)^2 + \frac{6.5}{1.2}(14) + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}\right]$$

$$= 0.9997$$

ii) For at least 3 calls to be busy of 6 calls we have the probability as.

$$P(r \geq 3) = 1 - P(r < 3)$$

$$= 1 - \left[6_{C_0}\left(\frac{14}{15}\right)^6 + 6_{C_1}\left(\frac{1}{15}\right)\left(\frac{14}{15}\right)^5 + 6_{C_2}\left(\frac{1}{15}\right)^2\left(\frac{14}{15}\right)^4\right]$$

$$= 0.0051$$

Example 11.43 : *20% of bolts produced by a machine are defective. Determine the probability that out of 4 bolts chosen at random. a) 1 is defective b) Zero is defective c) At most 2 bolts are defective.*

Solution : The probability of defective bolt is.

$$p = \frac{20}{100} = 0.2$$

$$\therefore \quad q = 0.8 \quad [p + q = 1]$$

b) Expected no. of families having 2 boys

$$= 2000\ 4_{C_2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 750$$

c) Expected no. of families having 1 or 2 girls. (i.e. having 3 boys or 2 boys)

$$= 2000\ [P\ (3) + P\ (2)]$$

$$= 2000\left[4_{C_3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + 4_{C_2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2\right]$$

$$= 1250$$

d) Expected no. of families having no girls (having 4 boys)

$$= 2000\ 4_{C_4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = 125$$

Example 11.45 : *Out of 800 families with 4 children each, how many families would be expected to have (I) 2 boys and 2 girls (II) at least one boy (III) no girl (IV) at most two girls ? Assume equal Probability for boys and girls.*

Solution : Here probability of boy and girl is equal.

$$\therefore \quad P = \frac{1}{2}, \quad q = \frac{1}{2}$$

$$n = 4, \quad N = 800$$

$\therefore$ Binomial distribution is $800\left(\frac{1}{2}+\frac{1}{2}\right)^4$

$\therefore$ I) P (2 boys and 2 girls)

$$= 800 \times 4_{C_2}\ p^2 q^{4-2}$$

$$= 800 \times \frac{4!}{2!2!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 300$$

II) P (at least 1 boy) $= P\ (1) + P\ (2) + P\ (3) + P\ (4)$

$$= 800 \times [1 - p\ (\text{no boy})]$$

$$= 800 \times [1 - p\ (0)]$$

$$= 800 \times \left[1 - 4_{C_n}\ p^0 q^{4-0}\right]$$

$$= 800 \times \left[1 - 1 \cdot \left(\frac{1}{2}\right)^4\right] = 750$$

III) Probability of at least 1 ball with X marks

$$= 1 - \text{probability of none}$$

$$= 1 - P(0)$$

$$= 1 - 6_{C_0} (P)^0 q^{6-0}$$

$$= 1 - \left(\frac{3}{5}\right)^6 = \frac{15625 - 729}{15625}$$

$$= \frac{14896}{15625}$$

$$= 0.953344$$

Note : Mean of binomial distribution is np and variance = npq.

➡ **Example 11.47 :** *Point out fallacy of the statement. The mean of binomial distribution is 3 and variance 5.*

Solution :

$$\text{Consider} = \frac{\text{Variance}}{\text{Mean}}$$

$$= \frac{npq}{np}$$

$$= q.$$

$$npq = 5, \quad np = 3$$

$$3 \times q = 5$$

$$q = 5/3$$

which is not possible

➡ **Example 11.48 :** *Mean and variance of binomial distribution are 6 and 2 respectively. Find $P(r \geq 1)$.*

Solution :

$$\text{Mean} = np = 6$$

$$\text{Variance} = npq = 2$$

$$6 \times q = 2$$

$$q = 1/3$$

$$P = 1 - \frac{1}{3} = \frac{2}{3}$$

As $np = 6 \Rightarrow n \cdot \frac{2}{3} = 6 \Rightarrow n = 9$

$$= z \cdot e^{-z}\left[1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots\right]$$

$$= z \cdot e^{-z} \cdot e^{z}$$

as $\quad e^z = 1+z+\frac{z^2}{2!}+\ldots.$

$$= z = np$$

$\therefore$ Mean $\quad \mu = np$

Variance $\quad \sigma^2 = \sum_{r=0}^{\infty} r^2 \, P(r) - \mu^2$

$$= \sum_{r=0}^{\infty} r^2 \cdot \frac{e^{-x} \, z^r}{r!} - z^2$$

$$= e^{-z}\left[\frac{1^2 \cdot z}{1!}+\frac{2^2 \cdot z^2}{2!}+\frac{3^2 z^3}{3!}+\ldots.\right] - z^2$$

$$= z \cdot e^{-z}\left[1+\frac{2z}{1!}+3\frac{z}{2!}+\ldots.\right] - z^2$$

$$= z \cdot e^{-z}\left[1+(1+1)\frac{z}{1!}+\frac{(1+2)\, z^2}{2!}+\frac{(1+3)\, z^3}{3!}+\ldots.\right] - z^2$$

$$= z \cdot e^{-z}\left[\left\{1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^2}{3!}+\ldots\right\}+\left\{\frac{z}{1!}+\frac{2z^2}{2!}+\frac{3z^3}{3!}+\ldots\right\}\right] - z^2$$

$$= z \cdot e^{-z} \cdot \left[e^z + z\left(1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots\right)\right] - z^2$$

$$= z \cdot e^{-z} \cdot \left[e^z + z \cdot e^z\right] - z^2$$

$$= z\,(1+z) - z^2 = z$$

$\therefore$ Variance $\quad \sigma^2 = \lambda = nP$

$\therefore$ Mean and Variance of Poisson's distribution = nP

11.16 Recurrence Formula for Poisson's Distribution

We have $\quad P(r) = \frac{e^{-\lambda} \lambda^r}{r!}$

$\therefore \quad P(r+1) = \frac{e^{-\lambda} \lambda^{r+1}}{(r+1)!}$

Example 11.53 : *In a Poisson's distribution. P (1) = 2P (2), find P (3).*

Solution :

$$P(3) = \frac{z^3 e^{-z}}{3!}$$

$$P(1) = 2P(2)$$

$$\frac{z \cdot e^{-z}}{1!} = \frac{2 \cdot z^2 e^{-z}}{2!}$$

$$z^2 = z$$

$$z = 1$$

$$\therefore \quad P(3) = \frac{z^3 e^{-z}}{3!} = \frac{1 \cdot e^{-1}}{3!} = 0.0613$$

Example 11.54 : *The accidents per shift in factory are given by,*

Acc / Shift	*0*	*1*	*2*	*3*	*4*	*5*
Frequency	*142*	*158*	*67*	*27*	*5*	*1*

Find a Poisson distribution.

Solution :

x_i	f_i	$f_i x_i$	
0	142	0	
1	158	158	Mean = $\frac{\sum f_i x_i}{\sum f_i}$
2	67	134	$= \frac{398}{400}$
3	27	81	
4	5	20	
5	1	5	z = 0.995
	400	398	

$$\begin{aligned} P\ (\text{at least one fatal accident}) &= 1 - \text{Probability (none)} \\ &= 1 - P\ (0) \\ &= 1 - \frac{e^{-.083} \times (.083)^0}{0\,!} \\ &= 1 - .92 \\ &= .08 \end{aligned}$$

Example 11.60 : *A manufacturer knows that the razor blades he makes contain on the average 0.5 % of defectives. He packs them in packets of 5. What is the probability that a packet picked at random will contain 3 or more faulty blades ?*

Solution : Here $p = 0.5\ \% = \dfrac{0.5}{100} = 0.005$

$n = 5, \quad z = np = 5 \times 0.005 = 0.025$

Then $\qquad P\ (r) = \dfrac{e^{-z} \cdot z^r}{r\,!}$

gives

$$\begin{aligned} P\ (r \geq 3) &= P\ (3) + P\ (4) + P\ (5) \\ &= \frac{e^{-0.025}\ (0.025)^3}{3\,!} + \frac{e^{-0.025}\ (0.025)^4}{4\,!} + \frac{e^{-0.025} (0.025)^5}{5\,!} \\ &= \frac{e^{-0.025}\ (0.025)^3}{5\,!} \left[20 + 5\ (0.025) + (0.025)^2\right] \\ &= \frac{0.975 \times 0.000015625 \times 20.125625}{120} \\ &= 0.000002555 \end{aligned}$$

Example 11.61 : *If the Probability that an individual suffers a bad reaction from a certain injection is 0.001, determine the Probability out of 2000 individuals. I) Exactly 3 II) More than 2 will suffer a bad reaction.*

Solution : Here

$$\begin{aligned} P &= .001 \\ n &= 2000 \\ \therefore \quad \lambda &= nP = .001 \times 2000 = 2 \\ \therefore \quad P\ (r) &= \frac{e^{-\lambda} \lambda^r}{r\,!} = \frac{e^{-2}\ 2^r}{r\,!} \\ \therefore \text{I)} \quad P\ (3) &= \frac{e^{-2}\ (2)^3}{3\,!} = 0.136 \times \frac{8}{6} \end{aligned}$$

The graph of the normal distribution is called the normal curve (some times known as normal probability curve or normal curve of errors). It is bell - shaped and symmetrical about the mean 'μ' as shown in the figure. The two tails of the curve extend to $+\infty$ and $-\infty$ towards the positive and negative directions of the X-axis respectively and gradually approach the X-axis without ever meeting it. The line $x = \mu$ divides the area under the normal curve above, X-axis into two equals parts. The area under the normal curve between any two given ordinates $x = x_1$ and $x = x_2$ represents the probability of values failing into the given interval. The total area under the normal curve above the x-axis is '1' i.e.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Thus $$P(x_1 < x < x_2) = \int_{x_1}^{x_2} f(x)\,dx$$

$$= \int_{x_1}^{x_2} \frac{1}{\sigma\cdot\sqrt{2\pi}}\cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ds$$

11.19 Standard Form of the Normal Distribution

If 'X' is a normal random variable with mean 'μ' and standard deviation σ, then the random variable $Z = \dfrac{X-\mu}{\sigma}$ has the normal distribution with mean '0' and standard deviation 1. The random variable Z is called the Standard normal random variable.

Thus probability density function or the normal distribution in standard form is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad (-\infty < z < \infty)$$

It is free from any parameter. This is useful to compute areas under the normal probability curve by making use of standard tables.

Area Under the Normal Curve :

The area under the normal curve is divided into two equal parts by $z = 0$. Left hand side area and right hand side area to $Z = 0$ is 0.5.

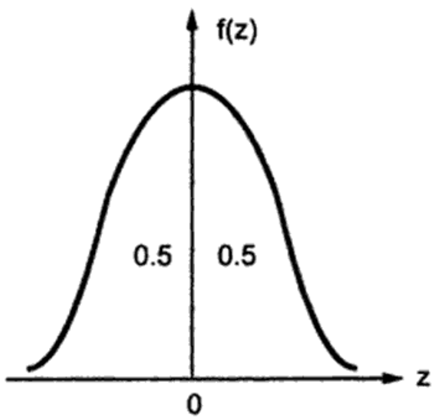


Fig. 11.2

11.20 Area Property (Normal Probability Integral)

The probability that random value x will be between $x = \mu$ and $x = x_1$ is given by

Table 11.1

Table of Area

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5259
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8906	0.8925	0.8943	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9494	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9653	0.9664	0.9871	0.9678	0.9686	0.9693	0.9699	0.9708
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9883	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9634	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9809	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9988	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9998	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9898

In each row and each column 0.5 to be subtracted

3) $$P(\mu < x < x_1) = \int_{\mu}^{x_1} y \, dx$$

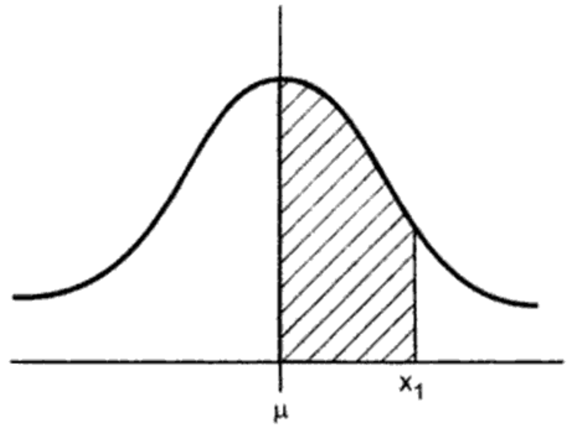


Fig. 11.5

Put $$\frac{x-\mu}{\sigma} = z, \quad \frac{dx}{\sigma} = dz$$

$$P(0 < z < z_1) = \int_0^{z_1} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$= \int_0^{z_1} (z) \, dz$$

is known as normal integral gives the area under the standard normal curve between $z = 0$ and $z = z_1$.

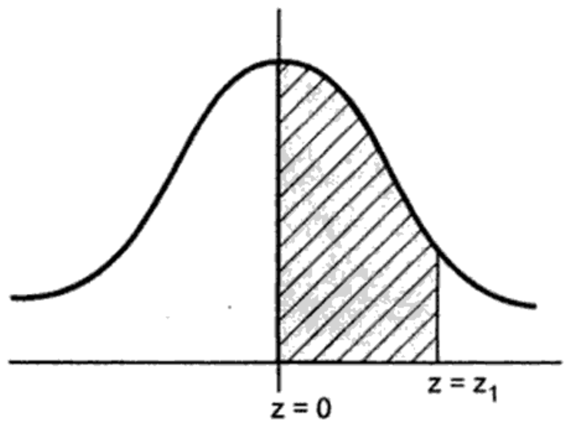


Fig. 11.6

4) $P(z_1 < z < z_2) = A(z_2) - A(z_1)$

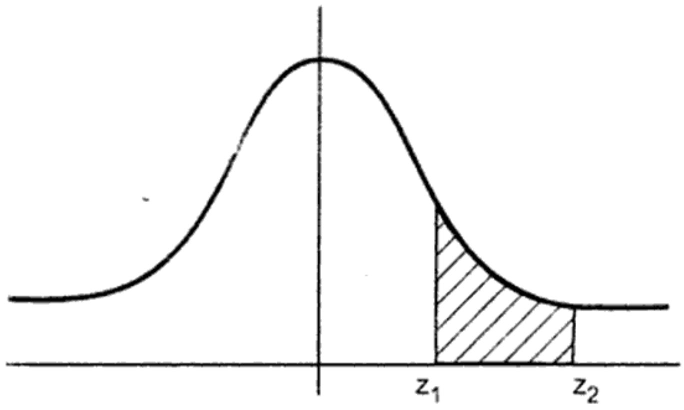


Fig. 11.7

5) $P(z > z_1) = 0.5 - A(z_1)$

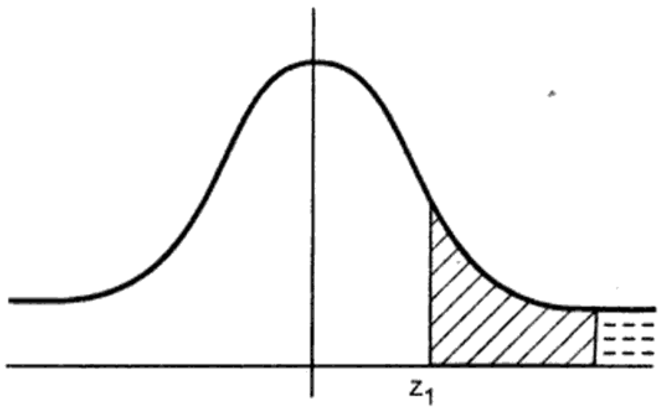


Fig. 11.8

6) $P(z < -z_1) = 0.5 - A(z_1)$

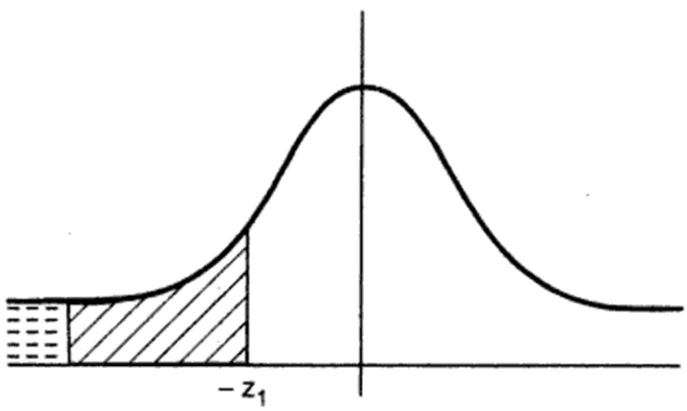


Fig. 11.9

Example 11.69 : *A sample of 100 dry battery cells tested to find the length of life produced the following results.*

Mean μ = 12, hours, Standard deviation σ = 3 hours.

Assuming the data to be normally distributed, what percentage of battery cells are expected to have life ?

i) More than 15 hours ii) Less than 6 hours iii) between 10 and 14 hours ?

Solution : Here X denotes length of life of dry battery cells.

Also $$Z = \frac{x-\mu}{\sigma} = \frac{x-12}{3}$$

i) Here x = 15,

$$\therefore \quad Z = \frac{15-12}{3} = 1$$

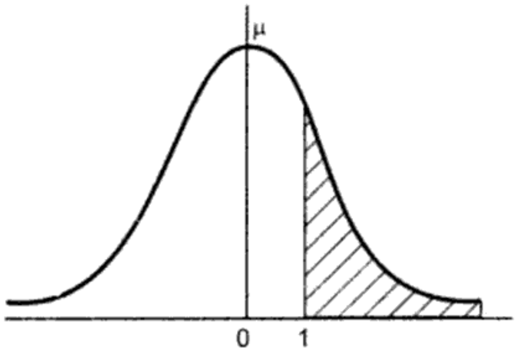


Fig. 11.18

$$\therefore \quad P(x > 15) = P(z > 1)$$
$$= P(0 < z < \infty) - P(0 < z < 1)$$
$$= 0.5 - 0.3413 = 0.1587$$
$$= 15.87\ \%$$

ii) for x = 6, $$Z = \frac{6-12}{3} = -2$$

$$\therefore \quad (x < 6) = (z < -2)$$
$$= P(z > 2)$$
$$= P(0 < z < \infty) - P(0 < z < z)$$
$$= 0.5 - 0.4772$$
$$= 0.0228 = 2.28\ \%$$

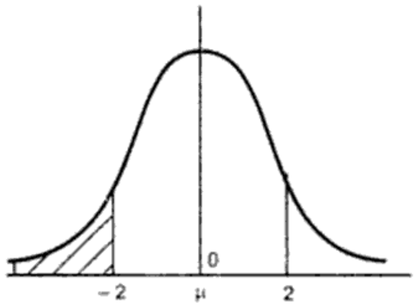


Fig. 11.19

iii) For x = 10, $$Z_1 = \frac{10-12}{3} = -\frac{2}{3} = -0.67$$

For x = 14, $$Z_1 = \frac{14-12}{3} = +\frac{2}{3} = 0.67$$

$$\therefore \quad P(10 < x < 14) = P(z_1 < z < z_2)$$
$$= P(-0.67 < z < 0.67)$$
$$= 2\,P(0 < z < 0.67)$$
$$= 2 \times 0.2487$$
$$= 0.4974 = 49.74\ \%$$

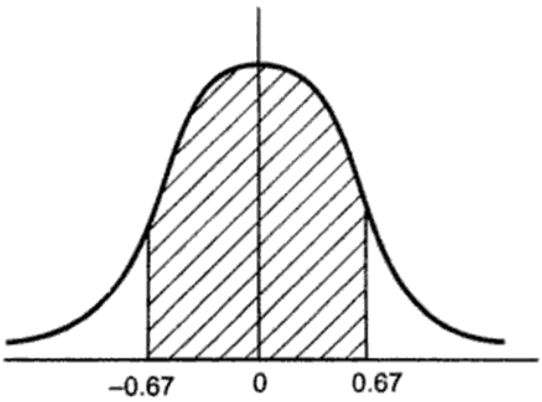


Fig. 11.20

Example 11.71 : *Five thousand students appeared in an examination carrying a maximum of 100 marks. It was found that the marks were normally distributed with mean = 39.5 and standard deviation = 12.5. Determine approximately the number of candidates who secured a first class for which a minimum of 60 marks is necessary. The following table gives deviation from mean and proportion of whole area of normal curve lying to the left of the ordinate at the deviation* $\frac{x}{\sigma}$.

$\frac{x}{\sigma}$	1.5	1.6	1.7	1.8
A	.93319	.94520	0.95543	.96407

Solution : Here $\mu = 39.5$

$\sigma = 12.5$

For x = 60 $z = \frac{60 - 39.5}{12.5} = 1.64$

$\therefore$ Required area is for z = 1.64

From table

area for z = 1.6 is .94520

area for z = 1.7 is .99543

$\therefore$ Difference for 0.1 is 0.01023

$\therefore$ Difference for 0.04 = .004092

$\therefore$ Area for z = 1.64 = 0.94520 + .004092

= 0.949292

This is total area lying to left of z = 1.64

$\therefore$ Area right to z = 1.64 = 1 − 0.949292

= .050708

(As total area under normal curve is 1)

$\therefore$ Required Number of students = 5000 × .050708

= 253.540

= **253 students approximately**

Example 11.72 : *Of the 1000 students of a college, the mean height is 68 inches, with a standard deviation of 5 inches. Between what limits will the middle 60 % of the heights lie ? Assume the distribution to be normal.*

Solution : The middle 60 % of the height means 30 % on either side of the mean. Now the probability 0.30 will lie between the values 0.8 and 0.9 from z $\left(=\frac{x}{\sigma}\right)$ tables. By interpolation or from accurate tables, we have $z = \left(\frac{x}{\sigma}\right) = 0.8418$ for a probability of 0.30.

$$\therefore \quad x = \sigma \times 0.8418$$
$$= 5 \times 0.8418$$
$$= 4.2090 \text{ inches}$$

And the limits shall be 68 + 4.2090 and 68 − 4.2090 i.e. 72.2090 inches and 63.7910 inches.

Note : Here the given number 1000 has not been used.

Example 11.73 : *For a normal distribution when mean $\bar{x} = 1$, S.D. = 3. Find the probabilities for the intervals :*

i) $3.43 \le x \le 6.19$

ii) $-1.43 \le x \le 6.19$

Solution : We have $z\left(=\frac{x}{\sigma}\right) = \frac{x - \bar{x}}{\sigma} = \frac{x-1}{3}$

i) When x = 3.43,

$$z_1 = \frac{-1.43 - 1}{3} = -0.81$$

when x = 6.19 $\quad z_2 = \frac{6.19 - 1}{3} = 1.73$

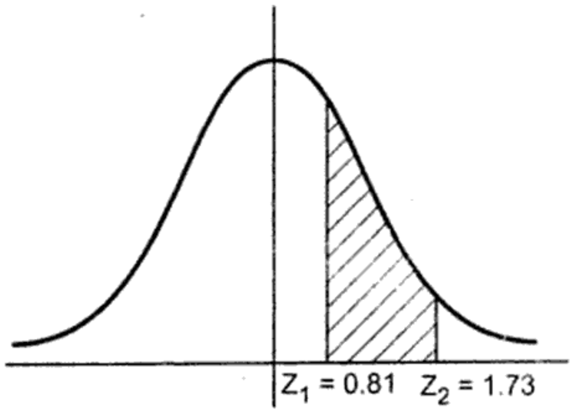


Fig. 11.23

Required probability P (− 3.43 ≤ x ≤ 6.19)

$= P(0 < z_1 < -1.73) - P(0 < z_2 < 0.81)$

$= 0.4582 + 0.2910$ (see table) $= 0.1672$

ii) when x = − 1.43

$$z_1 = \frac{1.43 - 1}{3} = -0.81$$

when x = 6.19, $\quad z_2 = \frac{6.19 - 1}{3} = 1.73$

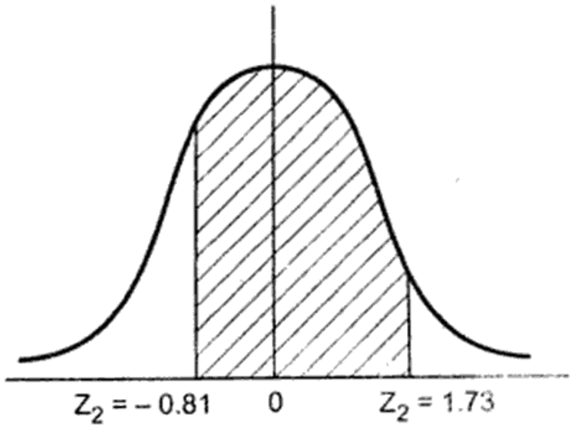


Fig. 11.24

Required probability P (− 1.43 ≤ x ≤ 6.19)

$$= P(0 < z_1 < -0.81) - P(0 < z_2 < 1.73)$$
$$= 0.2910 + 0.4582 = 0.7492$$

Example 11.74 : *The mean inside diameter of a sample of 200 washers produced by a machine is 0.502 cm and standard deviation is 0.005 cm. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 0.496 to 0.508 cm otherwise washers are considered defective. Determine the percentage of defective washers produced by the machine, assuming the diameters are normally distributed.*

Solution : Here $\mu = 0.502, \ \sigma = 0.005$

For $x = 0.496$ $\quad z_1 = \dfrac{0.496 - 0.502}{0.005} = -1.2$

For $x = 0.508$ $\quad z_2 = \dfrac{0.508 - 0.502}{.005} = 1.2$

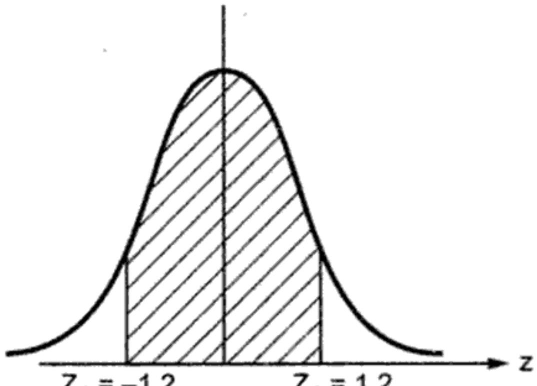


Fig. 11.25

$$
\begin{aligned}
\therefore P(0.496 < x < 0.508) &= P(z_1 < z < z_2) \\
&= P(-1.2 < z < 1.2) \\
&= 2\,P(0 < z < 1.2) \\
&= 2 \times 0.3849 \\
&= 0.7698 \\
&= 75.98\ \%
\end{aligned}
$$

$\therefore$ Percentage of defective washers

$$
\begin{aligned}
&= 100 - 76.98 \\
&= 23.02\ \%
\end{aligned}
$$

Example 11.75 : *Local authorities in a certain city install 2000 electric bulbs on the streets. If the bulbs have average life of 1000 burning hours with standard deviation of 200 hours. Assuming normality find*

i) What number of bulbs might be expected to fail in first 700 hours.

ii) After what period of burning hours would it be expected that 10 % of bulbs would have failed.

Solution : Here $\mu = 1000, \quad \sigma = 200$

For $x = 700$, $\quad z = \dfrac{700 - 1000}{200} = -1.5$

$$
\begin{aligned}
\therefore \quad P(x < 700) &= P(z < -1.5) \\
&= P(z > 1.5) \\
&= 0.5 - A(1.5)
\end{aligned}
$$

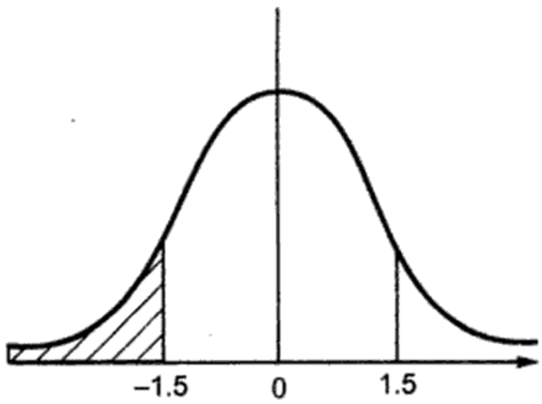


Fig. 11.26

$$\therefore \qquad \text{Mean } \mu = a + \frac{\sum f_i d_i}{\sum f_i} = 8.56 + \frac{0.11}{42} = 8.56262$$

$$\text{Standard deviation } \sigma = \sqrt{\frac{1}{N}\sum f_i d_i^2 - \left(\frac{\sum f_i d_i}{N}\right)^2}$$

$$= \sqrt{\frac{.0133}{42} - \left(\frac{0.11}{42}\right)^2}$$

$$\sigma = 0.0176$$

$\therefore$ Equation of normal curve

$$f(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty \le x \le \infty$$

where $\mu = 8.56262, \ \sigma = 0.0176$

Exercise 11.2

Problems on Binomial Distribution

1. *The probability that a man aged 60 years will live for 70 years is $\frac{1}{10}$. Find the probability that out of 5 men selected at random 2 will live for 70 years.*

   Hint : $p = \frac{1}{10}$; $q = \frac{9}{10}$, $n = 5$, $r = 3$, *then use* $n_{C_r}(p)^r (q)^{n-r}$ **[Ans. : 0.054675]**

2. *During war, one ship out of nine was sunk on an average in making a certain voyage. What was the probability that exactly '3' out of a convoy of 6 ships would arrive safely ?* **[Ans. : $\frac{10240}{9^6}$]**

3. *Eight dice are rolled. Calling a '5' or a '6' as success, find the probability of getting :*

   i) 3 successes **[Ans. : $\frac{8!}{5!\,3!}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^5$]**

   ii) at most 3 successes **[Ans. : $\frac{1}{3^3}\left[2^8 + 8(2)^7 + 28(2)^6 + 56(2)^5\right]$]**

4. *A bag contains 10 balls, each marked with one of the numbers 0 to 9. If four balls are drawn from the bag. Find the probability that none is marked '0'.* **[Ans. : $\left(\frac{9}{10}\right)^4$]**

Problems on Poisson Distribution

16. *Fit a Poisson distribution to the following :*

x	0	1	2	3	4
f	192	100	24	3	1

Hint : *Mean* $z = \dfrac{\sum fx}{\sum f} = \dfrac{|6|}{320} = 0.5$ *(Approximately)*

Then use $p\,(r) = \dfrac{e^{-z} \cdot z^r}{r!}$ **[Ans. :** $\dfrac{e^{-0.5}\,(0.5)^r}{r\,!}$**]**

17. *Using Poisson distribution, find the probability that the ace of spades will be drawn from a pack of well shuffled cards at least once in 104 consecutive trials.*

Hint : $p = \dfrac{1}{52}$ *(Probability of ace of spades),* $n = 104 \;\therefore\; z = np = 2$, *use* $p\,(r) = \dfrac{e^{-z} \cdot x^r}{r!} = \dfrac{1}{e^2} \cdot \dfrac{(2)^r}{r!}$

P (at least once) $= P\,(1) + P\,(2) + \ldots + P\,(104)$

$$= 1 - P\,(0) = 1 - \frac{1}{e^2} = 0.864$$

18. *Fit a Poisson distribution to the following frequency distribution and compare the theoretical frequencies with observed frequencies.*

x	0	1	2	3	4	5
f	158	160	60	25	10	2

19. *A source of liquid is known to contain bacteria with the mean number of bacteria per cubic centimeter equal to 2. Five 1 c.c. test tubes are filled with the liquid, assuming that Poisson distribution is applicable, calculate the probability that all test tubes show growth.* **[Ans. : 0.036]**

20. *An insurance company found that only 0.01 % of the population is involved in a certain type of accident each year. If 1000 policy holders are randomly selected from the population. What is the probability that not more than two of its clients are involved in such an accident next year.*

[Ans. : 0.9998]

21. *A book of 600 pages contains 40 printing mistakes. Assuming that errors are randomly distributed throughout the book and x, the number of errors per page has a Poisson distribution what is the probability that 10 pages selected at random will be free of errors ?* **[Ans. : 0.51]**

22. *The probability that a man aged 35 years will die before reaching the age of 40 years may be taken as 0.018. Out of a group of 400 men now aged 35 years, what is the probability that 2 men will die within next 5 years ?* **[Ans. : 0.01936]**

36. *The accidents per shift in a factory is given by the table :*

Accidents 'x' per shift	0	1	2	3	4	5
Frequency	142	158	67	27	5	1

Fit a Poisson distribution to the above and calculate theoretical frequencies.

Hint : $z = \dfrac{\sum f(x)}{\sum f} = 0.995$

$$P(r) = \frac{e^{-z}(z)^r}{r!}$$

Problems on Normal Distribution

37. *An aptitude test for selecting officers in a bank is conducted on 1000 candidates. The average score is 42 and the standard deviation of score is 24. Assuming normal distribution for the scores, find :*

    i) *the number of candidates whose scores exceed 60.* **[Ans. : 252]**

    ii) *the number of candidates whose scores lie between 30 and 60.* **[Ans. : 533]**

38. *The mean weight of 500 male students in a certain college is 151 lbs and the standard deviation is 15 lbs. Assuming the weights are normally distributed, in how many students weight between 120 and 155 lbs ?* **[Ans. : 300]**

39. *For a normal distribution, N = 300, μ = 75 and σ = 15. How many values lie between x = 60 and x = 70 ?*

    The area under the normal curve for various values of 2 is given as,

Z	Area
0.33	0.12930
0.34	0.13307
1.0	0.34134

[Ans. : 63 approximately]

40. *In a normal distribution N = 700, mean $\bar{x}$ = 95, σ = 15. How many values lie between 80 and 90 ?* **[Ans. : 149]**

41. *Assume that the weights of 1000 school children follow a normal law. The mean weight is 48 lbs, and the standard deviation is 6 lbs. Find the number of children having their weight :*

    i) *Between 36 lbs and 42 lbs* **[Ans. : 136]**

    ii) *Between 42 lbs and 57 lbs* **[Ans. : 624]**

    iii) *less than 30 lbs* **[Ans. : 1]**

    iv) *More than 60 lbs* **[Ans. : 23]**

11.22 Chi Square Distribution

Chi square distribution is used mainly in 1) Testing hypothesis 2) Testing independence of attributes 2) Testing the goodness of fit of a model.

The Chi square distribution variable or simply Chi square variable is denoted by X_n^2, Here n is the parameter of the distribution also called as "n degrees of freedom".

11.23 Definition of χ_n^2

The variate χ_n^2 is defined as the sum of squares of n independent standard normal variables [N (0, 1)]

i.e. If X_1, X_2 .... X_n be n independent N (0, 1) variables then the Chi-square distribution is given by

$$\boxed{Y = \chi_n^2 = \sum_{i=1}^{n} (X_i)^2}$$

where n = degrees of freedom.

11.24 Additive Property of Chi-square Distribution

If Y_1 and Y_2 are independent Chi-square variates with n_1 and n_2 degrees of freedom respectively, then $Y_1 + Y_2$ has chi-square distribution with $(n_1 + n_2)$ degrees of freedom.

11.25 Definition of Hypothesis

Hypothesis is the statement or assertion about the statistical distribution or unknown parameter of statistical distribution.

Hypothesis is a claim to be tested about the population parameters such as mean variance. For e.g. : 1) Proportion of fail students in two subjects in any college 2) Comparison of F.E. results in any two or more Engineering colleges. These claims stated in terms of population parameters or distribution are called hypothesis.

11.26 Null Hypothesis

According to the famous statistician R.A.Fisher Null hypothesis is denoted by H_0 and is defined as a hypothesis of "no difference".

For e.g. : 1) If there is no difference between the failure percentage of two subjects then $H_0 : \mu_1 = \mu_2$ i.e. there is no difference between two population means.

11.27 Alternative Hypothesis

Alternative hypothesis is denoted by H_1 and is defined as a complementary hypothesis to null hypothesis.

Table 11.1

Degrees of freedom	Distribution of χ^2	
	5 %	1 %
1	3.841	6.635
2	5.991	9.210
3	7.815	11.345
4	9.488	13.277
5	11.070	15.086
6	15.592	16.812
7	15.067	18.475
8	15.507	20.090
9	16.919	21.666
10	18.307	23.209
11	19.675	24.725
12	21.026	26.217
13	22.362	27.668
14	23.685	29.141
15	24.996	30.578
16	26.296	32.000
17	27.587	33.409
18	28.869	34.191
19	30.144	36.191
20	31.410	37.566
21	32.671	38.932
22	33.924	40.289
23	35.172	41.638
24	36.415	42.980
25	37.652	44.314
26	38.885	45.642

27	40.113	46.963
28	41.337	48.278
29	42.557	49.588
30	43.773	50.892
40	55.759	63.691
60	79.082	88.379
∞	–	–

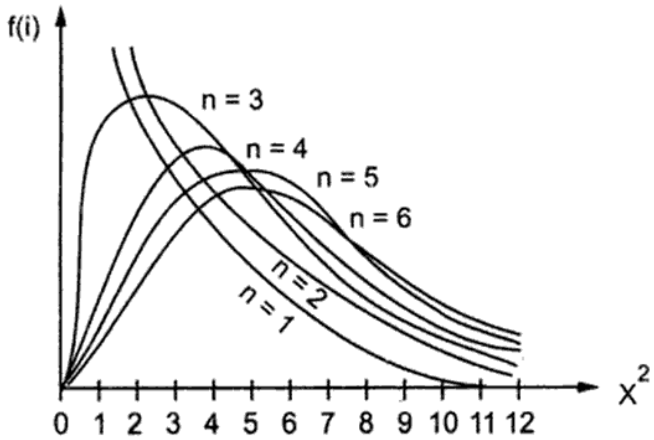


Fig. 11.29

f) Test for goodness of fit of χ^2 distribution :

Consider a frequency distribution, we try to fit some probability distribution.

Let H_0 : Fitting of the probability distribution to given data is proper.

As the test is based on χ^2 distribution. $\therefore$ known as χ^2 test of goodness of fit.

Suppose O_1 O_2 .... O_i .... O_k be the observed frequencies and e_1 e_2 ... e_i ... e_k be the expected frequencies or theoretical frequencies. There is no significant difference between observed and theoretical (expected) frequencies.

Let P = number of parameters estimated for fitting the probability distribution.

$$N = \sum_{i=1}^{k} O_i = \sum_{i=k}^{k} e_i$$

If H_0 is true then the static

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$= \sum_{i=1}^{k} \frac{O_i^2 - 2\,e_i\,O_i + e_i^2}{e_i}$$

Example 11.79 : *The table below gives the number of accidents that occurred in certain factory on various days of week.*

Days	Sun	Mon	Tue	Wed	Thurs	Fri	Sat
Accidents	6	4	9	7	8	10	12

Test at 5 % level of significance whether accidents are uniformly distributed over the days.

Solution : Let H_0 : The distribution of accidents over the days in uniform.

$$P_i = \frac{1}{7}$$

$$N = \sum O_i = 56$$

$$\therefore \quad E(x) = \text{Expected accidents}$$

$$= N \cdot P_i$$

$$= 56 \cdot \frac{1}{7}$$

$$= 8$$

x_i	O_i	e_i	$O_i - e_i$	$(O_i - e_i)^2$
1	6	8	– 2	4
2	4	8	– 4	16
3	9	8	1	1
4	7	8	– 1	1
5	8	8	0	0
6	10	8	2	4
7	12	8	4	16
				42

Here p = 0, k = 7

$$\therefore \quad \chi^2_{k-p-1} = \sum_{i=0}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$\chi^2_6 = \sum_{i=0}^{7} \frac{(O_i - e_i)^2}{e_i}$$

$$= \frac{42}{8}$$

$$= 5.25 \quad \text{(observed)}$$

As expected frequency $E(x_i) = N \cdot p_i$ thus we prepare the table.

x_i	O_i	P_i	$e_i = N \cdot P_i$	$O_i - e_i$
0	2	1/32	7	– 5
1	5	5/32	35	– 30
2	20	10/32	70	– 50
3	60	10/32	70	– 10
4	100	5/32	35	65
5	37	1/32	7	30
	224			

$k = 6,\ p = 0$

$$\therefore \quad \chi^2_{k-p-1} = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$\chi^2_5 = \frac{(-5)^2}{7} + \frac{(-30)^2}{35} + \frac{(-50)^2}{70} + \frac{(-10)^2}{70} + \frac{(+65)^2}{35} + \frac{(30)^2}{7}$$

$$= \frac{25}{7} + \frac{900}{35} + \frac{2500}{70} + \frac{100}{70} + \frac{4225}{35} + \frac{900}{7}$$

$$= 315.7142 \quad \text{(calculated)}$$

$$\chi^2_{5,\,0.05} = 11.07 \quad \text{(Table value)}$$

$$\therefore \quad \chi^2_5 > \chi^2_{5,\,0.05}$$

$\therefore$ H_0 is rejected.

Example 11.82 : *The following table of frequencies of seeds were observed in experiment on pea breeding*

Round and Green	*Wrinkled and Green*	*Round and Yellow*	*Wrinkled and Yellow*	*Total*
222	*120*	*32*	*150*	*524*

Theory predicts that the frequencies should be in proportion 8 : 2 : 2 : 1. Examine the correspondence between theory experiment.

Solution : Let H_0 : There is a good correspondence between theory and experiment.

Here N = 524

As the proportion is 8 : 2 : 2 : 1

$\therefore$ Their probabilities are

O_i	P_i	$e_i = N \cdot P_i$	$O_i - e_i$
34	9/16	36	– 2
10	3/16	12	– 2
20	4/16	16	4

Here k = 3, p = 0

$$\chi^2_{k-p-1} = \sum_{i=0}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$= \frac{(-2)^2}{36} + \frac{(-2)^2}{12} + \frac{(4)^2}{16}$$

$$\chi^2_2 = 1.444 \text{ (Calculated value)}$$

Now $\chi^2_{2,\,0.05} = 5.991$ (Table value)

as $\chi^2_2 < \chi^2_{2,\,0.05}$ we accept H_0

Example 11.84 : *One hundred samples were drawn from a production process each after 5 hours. The number of defectives were noted from these samples. A Poisson distribution by estimating the parameter m was fitted to these data. The results obtained are as follows*

Number of defectives	*Number of samples observed*	*Number of samples expected*
0	*63*	*60.65*
1	*28*	*30.33*
2	*6*	*7.58*
3	*2*	*1.26*
4	*1*	*0.16*
5 and above	*0*	*0.02*

Test the goodness of fit of Poisson distribution (Use 5 % level of significance)

Solution : Let H_0 : Fitting of Poisson distribution is good.

Here as the expected frequencies are less than five ∴ we pool the data

Thus

O_i	e_i	$O_i - e_i$
63	60.65	2.35
28	30.33	– 2.33
9	9.02	– 0.02

Here k = 5, p = 0

$$\chi^2_{k-p-1} = \sum_{i=0}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$\chi^2_4 = \left[\frac{(35-34.24)^2}{34.24} + \frac{(42-50)^2}{50} + \frac{(85-88.04)^2}{88.04}\right.$$

$$\left. + \frac{(48-45.11)^2}{45.11} + \frac{(40-32.61)^2}{32.61}\right] + \left[\frac{(28-28.76)^2}{28.76} + \frac{(50-42)^2}{42}\right.$$

$$\left. + \frac{(77-73.96)^2}{73.96} + \frac{(35-37.89)^2}{37.89} + \frac{(20-27.39)^2}{27.39}\right]$$

$$\chi^2_4 = 7.14 \text{ Calculated}$$

$$\chi^2_{4,0.05} = 9.488 \text{ (Table value)}$$

$$\therefore \quad \chi^2_4 < \chi^2_{4,0.05}$$

∴ We accept this hypothesis

i.e. the grade distribution for males is same as that of females.

Example 11.86 : *A random group of 40 people younger than 50 years was given a flu shot, and a second random group of 60 people 50 years or older was given the same flu shot. Each member of the groups was classified according to whether the member did not get the lfu (N), had a mild case of the flu (M), or had a severe case of the flu (S). The frequencies in each group are as indicated in the following table.*

		Reaction			
		N	*M*	*S*	*Totals*
Age	*Under 50 years*	30	6	4	40
	50 years or older	36	12	12	60
	Totals	66	18	16	100

Use a Chi-square random variable to test, at the 0.05 significance level, the hypothesis that the reactions to the shot are the same in each group.

Solution : By pooling the subjects under 50 years and those 50 years and over in each reaction group, we get the following estimated probabilities.

$$P_N = \frac{66}{100} 0.66, \; P_M = \frac{18}{100} = 0.18, \; P_S = \frac{16}{100} = 0.16$$

The expected frequencies mP_j for the m = 40 subjects under 50 years are

$mP_N = 40 \times 0.66 = 26.4$, $mP_N = 40 \times 0.18 = 7.2$, $mP_S = 40 \times 0.16 = 6.4$

and the expected frequencies nP_j for the n = 60 subjects 50 years and over are

$nP_N = 60 \times 0.66 = 39.6$, $nP_M = 60 \times 0.18 = 10.8$, $nP_S = 60 \times 0.16 = 9.6$

The corresponding Chi-square test value is

$$\chi^2 = \frac{(30-26.4)^2}{26.4} + \frac{(6-7.2)^2}{7.2} + \frac{(4-6.4)^2}{6.4} + \frac{(36-39.6)^2}{39.6} + \frac{(12-10.8)^2}{10.8} + \frac{(12-9.6)^2}{9.6}$$

$$= 2.65$$

Here k = 3 p = 0

$\therefore \quad \chi_2^2 = 2.65$ (Calculated)

$\chi_{2,0.05}^2 = 5.991$ (Table value)

As $\quad \chi_2^2 < \chi_{2,0.05}^2$

We accept the hypothesis that the reactions to the shot are same in each group.

Example 11.87 : *Salaries for 200 males and 300 females at a certain company are as indicated in the following frequency table, where the notation [a, b) means a salary greater than or equal to a but less than b.*

Salaries in thousands of dollars

		1 [20, 30)	2 [30, 40)	3 [40, 50)	4 [50, 60)	5 [60, –)	Totals
Gender	Male	20	34	46	60	40	200
	Female	45	78	90	62	25	300
	Totals	65	112	136	122	65	500

Use the Chi-square random variable to test, at the 0.05 significance level, the hypothesis the salary distributions are the same.

Solution : By pooling the male and female frequencies in each salary grade, we obtain the following estimated probabilities :

$$P_1 = \frac{65}{500} = 0.13,\ P_2 = \frac{112}{500} = 0.224,\ P_3 = \frac{136}{500} = 0.272,\ P_4 = \frac{122}{500} = 0.244,\ P_5 = \frac{65}{500} = 0.13$$

The expected frequencies mP_j for the m = 200 males are :

$mP_1 = 200 \times 0.13 = 26$, $mP_2 = 200 \times 0.224 = 44.8$, $mP_3 = 200 \times 0.272 = 54.4$,

$mP_4 = 200 \times 0.244 = 48.8$, $mP_5 = 200 \times 0.13 = 26$

and the expected frequencies nP_j for the n = 300 females are :

$nP_1 = 300 \times 0.13 = 39$, $nP_2 = 300 \times 0.224 = 67.2$, $nP_3 = 300 \times 0.272 = 81.6$,

$nP_4 = 300 \times 0.244 = 73.2$, $nP_5 = 300 \times 0.13 = 39$

The corresponding Chi-square test value is (Here k = 5, p = 0)

$$\chi^2_{k-p-1} = \frac{(20-26)^2}{26} + \frac{(34-44.8)^2}{44.8} + \frac{(46-54.4)^2}{54.4} + \frac{(60-48.8)^2}{48.8}$$

$$+ \frac{(40-26)^2}{26} + \frac{(45-39)^2}{39} + \frac{(78-67.2)^2}{67.2} + \frac{(90-81.6)^2}{81.6}$$

$$+ \frac{(62-73.2)^2}{73.2} + \frac{(25-39)^2}{39}$$

$$\chi^2_4 = 25.66 \text{ (Calculated)}$$

Now $\chi^2_{4,0.05} = 9.488$ (Table value)

$\therefore \quad \chi^2_4 > \chi^2_{4,0.05}$

$\therefore$ We reject the hypothesis that the salary distribution for males and females is same.

11.32 Chi-square Test for Independent Attributes

Contingency Table of Probabilities :

Let X and Y be attributes associated with individuals in population. Suppose that X can be classified into mutually disjoint categories $A_1, A_2, \ldots. A_m$ and Y can be classified into mutually disjoint categories $B_1, B_2 \ldots B_n$. The probability $P(A_i B_j)$ is denoted by P_{ij}. The following table 11.2 is an mxn contingency table of probabilities where P_{ij} is in the i^{th} and j^{th} column. Sum of all P_{ij}'s is 1.

	B_1	B_2	B_3	....	B_n	
A_1	P_{11}	P_{12}	P_{13}	....	P_{1n}	$P_{1.} = P(A_1)$
A_2	P_{21}	P_{22}	P_{23}	....	P_{2n}	$P_{2.} = P(A_2)$
A_3	P_{31}	P_{32}	P_{33}	....	P_{3n}	$P_{2.} = P(A_2)$
:	:	:	:		:	
:	:	:	:		:	
A_m	P_{m1}	P_{m2}	P_{m3}		P_{mn}	$P_{m.} = P(A_m)$
	$P_{.1}$ $= P(B_1)$	$P_{.2}$ $= P(B_2)$	$P_{.3}$ $P(B_3)$		$P_{.n}$ $P(B_n)$	

Here degrees of freedom = (m − 1) (n − 1).

Example 11.88 : *A random group of 300 males was cross-classified according to age and total cholesterol level, as indicated in the table below.*

Total cholesterol

		Under 200 Low	200-239 Medium	240 or higher High	Totals ($f_{i.}$)
	20-34	66	24	8	98
Age	35-54	54	48	22	124
	55-75	18	50	10	78
	Totals ($f_{.j}$)	138	122	40	300

Use the Chi-square random variable to test, at the 0.01 significance level, the hypothesis that the attributes of age and cholesterol level are independent.

Solution : The contingency table has m = 3 rows, where age bracket 20-34 corresponds to i = 1, 35-54 corresponds to i = 2, and 55-74 corresponds to i = 3. There are n = 5 columns, where j = 1 corresponds to low cholesterol level, j = 2 corresponds to medium, and j = 3 corresponds to high. The estimated row probabilities are :

$$P_{1.} = \frac{98}{300}, \; P_{2.} = \frac{124}{300}, \; P_{3.} = \frac{78}{300}$$

and the estimated column probabilities are :

$$P_{.1} = \frac{138}{300}, \; P_{.2} = \frac{122}{300}, \; P_{.3} = \frac{40}{300}$$

The expected cross-classification frequency estimates, where N = 300, are :

$N \times P_{1.} \times P_{.1} = 45.08,$ $\quad N \times P_{1.} \times P_{.2} = 39.853,$ $\quad N \times P_{1.} \times P_{.3} = 13.067,$

$N \times P_{2.} \times P_{.1} = 57.04,$ $\quad N \times P_{2.} \times P_{.2} = 50.427,$ $\quad N \times P_{2.} \times P_{.3} = 16.533,$

$N \times P_{3.} \times P_{.1} = 35.88,$ $\quad N \times P_{3.} \times P_{.2} = 31.72,$ $\quad N \times P_{3.} \times P_{.3} = 10.4$

The test value of the Chi-square statistic is :

$$\chi^2 = \frac{(66-45.08)^2}{45.08} + \frac{(24-39.853)^2}{39.853} + \frac{(8-13.067)^2}{13.067} + \frac{(54-57.04)^2}{57.04}$$

$$+ \frac{(48-50.427)^2}{50.427} + \frac{(22-16.533)^2}{16.533} + \frac{(18-35.88)^2}{35.88}$$

$$+ \frac{(50-31.72)^2}{31.72} + \frac{(10-10.4)^2}{10.4}$$

$$= 39.53 \quad \text{(Calculated)}$$

There are (m − 1) (n − 1) = 2×2 = 4 degrees of freedom.

Now $\chi^2_{4,\,0.05} = 9.488$

$\therefore \quad \chi^2_4 > \chi^2_{4,\,0.05}$

$\therefore$ We reject the hypothesis that the age and total cholesterol level are independent.

Example 11.89 : *A random group of 800 eligible voters was cross-classified according to annual income and party affiliation, as indicated in the following table. In the table, [20, 40) signifies income of at least $20,000 but less than $40,000; [40, 60) means at least $40,000 but less than $60,000 and [60,000, –) means $60,000 over. Apply a Chi-square test for independent of annual income and party affiliation at the 0.05 significance level.*

Annual income

		[20, 40)	[40, 60)	[60, –)	Totals ($f_{i.}$)
Party	Democratic	125	225	70	420
	Republican	60	200	120	380
	Totals ($f_{.j}$)	185	425	190	800

Solution : The contingency table has m = 2, where Democratic affiliation corresponds to i = 1 and Republican affiliation corresponds to i = 2, there are n = 3 columns, where j = 1 corresponds to the salary range [20, 40), j = 2 corresponds to [40, 60), and j = 3 corresponds to [60, –). The estimated row probabilities are :

$$P_{1.} = \frac{420}{800} = 0.525, \quad P_{2.} = \frac{380}{800} = 0.475$$

and the estimated column probabilities are :

$$P_{.1} = \frac{185}{800}, \; P_{.2} = \frac{425}{800}, \; P_{.3} = \frac{190}{800}$$

The expected frequency estimates are : N = 800

$$N \times P_{1.} \times P_{.1} = 800 \times 0.525 \times \frac{185}{800}, \quad N \times P_{1.} \times P_{.2} = 223.125, \quad N \times P_{1.} \times P_{.3} = 99.75$$

$$N \times P_{2.} \times P_{.1} = 87.875, \quad N \times P_{2.} \times P_{.2} = 201.875, \quad N \times P_{2.} \times P_{.3} = 90.25$$

The test value of the Chi-square statistic is :

$$\chi^2 = \frac{(125 - 97.125)^2}{97.125} + \frac{(225 - 223.125)^2}{223.125} + \frac{(70 - 99.75)^2}{99.75}$$

$$= + \frac{(60 - 87.875)^2}{87.875} + \frac{(200 - 201.875)^2}{201.875} + \frac{(120 - 90.25)^2}{90.25}$$

$$= 35.56 \text{ Calculated}$$

There are (m – 1) (n – 1) = 1×2 = 2 degrees of freedom.

$P(5) = \frac{4}{36} = P(9), \quad P(6) = \frac{5}{36} = P(8), \quad P(7) = \frac{6}{36}$

where $n = 360$

Sum	2	3	4	5	6	7	8	9	10	11	12
Expected Frequency	10	20	30	40	50	60	50	40	30	20	10

$$\chi^2 = \frac{(8-10)^2}{10} + \frac{(24-20)^2}{20} + \frac{(35-30)^2}{30} + \frac{(37-40)^2}{40} + \frac{(44-50)^2}{50} + \frac{(65-60)^2}{60} + \frac{(51-50)^2}{50}$$

$$+ \frac{(42-40)^2}{40} + \frac{(26-30)^2}{30} + \frac{(14-20)^2}{20} + \frac{(14-10)^2}{10} \approx 7.45 < \chi^2_{11\,0.05} \Rightarrow \text{ accepted}$$

2. *Over the years, the grades in a certain college professor's class are typically as follows : 10 percent As, 20 percent Bs, 50 percent Cs, 15 percent Ds, and 5 percent Fs. The grades for her current class of 100 are 16 As, 28 Bs, 46 Cs, 10 Ds, and 0 Fs. Test the hypothesis that the current class is typical by a Chi-square test at the 0.05 significance level.*

   Hint :

Grade	A	B	C	D	F
H_0 : probability =	0.1	0.2	0.5	0.15	0.05
Expected frequency	10	20	50	15	5
Actual frequency	16	28	46	10	0

The Chi-square test sum is

$$\chi^2_4 = \frac{(16-10)^2}{10} + \frac{(28-20)^2}{20} + \frac{(46-50)^2}{50} + \frac{(10-15)^2}{15} + \frac{(0-5)^2}{5} \approx 13.79 > \chi^2_{4,\,0.05} = 9.48$$

with $5 - 1 = 4$ *degrees of freedom we reject the hypothesis that the class is typical.*

3. *A bag is supposed to contain 20 percent red beans and 80 percent white beans. A random sample of 50 beans from the bag contains 16 red and 34 white. Apply the Chi-square test at the 0.05 significance level to either reject or not reject the hypothesis that the contents are as advertised.*

   Hint :

   If the contents are 20 percent red and 80 percent white, then P (red) = $p_1 = 0.2$, *and P (white) =* $p_2 = 0.8$; $np_1 = 50 \times 0.2 = 10$ *and* $np_2 = 50 \times 0.8 = 40$. *The test Chi-square value is*

$$\chi^2_1 = \frac{(16-10)^2}{10} + \frac{(34-40)^2}{40} = 4.5 > \chi^2_{1\,0.05} = 3.84$$

   $\therefore$ *rejected.*

4. *A coin is tossed 100 times, resulting in 60 heads (H) and 40 tails (T). Apply the Chi-square test at the 0.05 significance level to either reject or not reject the hypothesis that the coin is fair.*

Hint : $P(H) = p_1 = 0.5$, $P(T) = p_2 = 0.5$. We have $n = 100$, so $np_1 = 100 \times 0.5 = 50 = np_2$. The test Chi-square value is

$$\chi_1^2 = \frac{(60-50)^2}{50} + \frac{(40-50)^2}{50} = 4 > \chi^2_{1\,0.05} = 3.84$$

$\therefore$ rejected.

5. *The table below gives the number of accidents that occurred in the certain factory on the various days of a particular week.*

Days of week	Sun	Mon	Tues	Wed	Thurs	Fri	Sat
No. of accidents	6	4	9	7	8	10	12

Test at 5 % level whether accidents are uniformly distributed over the different days.

[Ans. : $\chi_6^2 = 5.25$; χ_6^2; 0.05 = 15.592, accept H_0]

6. *The following is a 2×2 contingency table :*

Eye colour in father	Eye colour in son	
	Not light	Light
Not light	23	15
Light	15	47

Test whether the eye colour in sone is associated with the eye colour in father.

[Ans. : $\chi_1^2 = 13.2$, $\chi^2_{1,\,0.05} = 3.841$, reject H_0]

7. *A die when tossed 300 times gave the following results :*

Score	1	2	3	4	5	6
Frequency	43	49	56	45	66	41

Are the data consistent at 5 % level of significance with the hypothesis that the die is true ?

[Ans. : $\chi_5^2 = 8.56$, $\chi^2_{5,\,0.05} = 11.07$, accept H_0]

8. *In 150 tosses of a coin 90 heads and 60 tails were observed. Test the hypothesis that the coin is fair by Chi-square test at 0.05 significance level.* **[Ans. : 6.00 > 3.84, rejected]**

9. *The table below gives the number of books issued from a certain library on the various days of a week.*

Days	Mon	Tues	Wed	Thurs	Fri	Sat
No. of books issued	120	130	110	115	135	110

Test at 5 % l.o.s whether the issuing of books is independent of a day. **[Ans. : 4.58 < 11.07, accept]**

10. *In a locality, 100 persons were randomly selected and asked for their educational achievements. The results are given as under.*

Sex	Education		
	Primary school	High school	College
Male	10	15	25
Female	25	10	15

Test whether education depends on sex at 1 % level of significance. **[Ans. : 9.929 > 5.991, reject]**

11. *In an experiment on pea breeding, a scientist obtained the following frequencies of seeds : 316 round and yellow, 102 wrinkled and yellow, 109 round and green and 33 wrinkled and green. Theory predicts that the frequencies of seeds should be in the proportion 9 : 3 : 3 : 1 respectively. Set a proper hypothesis and test it at 5 % l.o.s.* **[Ans. : 0.355 < 7.815, accept]**

12. *A newspaper publisher is interested in testing whether newspaper readership in the society is associated with reader's educational achievement. A related survey showed the followed results :*

Level of Education				
Type of readership	Post graduate	Graduate	Passed S.S.C	Not passed S.S.C
Never	9	12	30	60
Sometimes	25	20	15	20
Daily	68	48	40	10

Test whether type of newspaper readership depends on level of education. [Take $\alpha = 0.05$]

[Ans. : $\chi_6^2 = 97.65 > \chi_{6,0.05}^2 = 12.592$, rejected]

13. *From the information given below, test whether the type of occupation and attitude towards the social laws are independent. [Use 1 % l.o.s.]*

Attitude towards Social Laws			
Occupation	Favourable	Neutral	Opposite
Blue-collar	29	26	37
White-collar	25	32	56
Professional	34	21	42

[Ans. : $\chi_4^2 = 5.415 < \chi_{4,0.01}^2 = 9.488$, accepted]

Semesters of natural science

	1	2	3	4	Totals
Males	5	6	50	64	125
Females	8	14	34	44	100
Totals	13	20	84	108	225

[Ans. : $\chi_3^2 = 7.96 > \chi_{3, 0.05}^2$, rejected]

18. *Random samples of 200 first-year students and 150 transfer students at a given college resulted in the following frequency table for the number of high-school years in which a foreign language ws studied. Apply a Chi-square test at the 0.01 significance level to the hypothesis that first-year and transfer students have the same high-school foreign language backgrounds.*

Years of foreign-language study in high-school

	1	2	3	4	5	Totals
First-year students	10	11	75	61	43	200
Transfer students	20	18	54	30	28	150
Totals	30	29	129	91	71	350

[Ans. : $\chi_4^2 = 15.84 > \chi_{4, 0.01}^2$, rejected]

Chi-square Tests for Independent Attributes

19. *Seventy-five exercise programs were rated for quality of exercise and motivational value. Each attributes was classified as good, fair, or poor, and the cross-classification frequencies are indicated in the following table. Apply a Chi-square test at the 0.05 significance level to the hypothesis that quality of exercise and motivational value are independent.*

Motivational value

		Good	Fair	Poor	Totals
	Good	15	6	4	25
Exercise value	Fair	6	12	6	25
	Poor	5	8	12	25
	Totals	27	26	22	75

[Ans. : $\chi^2 = 13.01 > \chi_{4, 0.05}^2$, rejected]

20. *Sixty supermarket pizzas were rated for taste (fair, good, very good) and price (high, medium, low). The cross-classification results are indicated in the following frequency contingency table. Apply a Chi-square test at the 0.05 significance level to the hypothesis that taste and price are independent.*

3. *In a Poisson distribution if $p(r=1)=2p(r=2)$, find $p(r=3)$.* **[3 Marks]**

4. *In a telephone exchange, the probability that any one call is wrongly connected is 0.02. What is the minimum number of calls required to ensure a probability 0.1 that at least one call is wrongly connected ?* **[5 Marks]**

Dec. - 2002

1. *A box 'A' contains 2 white and 4 black balls. Another box 'B' contains 5 white and 7 black balls. A ball is transferred from box 'A' to box 'B'. Then a ball is drawn from box 'B'. Find the probability that it is white.* **[5 Marks]**

2. *Fit a Poisson distribution to the set of observations :* **[6 Marks]**

x	y
0	122
1	60
2	15
3	2
4	1

3. *A sample of 100 dry battery cells were tested to find the length of life. If the mean is 12 hours and standard deviation is 3 hours, assuming the data to be normally distributed, what number of battery cells are expected to have life :*

   i) More than 15 hours

   ii) Less than 6 hours

   iii) Between 10 and 14 hours. **[6 Marks]**

4. *On an average a box containing 10 articles is likely to have 2 defectives. If we consider a consignment of 100 boxes, how many of them are expected to have 3 or less defectives.* **[4 Marks]**

May - 2003

1. *A can hit a target 3 times in 5 shots, B 2 times in 5 shots and C 3 times in 4 shots. They fire a volley. What is the probability that :*

   i) Two shots hit ? ii) At least two shots hit ? **[6 Marks]**

2. *The accidents per shift in a factory are given by the table :* **[6 Marks]**

Accidents x per shift	f frequencies
0	142
1	158
2	67
3	27
4	5
5	1

3. *In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and standard deviation of 60 hours. Estimate the number of bulbs likely to burn for :*

   i) more than 2150 hours

   ii) less than 1950 hours and

   iii) more than 1920 hours and but less than 2160 hours. **[6 Marks]**

4. *In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts ?* **[5 Marks]**

Dec. - 2003

1. *The mean and variance of Binomial distribution are 6 and 2 respectively. Find* $P(r \geq 2)$ **[5 Marks]**

2. *In a telephone exchange, the probability that any one call is wrongly connected is 0.02. What is the minimum number of calls required to ensure a probability 0.1 that at least one call is wrongly connected.* **[5 Marks]**

3. *A problem in statistics is given to three students A, B and C whose chances of solving it are* $\frac{1}{2}, \frac{1}{3}$ *and* $\frac{1}{4}$ *respectively. What is the probability that the problem is solved ?* **[4 Marks]**

May - 2004

1. *If X is a Poisson variate such that*

   $P(X = 2) = 9\,P(X = 4) + 90\,P(X = 6),$

   find its standard deviation. **[5 Marks]**

2. *If A and B throw alternately with single Die, A having the first throw. The person who first throws ace is to win. What are their respective chances of winning ?* **[5 Marks]**

3. *Show that mean and S.D. of a Binomial distribution are respectively given by BP and* $\sqrt{npq}$ *and point out the fallacy of the statement "The mean of the Binomial distribution is 3 and variance 5".* **[6 Marks]**

4. *A die is thrown 264 times with the following results :* **[8 Marks]**

No. Appeared on the Die	Frequency
1	40
2	32
3	28
4	58
5	54
6	60

Show that the die is biased.

($\chi^2(Chi)^2$ = 11.7 for 5 d.f. at 5 % level.)

Dec. - 2004

1. *A manufacturer knows that the condensers he makes contain on an average 1 % of defectives. He packs them in the boxes of 100. What is the probability that a box picked at random will contain 4 or more faulty condensers ?* **[5 Marks]**
2. *If on an average one ship in every 10 is wrecked, find the probability that out of 5 ships expected to arrive at least 4 will arrive safely.* **[5 Marks]**
3. *A manufacturer of envelopes knows that the weight of envelopes is normally distributed with mean 1.9 gm and variance 0.01 gm. Find how many envelopes weighing*

   i) 2 gms or more

   ii) 2.1 gms or more

   can be expected in a given packet of 1000 envelopes.

   [Given Area for z = 1 is 0.3413 and Area for z = 2 is 0.4772]. **[4 Marks]**

May - 2005

1. *Prove that the following represents Poisson distribution :* **[6 Marks]**

X	f
0	109
1	65
2	22
3	3
4	1

2. *A set of five similar coins is tossed 210 times and the result is :* **[6 Marks]**

No. of Heads	Frequencies
0	2
1	5
2	20
3	60
4	100
5	31

Test the hypothesis that the data follow a Binomial distribution.

Given : at 5 % level of significance :

i) How many scores below 8 ?

ii) How many scores between 12 and 13 ?

iii) How many scores above 15 ?

Note :

i) Area for $z = 2.4$, is 0.4918

ii) Area for $z = 0.8$, is 0.2881

iii) Area for $z = 0.4$, is 0.1554 **[5 Marks]**

3. *A die is thrown 264 times with the following results :*

No. appear in the die	Frequency
1	40
2	32
3	28
4	58
5	54
6	60

Show that the die is biased. (χ^2 for 5 degrees of freedom at 5 % level of significance = 11.07).

[6 Marks]

❑❑❑

4) Addition and subtraction of vectors :

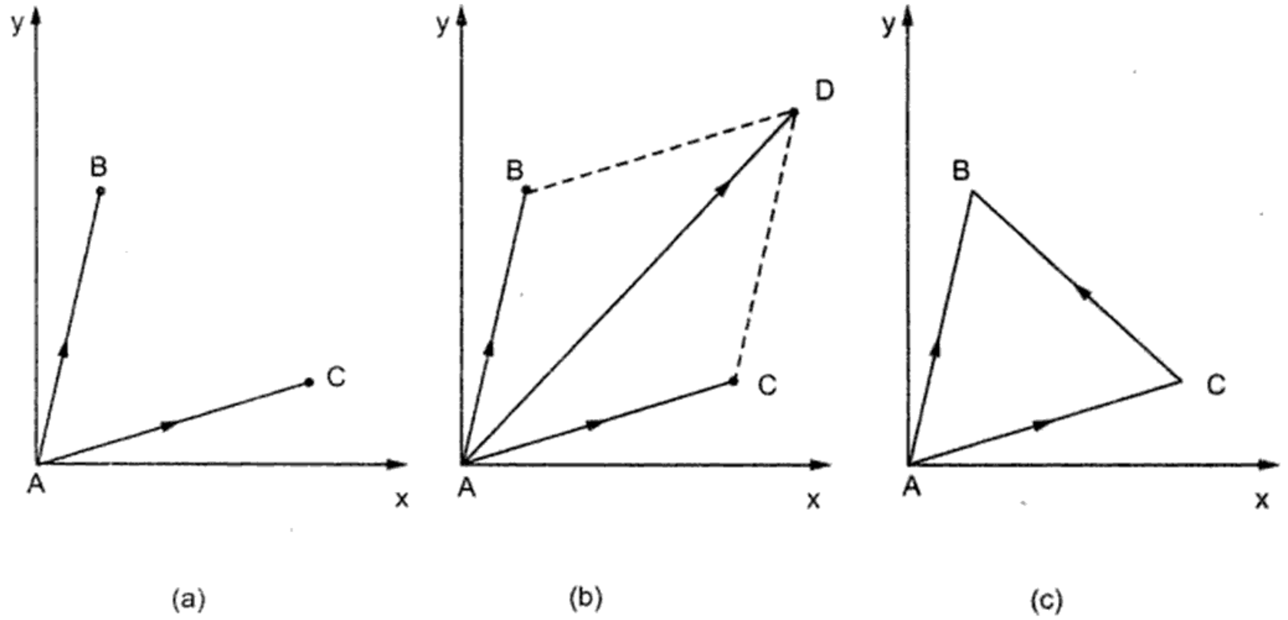


Fig. 12.2

Two vectors (non parallel) can be considered as having a common initial point such as A. [As shown in Fig. 12.2 (a)]. If $\overline{AB}$ and $\overline{AC}$ are sides of a parallogram [As shown in Fig. (b)] then the vector $\overline{AD}$ is the sum of $\overline{AB}$ and $\overline{AC}$ i.e. $\overline{AD} = \overline{AB} + \overline{AC}$.

$\overline{AD}$ is the principal diagonal (main diagonal) of the parallogram.

Also $\overline{AB} - \overline{AC}$ we can interprete as addition of $\overline{AB}$ with $-\overline{AE}$. We can also interpret the vector difference as the third side of a triangle with sides $\overline{AB}$ and $\overline{AC}$ [As shown in Fig.(c)].

5) Position vector : If $P(x, y, z)$ is any point is space and 0 is the origin then vector $\overline{OP}$ which indicates the position of the point P by a vector, is known as a position vector of that point P.

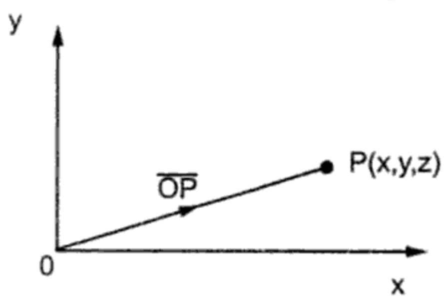


Fig. 12.3

Thus the position vector of $P(x, y, z)$ is

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

where $\hat{i}$, $\hat{j}$, $\hat{k}$ are the unit vectors in the direction of X, Y, Z axis respectively. Here x, y, z are scalar components of $\bar{r}$ along X, Y, Z axis respectively.

11) Geometrical meaning of $\bar{a}\cdot\bar{b}$

We know that $\bar{a}\cdot\bar{b} = ab\cos\theta$

$$\therefore \quad \bar{a}\cdot\bar{b} = \bar{a}\cdot b\hat{b} \qquad \text{As} \quad \bar{b} = b\hat{b}$$

$$= \left(\bar{a}\cdot\hat{b}\right) b$$

$$= (\text{Projection of } \bar{a} \text{ in the direction of } \bar{b}) \times \text{length of } \bar{b}$$

OR

$$\bar{a}\cdot\bar{b} = \begin{pmatrix}\text{Projection of } \bar{b} \text{ in} \\ \text{the direction of } \bar{a}\end{pmatrix} \times \text{length of } \bar{a}$$

12) Physical application of dot product.

i) Work done by a force $\bar{F}$ in displacing a particte from A to B. If $\bar{F}$ is a constant force acting on a particle at $\bar{A}$ causes a displacement $\overline{AB}$ $(= \bar{u})$ say then

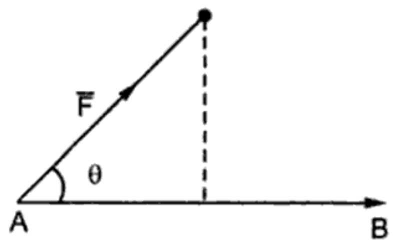


Fig. 12.4

$$\text{Work done} = (\text{Resolved component of } \bar{F} \text{ along AB}) \text{ (AB)}$$

$$\text{Work done} = (F\cos\theta)\, u$$

$$= |\bar{F}|\ |\bar{u}|\ \cos\theta$$

$$= \bar{F}\cdot\bar{u}$$

ii) Normal flux through a surface : Consider the flow of fluid through element δ_s of surface s, with velocity $\bar{v}$. Also $\hat{n}$ is the unit vector normal to δ_s.

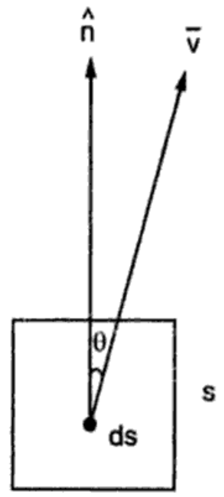


Fig. 12.5

ii) Angular velocity of a rigid body

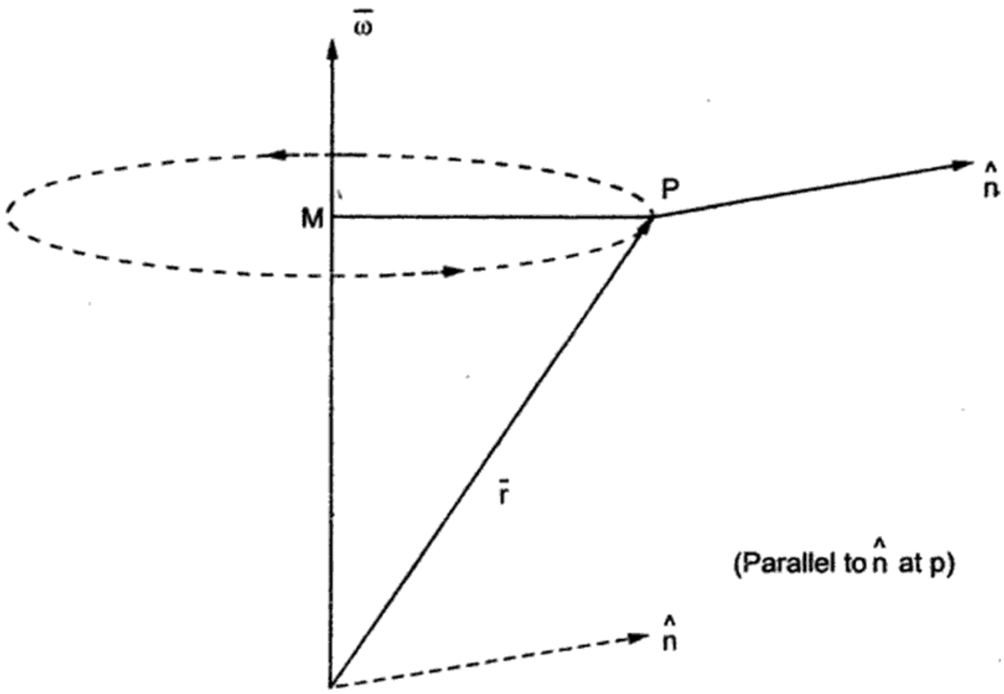


Fig. 12.10

Observe the above Fig. 12.10. Consider a rigid body rotating about an axis OM with angular velocity $\bar{\omega}$ radians per second. Let p be a point on the body such that $\overline{OP} = \bar{r}$. Let PM = perpendicular to OM $\hat{n}$ = unit vector normal to $\bar{\omega}$ and $\bar{r}$ (i.e. plane OPM).

Then
$$\begin{aligned} \bar{w} \times \bar{r} &= (\omega r \sin\theta)\,\hat{n} \\ &= \omega(\text{MP})\,\hat{n} \\ &= \omega\,(\text{radius of circular path})\,\hat{n} \\ &= (\text{speed of p})\,\hat{n} \\ &= V\,\hat{n} \\ &= \bar{V} \end{aligned}$$

Thus
$$\bar{V} = \bar{\omega} \times \bar{r}$$

c) Scalar Triple Product (STP) OR Box Product

It is a product of three vectors whose answer is a scalar.

The scalar triple product of $\bar{a}, \bar{b}, \bar{c}$ is represented as $(\bar{a} \times \bar{b}) \cdot \bar{c}$ and is denoted as $[\bar{a}\ \bar{b}\ \bar{c}]$.

If
$$\bar{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$$
$$\bar{b} = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$$

$$\bar{c} = c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k}$$

Then
$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i}(a_2 b_3 - a_3 b_2) - \hat{j}(a_1 b_3 - a_3 b_1) + \hat{k}(a_1 b_2 - a_2 b_1)$$

Thus
$$(\bar{a} \times \bar{b}) \cdot \bar{c} = c_1 (a_2 b_3 - a_3 b_2) - c_2 (a_1 b_3 - a_3 b_1) + c_3 (a_1 b_2 - a_2 b_1)$$

$$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{bmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note

i) STP is a scalar quantity.

ii) Cyclic changes are allowed in STP

i.e. $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{a}] = [\bar{c}\ \bar{a}\ \bar{b}]$

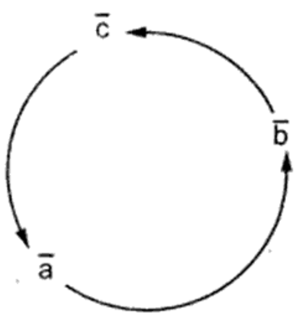


Fig. 12.11

3) Dot and cross are interchangeable in STP

$$\bar{a} \cdot (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \cdot \bar{c}$$

4) $[\bar{a}\ \bar{a}\ \bar{b}] = 0$ i.e. if two vectors are same in STP then STP = 0.

(If in a determinant two rows are equal then its value is zero).

5) Physical interpretation : STP $[\bar{a}\ \bar{b}\ \bar{c}]$ represents the volume of a parallopied with edges a, b, c.

6) If STP $[\bar{a}\ \bar{b}\ \bar{c}] = 0$ then the three vectors $\bar{a}\ \bar{b}\ \bar{c}$ lie in one plane i.e. $\bar{a}$, $\bar{b}$, $\bar{c}$ are coplaner.

7) $(\bar{a} \cdot \bar{b}) \times \bar{c}$ is not allowed as $\bar{a} \cdot \bar{b}$ is a scalar.

d) Vector triple product

It is a product of three vectors whose answer is a vector.

If $\bar{a}$, $\bar{b}$, $\bar{c}$ are three vectors, then the vector $(\bar{a} \times \bar{b}) \times \bar{c}$ is perpendicular to $\bar{a} \times \bar{b}$ and $\bar{c}$. Thus the vector $(\bar{a} \times \bar{b}) \times \bar{c}$ lies in the plane of $\bar{a}$ and $\bar{b}$.

i.e. $$\boxed{(\bar{1}\times\bar{2})\times\bar{3} = (\bar{1}\cdot\bar{3})\,\bar{2}-(\bar{1}\cdot\bar{2})\,\bar{3}}$$

Note

i) $(\bar{a}\times\bar{b})\times\bar{c} \neq \bar{a}\times(\bar{b}\times\bar{c})$

ii) $\bar{a}\times(\bar{b}\times\bar{c})+\bar{b}\times(\bar{c}\times\bar{a})+\bar{c}\times(\bar{a}\times\bar{b}) = 0$

e) **Vector quadrople product**

(i.e. product of four vectors)

i) **Scalar product of four vectors :**

If $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ be any four vectors

$$\text{Scalar product} = (\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})$$

Let $$\bar{a}\times\bar{b} = \bar{p}$$

$$\therefore \quad = \bar{p}\cdot(\bar{c}\times\bar{d})$$

Interchanging dot and cross in STP we get

$$= (\bar{p}\times\bar{c})\cdot\bar{d}$$

Substituting $\bar{p} = \bar{a}\times\bar{b}$ we get

$$= [(\bar{a}\times\bar{b})\times\bar{c}]\cdot\bar{d}$$

Using vector triple product formula

$$= [(\bar{a}\cdot\bar{c})\,\bar{b}-(\bar{b}\cdot\bar{c})\,\bar{a}]\cdot\bar{d}$$

$$= [(\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{d})-(\bar{b}\cdot\bar{c})(\bar{a}\cdot\bar{d})]$$

$$(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = \begin{vmatrix} \bar{a}\cdot\bar{c} & \bar{a}\cdot\bar{d} \\ \bar{b}\cdot\bar{c} & \bar{b}\cdot\bar{d} \end{vmatrix}$$

Note i) $$(\bar{a}\times\bar{b})^2 = (\bar{a}\times\bar{b})\cdot(\bar{a}\times\bar{b})$$

$$= \begin{vmatrix} \bar{a}\cdot\bar{a} & \bar{a}\cdot\bar{b} \\ \bar{b}\cdot\bar{a} & \bar{b}\cdot\bar{b} \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & \bar{a}\cdot\bar{b} \\ \bar{a}\cdot\bar{b} & b^2 \end{vmatrix}$$

$$= a^2\,b^2-(\bar{a}\cdot\bar{b})^2$$

ii) Vector product of four vectors

$$\text{Vector Product} = (\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})$$

Let $\bar{a}\times\bar{b} = \bar{p}$

$$= \bar{p}\times(\bar{c}\times\bar{d})$$

$$= (\bar{p}\cdot\bar{d})\,\bar{c}-(\bar{p}\cdot\bar{c})\,\bar{d}$$

Substituting $\bar{p} = \bar{a}\times\bar{b}$ we get

$$= [(\bar{a}\times\bar{b})\cdot\bar{d}]\,\bar{c}-[(\bar{a}\times\bar{b})\cdot\bar{c}]\,\bar{d}$$

$$= [\bar{a}\;\bar{b}\;\bar{d}]\,\bar{c}-[\bar{a}\;\bar{b}\;\bar{c}]\,\bar{d}$$

i.e. $$\boxed{(\bar{1}\times\bar{2})\times(\bar{3}\times\bar{4}) = [\bar{1}\,\bar{2}\,\bar{4}]\,\bar{3}-[\bar{1}\,\bar{2}\,\bar{3}]\,\bar{4}}$$

Similarly by substituting $\bar{c}\times\bar{d}$ as $\bar{p}$ we can show that

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\;\bar{c}\;\bar{d}]\,\bar{b}-[\bar{b}\;\bar{c}\;\bar{d}]\,\bar{a}$$

i.e. $$\boxed{(\bar{1}\times\bar{2})\times(\bar{3}\times\bar{4}) = [\bar{1}\,\bar{3}\,\bar{4}]\,\bar{2}-[\bar{2}\,\bar{3}\,\bar{4}]\,\bar{1}}$$

12.4 Vector Function of a Scalar Variable

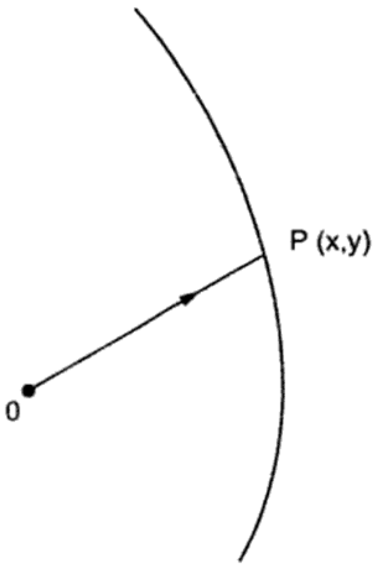


Fig. 12.12

Let t be a scalar variable. If $\bar{F}$ is defined for every value of t then it can be expressed as

$$\bar{F} = \bar{F}(t)$$

As $\bar{F}$ gives a vector for every value of scalar t, thus it is known as a vector function of a scalar variable t.

For example consider a curve C with parametric co-ordinates x = x (t) and y = y (t). As t varies the point P (x, y) moves along the curve C.

Every position of point P gives $\overline{OP}$ = position vector of point P. Thus C may be considered as the locus of the end points of the vectors $\bar{F}(t)$ i.e. the curve is defined by the vector $\bar{F}(t)$.

12.5 Differentiation of Vectors

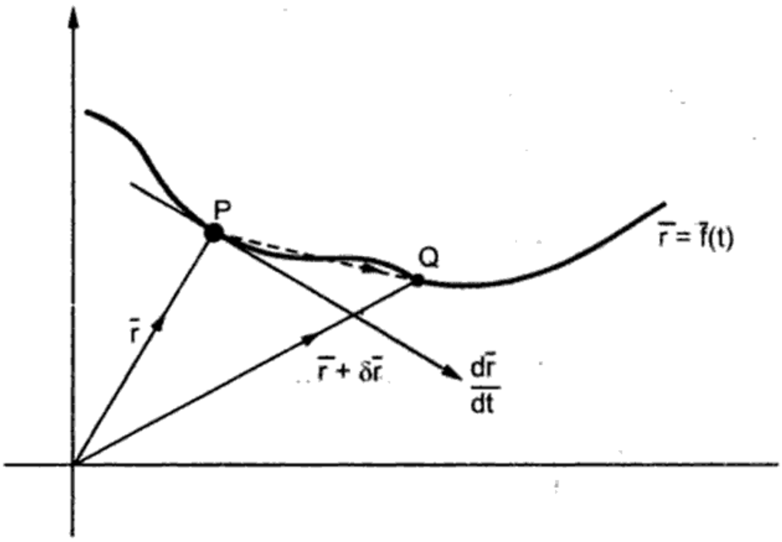


Fig. 12.13

Let $\bar{r} = \bar{f}(t)$ be a vector function of a scalar variable t

$$\text{Let} \quad \overline{OP} = \bar{r}$$

$$\overline{OQ} = \bar{r} + \delta\bar{r}$$

$$= \bar{f}(t + \delta t)$$

$$\therefore \quad \overline{PQ} = \overline{OQ} - \overline{OP}$$

$$= \bar{r} + \delta\bar{r} - \bar{r}$$

$$= \delta\bar{r}$$

$$= \bar{f}(t + \delta t) - \bar{f}(t)$$

$$\text{Thus} \quad \frac{\delta\bar{r}}{\delta t} = \frac{\bar{f}(t + \delta t) - \bar{f}(t)}{\delta t}$$

$$\therefore \quad \frac{d\bar{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta\bar{r}}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\bar{f}(t + \delta t) - \bar{f}(t)}{\delta t}$$

is called as vector derivative of $\bar{r}$ with respect to scalar t and is written as $\frac{d\bar{r}}{dt}$ if the limit exists.

Cross Product of Two Vectors

$\bar{a} \times \bar{b} = a\, b \sin\theta\, \hat{n}$

where $\hat{n}$ is the unit vector perpendicular to the plane of $\bar{a}$ and $\bar{b}$.

$$\hat{n} = \frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|}$$

- $\hat{i} \times \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \times \hat{k} = 0 \qquad \because \sin 0 = 0$

  $\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j} \qquad \because \sin 90 = \phi$

- $\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

- $\bar{a} \times \bar{b} = -\left(\bar{b} \times \bar{a}\right)$ (not commutative).

- Cross product of two vectors is a vector.

- If two vectors are parallel then $\bar{a} \times \bar{b} = 0$.

Scalar Triple Product

$$\bar{a} \cdot \left(\bar{b} \times \bar{c}\right) = \left[\bar{a}\ \bar{b}\ \bar{c}\right]$$

It is a product of three vectors whose answer is a scalar. It is also known as a box product.

$$\bar{a} \cdot \left(\bar{b} \times \bar{c}\right) = \left[\bar{a}\ \bar{b}\ \bar{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Scalar triple product gives the volume of parallelopiped with edges a, b, c.
- If $\left[\bar{a}\ \bar{b}\ \bar{c}\right] = 0$ then $\bar{a}$, $\bar{b}$, $\bar{c}$ are co-planer i.e. $\bar{a}$, $\bar{b}$, $\bar{c}$ lie in same plane.
- In S.T.P. cyclic changes are allowed $\left[\bar{a}\ \bar{b}\ \bar{c}\right] = \left[\bar{b}\ \bar{c}\ \bar{a}\right] = \left[\bar{c}\ \bar{a}\ \bar{b}\right]$.
- In S.T.P. dot and cross are interchangeable i.e. $\bar{a} \cdot \left(\bar{b} \times \bar{c}\right) = \left(\bar{a} \times \bar{b}\right) \cdot \bar{c}$

Vector Triple Product

It is a product of three vectors whose answer is a vector.

$$\bar{a} \times \left(\bar{b} \times \bar{c}\right) = (\bar{a} \cdot \bar{c})\, \bar{b} - \left(\bar{a} \cdot \bar{b}\right) \bar{c}$$

$$\bar{1} \times \left(\bar{2} \times \bar{3}\right) = \left(\bar{1} \cdot \bar{3}\right) \bar{2} - \left(\bar{1} \cdot \bar{2}\right) \bar{3}$$

Thus $\quad \dfrac{d^2 \bar{r}}{dt^2} = \bar{r}$ which proves (i)

Step 3 : As $\bar{r}$ is not given using $\hat{i}, \hat{j}, \hat{k}$ $\therefore$ we can't use the standard cross product formula.

$\therefore$ Consider

$$\frac{d\,\bar{r}}{dt} \times \frac{d^2\,\bar{r}}{dt^2} = \left(\bar{a} \cos ht + \bar{b} \sin ht\right) \times \left(\bar{a} \sin ht + \bar{b} \cos ht\right)$$

Step 4 : Use $\left(\bar{a}+\bar{b}\right) \times \left(\bar{c}+\bar{d}\right) = \bar{a} \times \bar{c} + \bar{a} \times \bar{d} + \bar{b} \times \bar{c} + \bar{b} \times \bar{d}$

$\therefore = \bar{a} \cos ht \times \bar{a} \sin ht + \bar{a} \cos ht \times \bar{b} \cos ht + \bar{b} \sin ht \times \bar{a} \sin ht + \bar{b} \sin ht \times \bar{b} \cos ht$

Step 5 : cos ht and sin ht are scalar terms $\therefore$ they are not involved in cross product i.e. cross product exists only for vectors.

Thus

$$\frac{d\,\bar{r}}{dt} \times \frac{d^2\,\bar{r}}{dt^2} = \cos ht \sin ht (\bar{a} \times \bar{a}) + \cos h^2 t \left(\bar{a} \times \bar{b}\right) + \sin h^2 t \left(\bar{b} \times \bar{a}\right)$$
$$+ \sin ht \cos ht \left(\bar{b} \times \bar{b}\right)$$

Step 6 : As $\bar{a} \times \bar{a} = 0, \quad \bar{b} \times \bar{b} = 0, \quad \bar{b} \times \bar{a} = -\left(\bar{a} \times \bar{b}\right)$

$$= 0 + \cosh^2 t \left(\bar{a} \times \bar{b}\right) - \left(\bar{a} \times \bar{b}\right) \sinh^2 t + 0$$

Step 7 : Take $\bar{a} \times \bar{b}$ common.

$\therefore = \left(\bar{a} \times \bar{b}\right)\left(\cosh^2 t - \sinh^2 t\right)$

Step 8 : As $\cosh^2 t - \sin^2 t = 1$

$$= \left(\bar{a} \times \bar{b}\right)(1)$$

Step 9 : $\bar{a}$ and $\bar{b}$ are constant vectors

$\therefore \quad \bar{a} \times \bar{b} =$ constant vector

Thus $\quad \dfrac{d\,\bar{r}}{dt} \times \dfrac{d^2 \bar{r}}{dt^2} =$ constant which proves (ii)

Step 10 : For (iii)

Consider $\bar{r} \cdot \left(\dfrac{d\,\bar{r}}{dt} \times \dfrac{d^2\,\bar{r}}{dt^2}\right)$

Step 11 : Put $\dfrac{d^2\,\bar{r}}{dt^2} = \bar{r}$

$$= \bar{r} \cdot \left(\frac{d\,\bar{r}}{dt} \times \bar{r}\right)$$

Step 12 : $\left[\bar{a} \cdot \left(\bar{b} \times \bar{c}\right)\right] = \left[\bar{a}\ \bar{b}\ \bar{c}\right]$

Note if two rows in a determinant are same then its value is zero i.e. $\left[\bar{a}\ \bar{b}\ \bar{a}\right] = 0.$

Thus $\left[\bar{r}\ \frac{d\bar{r}}{dt}\ \bar{r}\right] = 0$ which proves (iii)

Example 12.3 : *Prove that* $\frac{d}{dt}\left\{\bar{u}\cdot\left(\frac{d\bar{u}}{dt}\times\frac{d^2\bar{u}}{dt^2}\right)\right\} = \bar{u}\cdot\left(\frac{d\bar{u}}{dt}\times\frac{d^3\bar{u}}{dt^3}\right)$

Solution : Step 1 : Consider

$$\frac{d}{dt}\left\{\bar{u}\cdot\left(\frac{d\bar{u}}{dt}\times\frac{d^2\bar{u}}{dt^2}\right)\right\}$$

Step 2 : Use $\bar{a}\cdot\left(\bar{b}\times\bar{c}\right) = \left[\bar{a}\ \bar{b}\ \bar{c}\right]$

and $\frac{d}{dt}\left[\bar{a}\ \bar{b}\ \bar{c}\right] = \left[\frac{d\bar{a}}{dt}\ \bar{b}\ \bar{c}\right] + \left[\bar{a}\ \frac{d\bar{b}}{dt}\ \bar{c}\right] + \left[\bar{a}\ \bar{b}\ \frac{d\bar{c}}{dt}\right]$

$$\therefore\ \frac{d}{dt}\left[\bar{u}\ \frac{d\bar{u}}{dt}\ \frac{d^2\bar{u}}{dt^2}\right]$$

$$= \left[\frac{d\bar{u}}{dt}\ \frac{d\bar{u}}{dt}\ \frac{d^2\bar{u}}{dt^2}\right] + \left[\bar{u}\ \frac{d^2\bar{u}}{dt^2}\ \frac{d^2\bar{u}}{dt^2}\right] + \left[\bar{u}\ \frac{d\bar{u}}{dt}\ \frac{d^3\bar{u}}{dt^3}\right]$$

Step 3 : Use $\left[\bar{a}\ \bar{a}\ \bar{b}\right] = 0$

If two rows in a determinant are same then its value is zero.

$$\therefore\ \frac{d}{dt}\left[\bar{u}\ \frac{d\bar{u}}{dt}\ \frac{d^2\bar{u}}{dt^2}\right] = 0 + 0 + \left[\bar{u}\ \frac{d\bar{u}}{dt}\ \frac{d^3\bar{u}}{dt^3}\right]$$

Step 4 : Again use S.T.P. notation.

$$\therefore \frac{d}{dt}\ \bar{u}\cdot\left(\frac{d\bar{u}}{dt}\times\frac{d^2\bar{u}}{dt^2}\right) = \bar{u}\cdot\left(\frac{d\bar{u}}{dt}\times\frac{d^3\bar{u}}{dt^3}\right)$$

Example 12.4 : *If* $\frac{d\bar{a}}{dt} = \bar{c}\times\bar{a}$, $\frac{d\bar{b}}{dt} = \bar{c}\times\bar{b}$ *then prove that*

$$\frac{d}{dt}\left(\bar{a}\times\bar{b}\right) = \bar{c}\times\left(\bar{a}\times\bar{b}\right)$$

Solution : Step 1 : Consider RHS and use vector triple product formula

$$\bar{1}\times\left(\bar{2}\times 3\right) = \left(\bar{1}\cdot\bar{3}\right)\bar{2} - \left(\bar{1}\cdot\bar{2}\right)\bar{3}$$

$$\bar{c}\times\left(\bar{a}\times\bar{b}\right) = \left(\bar{c}\cdot\bar{b}\right)\bar{a} - \left(\bar{c}\cdot\bar{a}\right)\bar{b} \qquad \text{... (1)}$$

Step 2 : Consider L.H.S.

$$\frac{d}{dt}\left(\bar{a}\times\bar{b}\right) = \frac{d\bar{a}}{dt}\times\bar{b} + \bar{a}\times\frac{d\bar{b}}{dt}$$

Step 3 : Substitute the values of $\frac{d\bar{a}}{dt}$ and $\frac{d\bar{b}}{dt}$

$$= (\bar{c}\times\bar{a})\times\bar{b} + \bar{a}\times(\bar{c}\times\bar{b})$$

Step 4 : We can't use V.T.P. formula for first term

$\therefore$ Use $\quad \bar{a}\times\bar{b} = -(\bar{b}\times\bar{a})$

$$= -[\bar{b}\times(\bar{c}\times\bar{a})] + [\bar{a}\times(\bar{c}\times\bar{b})]$$

Step 5 : Now apply V.T.P. formula for both terms. $\bar{1}\times(\bar{2}\times\bar{3}) = (\bar{1}\cdot\bar{3})\bar{2}-(\bar{1}\cdot\bar{2})\bar{3}$

$$= -[(\bar{b}\cdot\bar{a})\bar{c}-(\bar{b}\cdot\bar{c})\bar{a}] + [(\bar{a}\cdot\bar{b})\bar{c}-(\bar{a}\cdot\bar{c})\bar{b}]$$

Step 6 : Simplify i.e. open the [ ]

$$= -(\bar{b}\cdot\bar{a})\bar{c} + (\bar{b}\cdot\bar{c})\bar{a} + (\bar{a}\cdot\bar{b})\bar{c} - (\bar{a}\cdot\bar{c})\bar{b}$$

Step 7 : Use $(\bar{a}\cdot\bar{b}) = (\bar{b}\cdot\bar{a})$

$$= -(\bar{a}\cdot\bar{b})\bar{c} + (\bar{c}\cdot\bar{b})\bar{a} + (\bar{a}\cdot\bar{b})\bar{c} - (\bar{c}\cdot\bar{a})\bar{b}$$

Step 8 : Simplify

$$= (\bar{c}\cdot\bar{b})\bar{a} - (\bar{c}\cdot\bar{a})\bar{b}$$

Step 9 : From V.T.P. $(\bar{1}\cdot\bar{3})\bar{2}-(\bar{1}\cdot\bar{2})\bar{3} = \bar{1}\times(\bar{2}\times\bar{3})$

$$= \bar{c}\times(\bar{a}\times\bar{b})$$

$$= \text{RHS}$$

Example 12.5 : $\bar{a} = \sin\theta\hat{i} + \cos\theta\hat{j} + \theta\hat{k}$

$$\bar{b} = \cos\theta\hat{i} - \sin\theta\hat{j} - 3\hat{k}$$

$$\bar{c} = \hat{i} + 2\hat{j} - 3\hat{k}$$

Find $\frac{d}{d\theta}\left\{\bar{a}\times(\bar{b}\times\bar{c})\right\}$ *at* $\theta = \pi/2.$

Solution : Step 1 : Consider $\bar{a}$, $\bar{b}$, $\bar{c}$ and find their derivatives.

$$\bar{a} = \sin\theta\,\hat{i} + \cos\theta\,\hat{j} + \theta\,\hat{k}$$

$$\frac{d\bar{a}}{d\theta} = \cos\theta\,\hat{i} - \sin\theta\,\hat{j} + \hat{k}$$

$$\bar{b} = \cos\theta\,\hat{i} - \sin\theta\,\hat{j} - 3\,\hat{k}$$

$$\frac{d\bar{b}}{d\theta} = -\sin\theta\,\hat{i} - \cos\theta\,\hat{j} - 0\,\hat{k}$$

$$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{1+9+4}} = \frac{\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{14}}$$

Step 3 : Consider $\bar{r}$ and find $\frac{d\bar{r}}{dt}$ and $\frac{d^2\bar{r}}{dt^2}$.

$$\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

i.e.
$$\bar{r} = 2t^2\hat{i}+(t^2-4t)\hat{j}+(3t-5)\hat{k}$$

$$\bar{v} = \frac{d\bar{r}}{dt} = 4t\hat{i}+(2t-4)\hat{j}+3\hat{k}$$

$$\bar{a} = \frac{d^2\bar{r}}{dt^2} = 4\hat{i}+2\hat{j}+0\hat{k}$$

Step 4 : Find velocity and acceleration at t = 1

$$(\bar{v})_{t=1} = 4\hat{i}-2\hat{j}+3\hat{k}$$

$$(\bar{a})_{t=1} = 4\hat{i}+2\hat{j}+0\hat{k}$$

Step 5 : Component of $\bar{v}$ in the direction of $\bar{u}$ is given by $\bar{v}\cdot\bar{u}$.

$$= (4\hat{i}-2\hat{j}+3\hat{k})\cdot\left(\frac{\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{14}}\right)$$

$$(\bar{a}\cdot\bar{b}) = a_1b_1+a_2b_2+a_3b_3$$

$$= \frac{4+6+6}{\sqrt{14}}$$

$$= \frac{16}{\sqrt{14}}$$

Step 6 : Component of $\bar{a}$ in the direction of $\bar{u}$ is given by $\bar{a}\cdot\hat{u}$.

$$= (4\hat{i}+2\hat{j}+0\hat{k})\cdot\left(\frac{\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{14}}\right)$$

$$= \frac{4-6+0}{\sqrt{14}} = \frac{-2}{\sqrt{14}}$$

Exercise 12.1

1. *A particle moves along a curve* $x = t^3 + 1$, $y = t^2$, $z = 2t - 5$. *Find the components of velocity and acceleration at* $t = 1$ *in the direction* $\hat{i} + \hat{j} + 3\hat{k}$. **[Ans. :** $\sqrt{11}, \frac{8}{\sqrt{11}}$**]**

2. *Show that tangent at any point on the curve* $x = e^{\theta}\cos\theta$, $y = e^{\theta}\sin\theta$, $z = e^{\theta}$ *makes constant angle with z axis. Hence show that the curve entirely lies on cone.*

3. *Find the velocity and acceleration vectors of a particle moving along a curve* $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$ *also find their magnitudes.* **[Ans. : 10, 18]**

4. *The acceleration of a particle at any time* $t \geq 0$ *is given by* $12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$. *The velocity and displacement are zero at* $t = 0$. *Find velocity and displacement at any time t.*
[Ans. : $\bar{v} = 6\sin 2t\,\hat{i} + 4(\cos 2t - 1)\,\hat{j} + 8t^2\,\hat{k}$

$\bar{r} = 3(1 - \cos 2t)\,\hat{i} + 2(\sin 2t - 2t)\,\hat{j} + \frac{8}{3}t^3\,\hat{k}$**]**

5. *If* $\bar{r} = \sec t\,\hat{i} + \tan t\,\hat{j}$ *find the velocity and acceleration at* $t = \frac{\pi}{6}$. **[Ans. :** $\frac{2i + 4j}{3}, \frac{2}{3\sqrt{3}}(5i + 4j)$**]**

6. *If* $\bar{a} = \sin u\,\hat{i} + \cos u\,\hat{j} + u\hat{k}$, $\bar{b} = \cos u\,\hat{i} - \sin u\,\hat{j} - 3\hat{k}$, $\bar{c} = 2\hat{i} + 3\hat{j} - \hat{k}$ *show that* $\frac{d}{du}\left\{\bar{a} \times (\bar{b} \times \bar{c})\right\}$ *at* $u = 0$ *is* $7\hat{i} + 6\hat{j} - 6\hat{k}$.

7. *Find the angle between the two vectors* $\bar{a} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\bar{b} = -\hat{i} + 5\hat{j} + \hat{k}$. **[Ans. :** $\theta = \cos^{-1}\left(\frac{\sqrt{42}}{9}\right)$**]**

8. *If* $\bar{a} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ *and* $\bar{b} = \hat{i} + \hat{j} + 2\hat{k}$ *find* $comp_b\, a$ *and* $comp_a\, b$. **[Ans. :** $\frac{-3}{\sqrt{6}}, \frac{-3}{\sqrt{29}}$**]**

   Hint : $comp_b\, a$ = *component of* $\bar{a}$ *in the direction of* $\bar{b}$

   $= \bar{a} \cdot \hat{b}$

   $comp_a\, b$ = *component of* $\bar{b}$ *in the direction of* $\bar{a}$

   $= \bar{b} \cdot \hat{a}$

9. *The position vector at any time t is given by* $\bar{r} = \cos(t-1)\,\hat{i} + \sinh(t-1)\,\hat{j} + \alpha t^3\hat{k}$. *Find the condition imposed on* α *such that acceleration in normal to the position vector at* $t = 1$. **[Ans. :** $\alpha = \pm\frac{1}{\sqrt{6}}$**]**

10. *Find the angle between the tangents to the curve* $x = t^2$, $y = 2t$, $z = -t^3$ *at the points* $t = 1$ *and* $t = -1$. **[Ans. :** $\cos^{-1}\left(\frac{9}{17}\right)$**]**

11. *A particle moves along the curve* $x = t^3$, $y = t^2$, $z = 2t + 3$. *Find the velocity and acceleration and their magnitudes. Also find the components of velocity and acceleration in the direction* $2\hat{i} + 3\hat{j} + 4\hat{k}$ *at* $t = 2$. **[Ans. :** $\frac{44}{\sqrt{29}}, \frac{30}{\sqrt{29}}$**]**

$\therefore \quad r = a \text{ (say)}$

which represents polar equation of sphere.

Example 12.10 : *If a particle moves always on the surface of sphere prove that i) $\bar{r} \cdot \bar{a} + \bar{v} \cdot \bar{v} = 0$ ii) $\bar{r} \cdot \bar{a} \leq 0$*

Solution : Step 1 : Polar equation of sphere is

$$r = \text{constant}$$

Thus $\quad |\bar{r}| = \text{constant}$

Step 2 : We know that $\bar{r} \cdot \bar{r} = r^2$

Step 3 : Differentiate w.r.t. t

$$\frac{d\bar{r}}{dt} \cdot \bar{r} + \bar{r} \cdot \frac{d\bar{r}}{dt} = 0$$

i.e. $\quad 2\left(\bar{r} \cdot \dfrac{d\bar{r}}{dt}\right) = 0$

Step 4 : We know $\dfrac{d\bar{r}}{dt} = \bar{v} = \text{velocity}$

$$\therefore \quad \bar{r} \cdot \bar{v} = 0$$

Step 5 : Differentiate w.r.t. t.

$$\frac{d\bar{r}}{dt} \cdot \bar{v} + \bar{r} \cdot \frac{d\bar{v}}{dt} = 0$$

Step 6 : $\quad \dfrac{d\bar{r}}{dt} = \bar{v}$ and $\dfrac{d\bar{v}}{dt} = \bar{a}$

$$\therefore \quad \bar{v} \cdot \bar{v} + \bar{r} \cdot \bar{a} = 0$$

which proves (i)

Step 7 : We know that $\bar{v} \cdot \bar{v} = v^2 \geq 0$

Thus from step 6 $\bar{r} \cdot \bar{a} \leq 0$ which proves (ii).

Example 12.11 : *Prove that the necessary and sufficient condition for a vector $\bar{F}$ to have constant direction is $\bar{F} \times \dfrac{d\bar{F}}{dt} = 0$.*

Solution : Step 1 : To prove the necessary part.

Let $\bar{F}$ be a vector with constant direction $\hat{F}$.

Step 2 : $\therefore$ $\hat{F}$ is a constant vector.

$$\hat{F} = \text{unit vector in the direction of } \bar{F}$$

Step 11 : Integrate w.r.t. t

$$\log F_1 = \log F_2 + \log C_1$$

$$\log F_1 = \log F_2 C_1$$

i.e. $$F_1 = F_2 C_1$$

or $$F_2 = \frac{F_1}{C_1}$$

i.e. $$F_2 = C_1' F_1$$

Thus we have expressed F_2 in terms of F_1

Step 12 : Similarly we can express F_3 in terms of F_1

$$F_3 = C_2' F_1$$

Step 13 : Substitute in $\overline{F}$

$$\overline{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

$$= F_1 \hat{i} + F_1 C_1' \hat{j} + F_1 C_2' \hat{k}$$

Step 14 : Take F_1 common

$$\overline{F} = F_1 \left(\hat{i} + C_1' \hat{j} + C_2' \hat{k}\right)$$

Step 15 : Thus $\overline{F}$ is a vector with constant direction which proves the sufficient part.

Example 12.12 : *A vector $\bar{r}$ satisfies the equation*

$$m \frac{d^2 \bar{r}}{dt^2} = e\overline{E} + \frac{e}{C} \frac{d\bar{r}}{dt} \times \overline{H} \quad \text{where } \overline{E} = E\hat{j} \text{ and } \overline{H} = H\hat{k}$$

e, E, m, C H are constants. Find solution under condition $\bar{r} = 0$, $\dfrac{d\bar{r}}{dt} = 0$ when $t = 0$.

Solution :

Step 1 : Given $\bar{r} = 0$ and $\dfrac{d\bar{r}}{dt} = 0$ at $t = 0$.

i.e. $x\,i + y\,j + z\,k = 0$ and $\dfrac{dx}{dt} i + \dfrac{dy}{dt} j + \dfrac{dz}{dt} k = 0$ at $t = 0$

i.e. $x = 0$, $y = 0$, $z = 0$ and $\dfrac{dx}{dt} = 0$, $\dfrac{dy}{dt} = 0$, $\dfrac{dz}{dt} = 0$ at $t = 0$

Step 2 : Consider the equation and divide by m, substitute $\overline{E}$ and $\overline{H}$.

$$\frac{d^2 \bar{r}}{dt^2} = \frac{e}{m} E\hat{j} + \frac{e}{mC} \frac{d\bar{r}}{dt} \times H\hat{k}$$

Step 7 : Take magnitudes of both sides and substitute $\hat{T} = \frac{\bar{v}}{|\bar{v}|}$

$$\therefore \quad \left|\bar{a} \times \frac{\bar{v}}{|\bar{v}|}\right| = a_N (1)$$

Thus
$$a_N = \frac{|\bar{a} \times \bar{v}|}{|\bar{v}|}$$

which gives the normal component.

Example 12.14 : *A particle moves along the curve $\bar{r} = t^2\hat{i} - t^3\hat{j} + t^4\hat{k}$ find magnitudes of tangential and normal components of acceleration at $t = 1$.*

Solution : Step 1 : Consider $\bar{r}$ and find $\frac{d\bar{r}}{dt}$ and $\frac{d^2\bar{r}}{dt^2}$.

$$\bar{r} = t^2\hat{i} - t^3\hat{j} + t^4\hat{k}$$

$$\frac{d\bar{r}}{dt} = 2t\hat{i} - 3t^2\hat{j} + 4t^3\hat{k}$$

$$\frac{d^2\bar{r}}{dt^2} = 2\hat{i} - 6\hat{j} + 12t^2\hat{k}$$

Step 2 : Find $\bar{v}$ and $\bar{a}$ at $t = 1$

$$\bar{v} = 2\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\bar{a} = 2\hat{i} - 6\hat{j} + 12\hat{k}$$

Step 3 : Find $v = |\bar{v}|$ and $a = |\bar{a}|$ and $\bar{v} \cdot \bar{a}$

$$v = |\bar{v}| = \sqrt{4+9+16} = \sqrt{29}$$

$$a = |\bar{a}| = \sqrt{4+36+144} = \sqrt{184}$$

$$\bar{v} \cdot \bar{a} = 4 + 18 + 48 = 70$$

Step 4 :
$$a_T = \frac{\bar{v} \cdot \bar{a}}{|\bar{v}|}$$

$$= \frac{70}{\sqrt{29}}$$

Step 5 : We know that $a^2 = a_T^2 + a_N^2$

$$\therefore \quad a_N^2 = a^2 - a_T^2$$

$$\therefore \quad a_N^2 = 184 - \left(\frac{70}{\sqrt{29}}\right)^2$$

$$a_N^2 = \frac{436}{29}$$

$$a_N = \sqrt{\frac{436}{29}}$$

$$\bar{v} \cdot \bar{a} = a^2 (1+\cos t)(-\sin t) + a^2 \cos t \cdot \sin t$$

$$= a^2 [-\sin t]$$

$$= -a^2 \cdot 2 \sin\frac{t}{2} \cos\frac{t}{2}$$

Step 3 :

$$a_T = \frac{\bar{v} \cdot \bar{a}}{|\bar{v}|}$$

$$= -\frac{a^2 \cdot 2 \cdot \sin t/2 \cdot \cos t/2}{2a \cdot \cos t/2}$$

$$= -a \sin\frac{t}{2}$$

Step 4 :

$$a_N^2 = a^2 - a_T^2$$

$$a_N^2 = a^2 - a^2 \sin^2\frac{t}{2}$$

$$a_N^2 = a^2 \cos^2 t/2$$

$$\therefore \quad a_N^2 = a \cos t/2$$

Exercise 12.2

1. *Find the magnitudes of tangential and normal components of acceleration for a particle moving on the curve,*

   i) $x = t^3 + 1$, $y = t^2$, $z = t$ *at* $t = 1$ **[Ans. : i)** $a_T = \frac{22}{\sqrt{14}}$ $a_N = \sqrt{\frac{38}{7}}$**]**

   ii) $x = \cos t + t \sin t$, $y = \sin t - t \cos t$ **[Ans. :** $a_T = 1$, $a_N = t$**]**

2. *If* $x = \log(t^2+1)$, $y = t - 2\tan^{-1}t$ *find velocity and acceleration of a particle at time t. Also find the tangential and normal components of the acceleration.* **[Ans. :** $a_T = 0$, $a_N = \frac{2}{1+t^2}$**]**

3. *A particle moves along a curve* $\bar{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$ *find at* $t = 2$

   i) The magnitude of tangential and normal components of acceleration **[Ans. :** $a_T = 16$, $a_N = 2\sqrt{73}$**]**

   ii) Components of velocity and acceleration in the direction $\hat{i} - 2\hat{j} + 2\hat{k}$. **[Ans. :** $-\frac{16}{3}$, $-\frac{32}{3}$**]**

4. *Find the tangential and normal components of acceleration at any time t for the curve* $\bar{r} = at\cos t\,\hat{i} + at\sin t\,\hat{j}$. **[Ans. :** $a_T = \frac{at}{\sqrt{1+t^2}}$, $a_N = \frac{a(t^2+2)}{\sqrt{1+t^2}}$**]**

5. *Find tangential and normal components of acceleration of a particle whose position at any time t is given by* $x = e^t \cos t$, $y = e^t \sin t$. **[Ans. :** $a_T = \sqrt{2}\, e^t$, $a_N = \sqrt{2}\, e^t$**]**

12.9 Radial and Transverse Components of Velocity and Acceleration of a Particle Describing a Plane Curve in Polar Co-ordinates

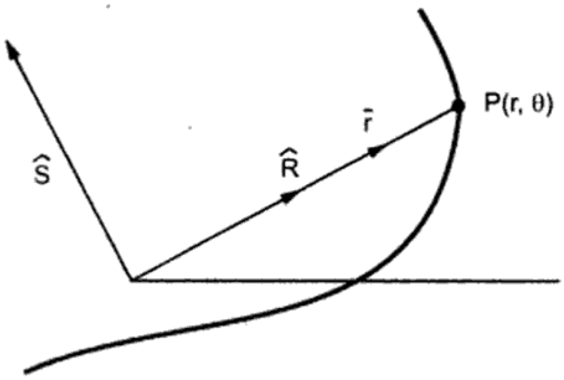


Fig. 12.14

Step 1 : Let $r = f(\theta)$ be the equation of curve and $p(r,\theta)$ be the position of a particle at time t.

Let $\hat{R}$ = unit vector in the radial direction

$\hat{s}$ = unit vector in the transverse direction

Step 2 : Consider $\hat{R}$ and $\hat{S}$. From Fig. 12.14

$$\hat{R} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{S} = \cos(\pi/2+\theta)\,\hat{i} + \sin(\pi/2+\theta)\,\hat{j}$$

$$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

Step 3 : Differentiate $\hat{R}$ w.r.t. t

$$\therefore \quad \frac{d\hat{R}}{dt} = -\sin\theta\frac{d\theta}{dt}\,\hat{i} + \cos\theta\frac{d\theta}{dt}\,\hat{j}$$

$$= \left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)\frac{d\theta}{dt}$$

$$\boxed{\frac{d\hat{R}}{dt} = \hat{S}\,\frac{d\theta}{dt}}$$

Step 4 : Differentiate $\hat{s}$ w.r.t. t

$$\frac{d\hat{s}}{dt} = -\cos\theta\frac{d\theta}{dt}\,\hat{i} - \sin\theta\frac{d\theta}{dt}\,\hat{j}$$

$$= -\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)\frac{d\theta}{dt}$$

$$\boxed{\frac{d\hat{S}}{dt} = -\hat{R}\,\frac{d\theta}{dt}}$$

Step 5 : To express $\bar{v}$ = (Rad vel) $\hat{R}$ + (Trans vel) $\hat{S}$

Consider $\hat{R}$,

$\hat{R}$ is the unit vector in the radial direction i.e.

$$\hat{R} = \frac{\bar{r}}{r}$$

Thus $$\bar{r} = r\,\hat{R}$$

Step 6 : Differentiate w.r.t. t

$$\frac{d\bar{r}}{dt} = \frac{dr}{dt}\hat{R} + r\frac{d\hat{R}}{dt}$$

$$= \left(\frac{dr}{dt}\right)\hat{R} + r\left(\hat{S}\frac{d\theta}{dt}\right) \quad \text{from step 3}$$

$$= \left(\frac{dr}{dt}\right)\hat{R} + \left(r\frac{d\theta}{dt}\right)\hat{S}$$

Step 7 : Comparing with $\bar{v}$ = (Rad vel) $\hat{R}$ + (trans vel) $\hat{S}$ we get

$$\text{Radial velocity} = \frac{dr}{dt}$$

$$\text{Trans. velocity} = r\frac{d\theta}{dt}$$

Step 8 : To express $\bar{a}$ = (Rad acceleration) $\hat{R}$ + (Trans acceleration) $\hat{S}$

Consider $\bar{v} = \left(\frac{dr}{dt}\right)\hat{R} + \left(r\frac{d\theta}{dt}\right)\hat{S}$

Step 9 : Differentiate w.r.t. t.

$$\frac{d\bar{v}}{dt} = \left(\frac{d^2r}{dt^2}\right)\hat{R} + \left(\frac{dr}{dt}\right)\frac{d\hat{R}}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{S} + \left(r\frac{d\theta}{dt}\right)\frac{d\hat{S}}{dt}$$

Step 10 : We know $\frac{d\hat{R}}{dt} = \left(\frac{d\theta}{dt}\right)\hat{S}$ and $\frac{d\hat{S}}{dt} = -\left(\frac{d\theta}{dt}\right)\hat{k}$

$$\therefore \quad \bar{a} = \frac{d^2r}{dt^2}\hat{R} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{S} + \left(\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{S} + r\frac{d\theta}{dt}\left(-\frac{d\theta}{dt}\hat{R}\right)$$

Step 11 : Collecting the terms.

$$\bar{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{R} + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\hat{S}$$

Thus Radial acceleration $= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

$$\text{Trans. acceleration} = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}$$

Note : i) For convenience, lets dots denote differentiation w.r.t. the time t. Then

$$\text{radial velocity} = \dot{r}$$

$$\text{Trans. velocity} = r\dot{\theta}$$

$$\dot{r} = \frac{1}{2\sqrt{a^2+b^2 t^2}} \cdot 2b^2 t$$

$$\dot{r} = \frac{b^2 t}{r}$$

Thus $$\ddot{r} = b^2\left[\frac{r(1)-t(\dot{r})}{r^2}\right]$$

$$= b^2\left[\frac{r-t\left(\frac{b^2 t}{r}\right)}{r^2}\right]$$

$$= \frac{b^2}{r^2}\left[r-\frac{b^2 t^2}{r}\right]$$

$$= \frac{b^2}{r^2}\left[\frac{r^2-b^2 t^2}{r}\right]$$

$$= \frac{b^2}{r^2}\left[\frac{a^2+b^2 t^2-b^2 t^2}{r}\right]$$

$$= \frac{b^2}{r^3}\left[a^2\right]$$

$$= \frac{a^2 b^2}{r^3}$$

Step 3 : Substitute all values

$\therefore$ Radial velocity $= \dot{r} = \dfrac{b^2 t}{\sqrt{a^2+b^2 t^2}} = \dfrac{b^2 t}{r}$

Trans velocity $= r\dot{\theta} = \dfrac{1}{\sqrt{a^2+b^2 t^2}}\, w$

Radial acceleration $= \ddot{r} - r\dot{\theta}^2$

$$= \frac{a^2 b^2}{r^3} - r w^2$$

Trans acceleration $= 2\dot{r}\dot{\theta} + r\ddot{\theta}$

$$= 2\,\frac{b^2 t}{r}\,\overline{w} + 0$$

Example 12.19 : *Find radial and transverse components of velocity and acceleration of a particle describing with constant angular velocity the curve $r = a(1+\cos\theta)$. If the particle moves with constant speed v show that $\frac{d\theta}{dt} \propto r^{-1/2}$.*

Solution : Step 1 : Consider r and find $\dot{r}$, $\ddot{r}$.

Given $\dot{\theta} = \frac{d\theta}{dt} = w = \text{constant}$

$\therefore \quad \ddot{\theta} = 0$

Now

$$r = a(1+\cos\theta)$$

$$\frac{dr}{dt} = a(-\sin\theta)\frac{d\theta}{dt}$$

$$\therefore \quad \dot{r} = -a\sin\theta\, w$$

Also

$$\ddot{r} = -a\cos\theta\frac{d\theta}{dt}\, w$$

$$= -a w^2 \cos\theta$$

Step 2 : Substitute all values

$$\text{Radial velocity} = \dot{r} = -a w \sin\theta$$

$$\text{Trans velocity} = r\theta = r w$$

$$= a(1+\cos\theta)\, w$$

$$\text{Radial acceleration} = \ddot{r} - r\dot{\theta}^2$$

$$= -a w^2 \cos\theta - a(1+\cos\theta) w^2$$

$$= -a w^2 [\cos\theta + 1 + \cos\theta]$$

$$= -a w^2 [1 + 2\cos\theta]$$

$$\text{Trans acceleration} = 2\dot{r}\theta + r\ddot{\theta}$$

$$= 2(-a w \sin\theta) w + 0$$

$$= -2 a w^2 . \sin\theta$$

Step 3 : We know that $v^2 = (\text{Radial Vel})^2 + (\text{Trans Vel})^2$

$$\therefore \quad v^2 = (-a w \sin\theta)^2 + (a w (1+\cos\theta))^2$$

$$v^2 = a^2 w^2 \left[\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta\right]$$

$$= a^2 w^2 [2 + 2\cos\theta]$$

$$= a^2 w^2 \cdot 2(1+\cos\theta)$$

$$= 2 w^2 \cdot a \cdot a(1+\cos\theta)$$

$$v^2 = 2 w^2 \cdot a \cdot r$$

Thus $$\frac{v^2}{2\,ar} = w^2$$

i.e. $$w^2 = \frac{v^2}{2\,a}\, r^{-1}$$

Take square root

$$w = \frac{v}{\sqrt{2a}}\, r^{-1/2}$$

i.e. $$\frac{d\theta}{dt} = \frac{v}{\sqrt{2a}}\, r^{-1/2}$$

Thus $$\frac{d\theta}{dt} \propto r^{-1/2}$$

Example 12.20 : *i) A particle P moves in a plane with constant angular velocity w about 0. If the rate of growth of acceleration is parallel to OP (i.e. wholly radial) prove that* $\frac{d^2r}{dt^2} = \frac{1}{3} r w^2$

ii) If initially the particle is at A and starts with velocity aw perpendicular to OA. Show that $(\dot{r})^2 = \frac{w^2}{3}\left(r^2 - a^2\right)$

iii) Also find r as a function of t.

Solution : Step 1 : Let $\hat{R}$ = Unit vector along the radial direction

$\hat{S}$ = unit vector along the transverse.

We know that $$\frac{d\hat{R}}{dt} = w\,\hat{S}$$

and $$\frac{d\hat{S}}{dt} = -w\,\hat{R}$$

where $$w = \frac{d\theta}{dt}$$

Step 2 : We know that

$$\bar{a} = (\text{Radial acceleration})\,\hat{R} + (\text{Trans acceleration})\,\hat{S}$$

i.e. $$\bar{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{R} + \left(2\,\dot{r}\,\dot{\theta} + r\ddot{\theta}\right)\hat{S}$$

Step 10 : Integrate w.r.t. t

$$(\dot{r})^2 = \frac{w^2}{3} r^2 + C_1$$

Step 11 : Initially i.e. at t = 0 the particle is at A and starts with velocity aw perpendicular to OA.

i.e. at t = 0 the tran vel = a w

i.e. at t = 0 $\quad rw = aw$

i.e. at t = 0 $\quad r = a$

i.e. at t = 0 $\quad \dot{r} = 0$

$\therefore$ at $\quad t = 0$

$$(0)^2 = \frac{w^2}{3} a^2 + C_1$$

$$\Rightarrow \quad C_1 = -\frac{w^2 a^2}{3}$$

Step 12 : Substituting c_1 we get

$$(\dot{r})^2 = \frac{w^2}{3} r^2 - \frac{a^2 w^2}{3}$$

i.e.

$$(\dot{r})^2 = \frac{w^2}{3} (r^2 - a^2)$$

which proves the second part

Step 13 : Take square root

$$\dot{r} = \frac{w}{\sqrt{3}} \sqrt{r^2 - a^2}$$

i.e.

$$\frac{dr}{dt} = \frac{w}{\sqrt{3}} \sqrt{r^2 - a^2}$$

$$\frac{dr}{\sqrt{r^2 - a^2}} = \frac{w}{\sqrt{3}} dt$$

Step 14 : Integrate

$$\log\left(r + \sqrt{r^2 - a^2}\right) = \frac{wt}{\sqrt{3}} + C_2$$

Step 15 : At t = 0, r = a, $\dot{r} = 0$ from step 11.

$$\therefore \quad \log(a) = 0 + C_2$$

12.10 Law of Central Orbits (Orbital Motion)

Let $r = f(\theta)$ be the polar curve. Let $P(r, \theta)$ be any particle moving along the curve under the action of a force which is always directed towards a fixed centre '0'.

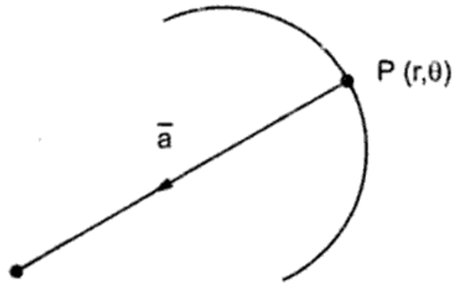


Fig. 12.16

We know that

$$\bar{a} = \left[\ddot{r} - r\left(\dot{\theta}\right)^2\right]\hat{R} + \left[2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right]\hat{S} \qquad \text{... (i)}$$

As the acceleration is directed towards the centre.

$\therefore$ its transverse component must be zero.

$$\therefore\ 2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta} = 0 \qquad \text{... (ii)}$$

Multiply by r

$$\therefore\ 2\,r\,\dot{r}\,\dot{\theta} + r^2\,\ddot{\theta} = 0$$

As $\dot{\theta} = \omega =$ constant $\therefore\ \ddot{\theta} = 0$

$$\therefore\ 2\,r\,\dot{r}\,\omega + 0 = 0$$

$$\therefore\ \frac{d}{dt}\left(r^2\,\omega\right) = 0$$

Integrating we get

$$r^2\,\omega = \text{constnat} = h \text{ (say)}$$

Let $\quad u = \dfrac{1}{r}$ for convenience.

Thus $\quad r^2\,\omega = h$ gives

$$\frac{\omega}{u^2} = h$$

$$\therefore\ \omega = h\,u^2 \quad \text{i.e.} \quad \frac{d\theta}{dt} = h\,u^2$$

We know that

$$\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= \frac{dr}{d\theta}\cdot h\,u^2$$

$$= \frac{d}{d\theta}\left(\frac{1}{u}\right)\cdot h\,u^2$$

Solution :

i) **Step 1 :** Write $u = \frac{1}{r}$

$$\text{Given} \qquad \frac{l}{r} = 1 + e\cos\theta$$

$$\therefore \qquad \frac{1}{r} = \frac{1 + e\cos\theta}{l}$$

$$\therefore \qquad u = \frac{1 + e\cos\theta}{l}$$

Step 2 : Differentiate w.r.t. θ

$$\frac{du}{d\theta} = 0 + \frac{e(-\sin\theta)}{l}$$

$$\frac{du}{d\theta} = -\frac{e}{l}\sin\theta$$

Step 3 : Here we can't express $\frac{du}{d\theta}$ in terms of θ.

Again differentiate w.r.t. θ.

$$\frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta$$

Step 4 : Consider $f = h^2 u^2 \left(\frac{d^2 u}{d\theta^2} + u\right)$ and substitute

$$f = h^2 u^2 \left[-\frac{e\cos\theta}{l} + \frac{1 + e\cos\theta}{l}\right]$$

$$f = \frac{h^2 u^2}{l}$$

$$\therefore \qquad f \propto u^2$$

$$\therefore \qquad f \propto r^{-2}$$

ii) **Step 1 :** Write $u = \frac{1}{r}$

$$\text{Given} \qquad r = a(1 + \cos\theta)$$

$$= a\left(2\cos^2\frac{\theta}{2}\right)$$

$$r = 2a\cos^2\frac{\theta}{2}$$

$$\therefore \quad u = \frac{1}{r}$$

$$= \frac{1}{2a\cos^2\theta/2}$$

$$u = \frac{1}{2a}\sec^2\frac{\theta}{2}$$

Step 2 : Differentiate w.r.t. θ.

$$\frac{du}{d\theta} = \frac{1}{2a}2\sec\frac{\theta}{2}\cdot\sec\frac{\theta}{2}\cdot\tan\frac{\theta}{2}\cdot\frac{1}{2}$$

$$= \frac{1}{2a}\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}$$

$$\frac{du}{d\theta} = u\tan\frac{\theta}{2} \qquad \left(\text{As } u = \frac{1}{2a}\sec^2\frac{\theta}{2}\right)$$

Step 3 : Again differentiate w.r.t. θ.

$$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\tan\frac{\theta}{2} + u\cdot\frac{1}{2}\sec^2\frac{\theta}{2}$$

Step 4 : Put the value of $\frac{du}{d\theta}$

$$\frac{d^2u}{d\theta^2} = u\tan\frac{\theta}{2}\cdot\tan\frac{\theta}{2} + \frac{u}{2}\sec^2\frac{\theta}{2}$$

$$= u\tan^2\frac{\theta}{2} + \frac{u}{2}\sec^2\frac{\theta}{2}$$

Step 5 : Consider $f = h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)$

$$f = h^2u^2\left[u\tan^2\frac{\theta}{2} + \frac{u}{2}\sec^2\frac{\theta}{2} + u\right]$$

Step 6 : Take u common

$$f = h^2u^3\left[\tan^2\frac{\theta}{2} + 1 + \frac{1}{2}\sec^2\frac{\theta}{2}\right]$$

Step 7 : We know $1+\tan^2 x = \sec^2 x$

$$f = h^2u^3\left[\sec^2\frac{\theta}{2} + \frac{1}{2}\sec^2\frac{\theta}{2}\right]$$

$$= h^2u^3\frac{3}{2}\sec^2\frac{\theta}{2}$$

From Fig. 12.17 (a)

$$\hat{T} = \cos\psi\,\hat{i} + \sin\psi\,\hat{j}$$

and

$$\hat{N} = \cos(\pi/2+\psi)\,\hat{i} + \sin(\pi/2+\psi)\,j$$

$$= -\sin\psi\,\hat{i} + \cos\psi\,\hat{j}$$

We can also write $\hat{N} = \dfrac{d\hat{T}}{d\psi}$

where ψ is the angle at point P made by the tangent with some fixed straight line. Let S be the distance of point P from some reference point on the curve.

$$\therefore \quad \frac{dS}{d\psi} = \text{radius of curvature} = p$$

Also we know that

$$\hat{T} = \text{unit vector along the tangent}$$

(i.e. in the direction of velocity)

$$= \frac{\bar{V}}{|\bar{V}|}$$

$$\hat{T} = \frac{\bar{V}}{V}$$

Thus

$$\bar{V} = V\hat{T}$$

$$\therefore \quad \frac{d\bar{V}}{dt} = \frac{dv}{dt}\hat{T} + V\frac{d\hat{T}}{dt}$$

$$\therefore \quad \bar{a} = \frac{dv}{dt}\hat{T} + V\frac{d\hat{T}}{d\psi}\frac{d\psi}{ds}\frac{ds}{dt} \qquad \text{by chain rule}$$

$$= \frac{dv}{dt}\hat{T} + V\hat{N}\left(\frac{1}{\rho}\right)V$$

$$= \frac{dv}{dt}\hat{T} + \frac{V^2}{\rho}\hat{N}$$

Thus tangential component of acceleration $= \dfrac{dv}{dt}$

i.e.

$$a_T = \frac{d^2s}{dt^2}$$

and normal component of acceleration $= \dfrac{v^2}{\rho}$

i.e.

$$a_N = \frac{(ds/dt)^2}{(ds/d\psi)}$$

12.12 Illustrations

Example 12.22 : *A particle moving along a curve with velocity $\bar{v}$ and acceleration $\bar{a}$. Show that $\rho = \dfrac{v^3}{|\bar{v} \times \bar{a}|}$. Also find the radius of curvature ρ and curvature k if $\bar{v} = p\,\hat{i} + q\,\hat{j}$ and $\bar{a} = r\,\hat{i} + s\,\hat{j}$.*

Solution : Step 1 : We know that

$$\bar{v} = V\hat{T} \text{ and}$$

$$\bar{a} = \frac{dv}{dt}\hat{T} + \frac{V^2}{\rho}\hat{N}$$

Step 2 : Consider $\bar{v} \times \bar{a}$

$$\bar{v} \times \bar{a} = v\,\hat{T} \times \left[\left(\frac{dv}{dt}\right)\hat{T} + \frac{v^2}{\rho}\hat{N}\right]$$

$$= v\frac{dv}{dt}\left(\hat{T} \times \hat{T}\right) + \frac{v^3}{\rho}\left(\hat{T} \times \hat{N}\right)$$

Step 3 : Put $\hat{T} \times \hat{T} = 0$ and $\hat{T} \times \hat{N} = \hat{W}$ = some unit vector.

$$\therefore \quad \bar{v} \times \bar{a} = 0 + \frac{v^3}{\rho}\left(\hat{W}\right)$$

Step 4 : Taking magnitudes on both sides.

$$|\hat{W}| = 1$$

$$\therefore \quad |\bar{v} \times \bar{a}| = \frac{v^3}{\rho}(1)$$

$$\therefore \quad \rho = \frac{V^3}{|\bar{v} \times \bar{a}|}$$

Thus $$\boxed{\rho = \frac{v^3}{|\bar{v} \times \bar{a}|}}$$

Note : We can use this as a standard formula for calculating ρ.

Step 5 : Given $\bar{v} = p\,\hat{i} + q\,\hat{j}$, $\bar{a} = r\,\hat{i} + s\,\hat{j}$

$$\therefore \quad |\bar{v}| = \sqrt{p^2 + q^2}$$

and $$\bar{v} \times \bar{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & 0 \\ r & s & 0 \end{vmatrix}$$

$$= \hat{k}\,(p\,s - r\,q)$$

Example 12.24 : *A particle moves along a curve $\bar{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + amt\,\hat{k}$ find the radius of curvature at $t = 0$.*

Solution : Step 1 : Consider $\bar{r}$ and find $\bar{v}$ and $\bar{a}$.

$$\bar{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + amt\,\hat{k}$$

$$\therefore \quad \bar{v} = \frac{d\bar{r}}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + am\,\hat{k}$$

$$\bar{a} = \frac{d^2\bar{r}}{dt^2} = -a\cos t\,\hat{i} - a\sin t\,\hat{j} + 0\,\hat{k}$$

Step 2 : Find $\bar{v}$ and $\bar{a}$ at $t = 0$

$$\bar{v} = 0\,\hat{i} + a\,\hat{j} + am\,\hat{k}$$

$$\bar{a} = -a\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$$

Step 3 : Find $\bar{v} \times \bar{a}$

$$\bar{v} \times \bar{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & a & am \\ -a & 0 & 0 \end{vmatrix}$$

$$= i(0-0) - j\left(0 + a^2 m\right) + k\left(0 + a^2\right)$$

$$= -a^2 m\,\hat{j} + a^2\,\hat{k}$$

Step 4 : Find $|\bar{v}|$ and $|\bar{v} \times \bar{a}|$

$$v = |\bar{v}| = \sqrt{0 + a^2 + a^2 m^2}$$

$$= a\sqrt{1 + m^2}$$

$$|\bar{v} \times \bar{a}| = \sqrt{a^4 m^2 + a^4}$$

$$= a^2\sqrt{1 + m^2}$$

Step 5 :

$$\rho = \frac{v^3}{|\bar{v} \times \bar{a}|}$$

$$\therefore \quad \rho = \frac{a^3\left(1 + m^2\right)^{3/2}}{a^2\left(1 + m^2\right)^{1/2}}$$

$$\rho = a\left(1 + m^2\right)$$

Example 12.25 : *Find ρ if $\bar{r} = e^t \hat{i} + e^{-t} \hat{j} + \sqrt{2} t \hat{k}$.*

Solution : **Step 1 :** Consider $\bar{r}$ and find $\bar{v}$ and $\bar{a}$.

$$\bar{r} = e^t \hat{i} + e^{-t} \hat{j} + \sqrt{2}\ t\ \hat{k}$$

$$\bar{v} = e^t \hat{i} - e^{-t} \hat{j} + \sqrt{2}\ \hat{k}$$

$$\bar{a} = e^t \hat{i} + e^{-t} \hat{j} + 0\ \hat{k}$$

Step 2 : Find $\bar{v} \times \bar{a}$

$$\bar{v} \times \bar{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ e^t & -e^{-t} & \sqrt{2} \\ e^t & e^{-t} & 0 \end{vmatrix}$$

$$= i\left(0 - \sqrt{2}\ e^{-t}\right) - \hat{j}\left(0 - \sqrt{2}\ e^t\right) + \hat{k}\ (1+1)$$

$$= -\sqrt{2}\ e^{-t}\ \hat{i} + \sqrt{2}\ e^t\ \hat{j} + 2\hat{k}$$

Step 3 : Find $|\ \bar{v}\ |$ and $|\bar{v} \times \bar{a}|$

$$|\ \bar{v}\ | = \sqrt{e^{2t} + e^{-2t} + 2}$$

$$= \sqrt{\left(e^t + e^{-t}\right)^2}$$

$$= e^t + e^{-t}$$

and

$$|\bar{v} \times \bar{a}| = \sqrt{2\ e^{-2t} + 2\ e^{2t} + 4}$$

$$= \sqrt{2\left(e^{2t} + e^{-2t} + 2\right)}$$

$$= \sqrt{2\left(e^t + e^{-t}\right)^2}$$

$$= \sqrt{2}\ \left(e^t + e^{-t}\right)$$

Step 4 :

$$\rho = \frac{v^3}{|\bar{v} \times \bar{a}|}$$

$$\rho = \frac{\left(e^t + e^{-t}\right)^3}{\sqrt{2}\left(e^t + e^{-t}\right)}$$

$$\rho = \frac{\left(e^t + e^{-t}\right)^2}{\sqrt{2}}$$

$$= \frac{1}{a \sec\psi} (a\,\omega \sec\psi)^2$$

$$= a\,\omega^2 \sec\psi$$

Step 4 : The resultant acceleration is given by

$$|\bar{a}| = \sqrt{a_T^2 + a_N^2}$$

Squaring we get

$$|a^2|^2 = a_T^2 + a_N^2$$

Step 5 : Substituting the values we get

$$|\bar{a}|^2 = (a\,\omega^2 \sec\psi \tan\psi)^2 + (a\,\omega^2 \sec\psi)^2$$

$$|\bar{a}|^2 = a^2\,\omega^4 \sec^2\psi\,[\tan^2\psi + 1]$$

$$= a^2\,\omega^4 \sec^4\psi$$

$$\therefore \quad |\bar{a}| = a\,\omega^2 \sec^2\psi$$

As $\quad \rho = a \sec\psi$

$$\therefore \quad |\bar{a}| = \frac{\omega^2}{a}\rho^2$$

$$\therefore \quad |\bar{a}| \propto \rho^2$$

Thus acceleration varies as square of radius of curvature.

Example 12.27 : *A particle moves along catenary $s = c\tan\psi$, the direction of acceleration of any point make equal angles with the tangent and normal to the path at that point. If the speed at the vertex $(\psi = 0)$ be v_0. Show that the magnitudes of velocity and acceleration at any point are given by $v_0 e^{\psi}$ and $\frac{\sqrt{2}}{c} v_0^2 e^{2\psi} \cos^2\psi$.*

Solution : Step 1 : Given that the direction of acceleration makes equal angles with tangent and normal.

$$\therefore \quad a_T = a_N$$

Step 2 : We know $\bar{a} = \frac{dv}{dt}\hat{T} + \frac{v^2}{\rho}\hat{N}$

$$\therefore \quad a_T = \frac{dv}{dt}, \quad a_N = \frac{v^2}{\rho}$$

Substituting we get

$$\frac{dv}{dt} = \frac{1}{\rho} V^2$$

Applying chain rule

$$\therefore \quad \frac{dv}{ds}\frac{ds}{dt} = \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2$$

$$\therefore \quad \frac{dv}{ds} = \frac{1}{\rho}\frac{ds}{dt}$$

$\therefore$ Substituting $\rho = \dfrac{ds}{d\psi}$

$$\therefore \quad \frac{dv}{ds} = \frac{1}{(ds/d\psi)} V$$

$$\therefore \quad \frac{dv}{v} = d\psi$$

Step 3 : Integrating both sides, we get

$$\int_{v_0}^{v} \frac{dv}{v} = \int_{0}^{\psi} d\psi$$

Step 4 : $[\log v]_{v_0}^{v} = \psi$

$$\log \frac{V}{V_0} = \psi$$

$$\therefore \quad \frac{V}{V_0} = e^{\psi}$$

$$V = V_0 e^{\psi} \qquad \text{... (i)}$$

Step 5 : Which proves the first part

Given $\quad s = c \tan \psi$

$$\therefore \quad \frac{ds}{d\psi} = c \sec^2 \psi$$

$$\therefore \quad \rho = c \sec^2 \psi$$

Step 6 : We know that

$$a_T = \frac{dv}{dt}, \quad a_N = \frac{v^2}{\rho}$$

and here $\quad a_T = a_N$

$$\therefore \quad |\bar{a}| = \sqrt{a_T^2 + a_N^2}$$

$$= \sqrt{2 a_N^2}$$

d) The operator ∇ or nabla : The vector differential operator (del)

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

12.14 Gradient of a Scalar

Point function : If $\phi(x, y, z)$ is a scalar point function and the del operator ∇ operates on a scalar point function $\phi(x, y, z)$. We get a vector function $\nabla\phi$ and is called as gradient of a scalar point function ϕ (OR Grad ϕ).

Thus

$$\begin{aligned}\text{Grad}\ \phi &= \nabla\phi \\ &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi \\ &= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\end{aligned}$$

i) Geometrical meaning of gradient :

Consider a general point $p(x, y, z)$ on a level surface $\phi(x, y, z) = c$.

$$\therefore \quad \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Consider $\nabla\phi \cdot d\bar{r}$

$$\begin{aligned} &= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right) \\ &= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \\ &= d\phi \\ &= 0 \qquad \left\{\begin{array}{ll}\text{As} & \phi(x, y, z) = c \\ \therefore & d\phi = 0\end{array}\right\}\end{aligned}$$

As $\nabla\phi \cdot d\bar{r} = 0$

$\therefore$ $\nabla\phi$ is perpendicular to $d\bar{r}$

As $d\bar{r}$ acts along the tangent at $p(x, y, z)$. Hence $\nabla\phi$ acts along the normal to the surface $\phi = c$ at $p(x, y, z)$.

Thus $\quad (\nabla\phi)_p$ = vector normal at point p

Step 4 : Take $\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}, \frac{\partial \phi}{\partial w}$ common

$$\nabla \phi = \frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}\right]$$

$$+ \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x}\hat{i} + \frac{\partial v}{\partial y}\hat{j} + \frac{\partial v}{\partial z}\hat{k}\right]$$

$$+ \frac{\partial \phi}{\partial w}\left[\frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial z}\hat{k}\right]$$

Step 5 : Use $\nabla\phi$ formula for ∇ u, ∇ v, ∇ w.

$$\nabla \phi = \frac{\partial \phi}{\partial u}\nabla u + \frac{\partial \phi}{\partial v}\nabla v + \frac{\partial \phi}{\partial \omega}\nabla \omega$$

Hence proved.

Example 12.33 : *Find the angle between the normals to the surface.* $xy = z^2$ *at* $(1, 4, 2)$ *and* $(-3, -3, 3)$.

Solution : Note : [Grad ϕ represents the vector normal to the level surface $\phi = C$].

Step 1 : Consider $\phi = 0$

Let $\phi = xy - z^2$

Step 2 : Find $\nabla \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

$$\nabla \phi = y\,\hat{i} + x\,\hat{j} - 2z\,\hat{k}$$

Step 3 : As $\nabla\phi$ represents the vector normal $\therefore$ Let $\overline{N}_1$ and $\overline{N}_2$ be the values of $\nabla\phi$ at the two given points i.e. substituting two points in $\nabla\phi$ we get two normals $\overline{N}_1$ and $\overline{N}_2$.

$$\overline{N}_1 = (\nabla\phi)_{(1,4,2)} = 4\,\hat{j} + \hat{j} - 4\,\hat{k}$$

$$\overline{N}_2 = (\nabla\phi)_{(-3,-3,3)} = -3\,\hat{i} - 3\,\hat{j} - 6\,\hat{k}$$

Step 4 : Use angle between two vectors formula

$$\cos\theta = \frac{\overline{N}_1 \cdot \overline{N}_2}{|\,\overline{N}_1\,|\,|\,\overline{N}_2\,|}$$

Step 5 : Use $\bar{a}\cdot\bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ and $|\,\bar{a}\,| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$$\cos\theta = \frac{-12 - 3 + 24}{\sqrt{16+1+16}\,\sqrt{9+9+36}}$$

Example 12.36 : *Find constant a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface. $4x^2y + z^3 = 4$ at $(1,-1,2)$.*

Step 1 : Consider $\phi_1 = 0$, $\phi_2 = 0$ as two surfaces.

$$\phi_1 = a x^2 - b y z - (a+z) x = 0$$

$$\phi_2 = 4x^2 y + z^3 - 4 = 0$$

Step 2 : Find $\nabla\phi_1$ and $\nabla\phi_2$

$$(\nabla\phi_1) = (2ax - a - 2) i - bz j - (by) k = 0$$

$$(\nabla\phi_2) = 8xy i + 4x^2 j + 3z^2 k - 0 = 0$$

Step 3 : Substituting the point in $\nabla\phi_1$ and $\nabla\phi_2$ we get

$$\overline{N}_1 = (\nabla\phi_1)_{(1,-1,2)} = (2a - a - 2) i - 2b j + b k = 0$$

$$\overline{N}_2 = (\nabla\phi_2)_{(1,-1,2)} = -8 i + 4 j + 12 k = 0$$

Step 4 : As two surfaces are orthogonal.

$$\overline{N}_1 \cdot \overline{N}_2 = 0$$

$$\cos\theta = 0 \qquad \theta = 90°$$

Consider

$$\overline{N}_1 \cdot \overline{N}_2 = -16a + 8a + 16 - 8b + 12b$$

$$0 = -8a + 4b + 16$$

$$0 = 4(-2a + b + 4)$$

$$\therefore \quad 2a = b + 4 \qquad \text{... (1)}$$

Step 5 : The point $(1,-1,2)$ lies on $a x^2 - b y z = (a+2) x$

The point will satisfies the equation

$$a x^2 - b y z = (a + 2) x$$

$$a(1)^2 - b(-1)(2) = (a + 2) 1$$

$$a + 2b = a + 2$$

$$b = 1 \qquad \text{... (2)}$$

Step 6 : Substitute the value of b in equation (1)

$$2a = b + 4$$

$$2a = 5$$

$$a = \frac{5}{2}$$

Exercise 12.6

1. *Find* $\nabla\phi$ *for i)* $\phi = x^2 + y^2 + z^2$ *at (1, 1, 1) ii)* $\phi = 2xz^4 - x^2y$ *at (2, – 2, 1) iii)* $\phi = r^m$ *where* $\bar{r} = xi + yj + zk$ *iv)* $\phi = \log\left(x^2 + y^2 + z^2\right)$ *v)* $\phi = \frac{r^3}{e^r}$ **[Ans. : i)** $\nabla\phi = 2\left(\hat{i} + \hat{j} + \hat{k}\right)$, **ii)** $10\,\hat{i} - 4\,\hat{j} + 6\,\hat{k}$, **iii)** $\nabla\phi = m\,r^{m-2}\,\bar{r}$, **iv)** $\frac{2\left(x\,\hat{i} + y\,\hat{j} + z\,\hat{k}\right)}{\left(x^2 + y^2 + z^2\right)}$, **v)** $\nabla\phi = \frac{\bar{r}}{e^r}\left(3r - r^2\right)$**]**

2. *If u, v, w are three scalar point functions such that u =* ϕ *(v, w) prove that*

$$\nabla u = \frac{\partial\phi}{\partial v}\nabla v + \frac{\partial\phi}{\partial w}\nabla w$$

3. *Show that* $\nabla\int f(u)\,du = f(u)\,\nabla u$

4. *Find the unit vector normal to the surface* $x^2 + y^2 + z^2 = a^2$ *at the point* $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$ **[Ans. :** $\pm\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$**]**

5. *Determine 'a' such that the tangent plane to the surface* $x^3 - 2xy + yz = a + 4$ *at the point (2, 1, a) will pass through origin. Hint :* $\nabla\phi\cdot\overline{op} = 0$ **[Ans. : a = – 8]**

6. *Find the constants a and b such that the surface* $ax^2 - byz = (a+1)x$ *is orthogonal to the surface* $xy^2 - yz^2 = 3$ *at (2, – 1, 1)* **[Ans. :** $a = \frac{9}{7}$, $b = \frac{-4}{7}$**]**

7. *Find the constants 'a' such that at any point of intersection of* $(x-a)^2 + y^2 + z^2 = 3$ *and* $x^2 + (y-1)^2 + z^2 = 1$ *their tangent planes will be perpendicular to each other.* **[Ans. :** $a = \pm\sqrt{3}$**]**

8. *Prove that*

   i) $\nabla\left[\bar{r}\cdot\left(\bar{a}\times\bar{b}\right)\right] = \bar{a}\times\bar{b}$

   ii) $\nabla\int r^n\,dr = r^{n-1}\bar{r}$

9. *If* $u = 3x^2y$, $v = xz^2 - 2y$ *then find grad* $[grad\cdot grad\,v]$ **[Ans. :** $\left(6yz^2 - 12x\right)\hat{i} + 6xz^2\hat{j} + 12xyz\,\hat{k}$**]**

10. *Find i)* $\nabla\left(e^{r^2}\right)$ *ii)* $\nabla(\log r)$ *iii)* $\nabla\left(r^2e^r\right)$ **[Ans. : i)** $2\,\bar{r}\,e^{r^2}$, **ii)** $\frac{\bar{r}}{r^2}$, **iii)** $(r+2)\,e^r\,\bar{r}$**]**

11. *Prove that* $\bar{r}\cdot\nabla v = 3v$ *where* $v = xyz$

12. *Prove that* $\nabla f(r)\times\bar{r} = 0$

13. *Find the angle between the surfaces* $x\log z = y^2 - 1$ *and* $x^2y = 2 - z$ *at (1, 1, 1).* **[Ans. :** $\cos^{-1}\frac{1}{\sqrt{30}}$**]**

14. *Find the constants a and b such that the surface* $ax^2 - byz = (a+1)x$ *is orthogonal to the* $xy^2 - yz^2 = 3$ *at (2, – 1, 1)* **[Ans. :** $a = \frac{13}{11}$, $b = \frac{-4}{11}$**]**

15. *Find a vector of magnitude* $\sqrt{22}$ *normal to* $xy^3z^2 = 4$ *at (– 1, – 1, 2)* **[Ans. :** $-\sqrt{2}\,(i + 3j - k)$**]**

Example 12.40 : *Find the rate of change of $f = xyz$ in the direction normal to the surface $x^2y + xy^2 + yz^2 = 3$ at $\left(\hat{i}+\hat{j}+\hat{k}\right)$.*

Solution : Step 1 : Consider ϕ

$$\phi = x\,y\,z$$

Step 2 : Find $\nabla\phi$

$$\nabla\phi = y\,z\,\hat{i} + x\,z\,\hat{j} + x\,y\,\hat{k}$$

Step 3 : Here $\hat{i}+\hat{j}+\hat{k}$ is given means the point is $(1,1,1)$.

$\therefore$ Substitute $(1,1,1)$

$$\therefore \qquad (\nabla\phi)_p = \hat{i}+\hat{j}+\hat{k}$$

Step 4 : Here $\bar{u}$ is not given. To find $\bar{u}$.

$\therefore$ Consider f as the given surface.

Let $$f = x^2\,y + x\,y^2 + y\,z^2 = 3$$

Step 5 : ∇f represents the vector normal to f $\therefore$ find $\nabla f = \hat{i}\dfrac{\partial f}{\partial x} + \hat{j}\dfrac{\partial f}{\partial y} + \hat{k}\dfrac{\partial f}{\partial z}$

$$\nabla f = \left(2xy + y^2\right)\hat{i} + \left(x^2 + 2xy + z^2\right)\hat{j} + (2yz)\,\hat{k}$$

Step 6 : Find ∇f at point $(1,1,1)$

$$(\nabla f)_{(111)} = 3\,\hat{i} + 4\,\hat{j} + 2\,\hat{k}$$

Step 7 : Thus the required direction is

$$\bar{u} = (\nabla f)_p$$

$$\therefore \qquad \bar{u} = 3\,\hat{i} + 4\,\hat{j} + 2\,\hat{k}$$

Step 8 : Find $\hat{u}$

$$\hat{u} = \frac{3\,\hat{i}+4\,\hat{j}+2\,\hat{k}}{\sqrt{9+16+4}} = \frac{3\,\hat{i}+4\,\hat{j}+2\,\hat{k}}{\sqrt{29}}$$

Step 9 :

$$\text{D.D.} = (\nabla\phi)_p \cdot \hat{u}$$

$$\text{D.D.} = \left(\hat{i}+\hat{j}+\hat{k}\right)\cdot\frac{(3\,\hat{i}+4\,\hat{j}+2\,\hat{k})}{\sqrt{29}}$$

$$= \frac{3+4+2}{\sqrt{29}} = \frac{9}{\sqrt{29}}$$

Example 12.41 : *Find the directional derivative of $\phi = x^2 - y^2 - 2z^2$ at the point $p(2, -1, 3)$ in the direction PQ where Q is $(5, 6, 4)$.*

Solution : Step 1 : Consider ϕ

$$\phi = x^2 - y^2 - 2z^2$$

Step 2 : Find $\nabla\phi$

$$\nabla\phi = 2x\,\hat{i} - 2y\,\hat{j} - 4z\,\hat{k}$$

Step 3 : Find $\nabla\phi$ at point p $(2, -1, 3)$

$$(\nabla\phi)_p = 4\,\hat{i} + 2\,\hat{j} - 12\,\hat{k}$$

Step 4 : Here $\bar{u}$ is not given.

To find
$$\begin{aligned} \bar{u} &= \overline{PQ} \\ &= \overline{Q} - \overline{P} \\ &= \left(5\,\hat{i} + 6\,\hat{j} + 4\,\hat{k}\right) - \left(2\,\hat{i} - \hat{j} + 3\,\hat{k}\right) \\ &= 3\,\hat{i} + 7\,\hat{j} + \hat{k} \end{aligned}$$

Step 5 : Find $\hat{u}$

$$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{3\,\hat{i} + 7\,\hat{j} + \hat{k}}{\sqrt{9 + 49 + 1}} = \frac{3\,\hat{i} + 7\,\hat{j} + \hat{k}}{\sqrt{59}}$$

Step 6 :
$$\begin{aligned} \text{D.D.} &= (\nabla\phi)_p \cdot \hat{u} \\ &= \left(4\,\hat{i} + 2\,\hat{j} - 12\,\hat{k}\right) \cdot \frac{3\,\hat{i} + 7\,\hat{j} + \hat{k}}{\sqrt{59}} \\ &= \frac{12 + 14 - 12}{\sqrt{59}} \\ &= \frac{14}{\sqrt{59}} \end{aligned}$$

Example 12.42 : *Find DD of $\phi = xy^2 + yz^3$ at $(2, -1, 1)$ along the line $2(x-2) = (y+1) = z - 1$.*

Solution : Step 1 : Consider ϕ

$$\phi = xy^2 + yz^3$$

Step 2 : Find $\nabla\phi$

$$\nabla\phi = y^2\,\hat{i} + \left(2xy + z^3\right)\hat{j} + \left(3xyz^2\right)\hat{k}$$

Step 3 : Find $\nabla\phi$ at $(2, -1, 1)$

$$(\nabla\phi)_p = \hat{i} - 3\,\hat{j} - 3\,\hat{k}$$

Step 4 : Here $\bar{u}$ is not given.

To find $\bar{u}$ consider equation of line joining two points.

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

i.e. $$\frac{x-x_1}{\alpha} = \frac{y-y_1}{\beta} = \frac{z-z_1}{\gamma}$$

where $$\alpha = x_2-x_1$$
$$\beta = y_2-y_1$$
$$\gamma = z_2-z_1$$

are known as the direction ratios of the line and its direction is given by $\alpha\,\hat{i}+\beta\,\hat{j}+\gamma\,\hat{k}$

$\therefore$ Consider the given equation of line

$$2(x-2) = y+1 = z-1$$

Divide by 2

$$\therefore \quad \frac{x-2}{1} = \frac{y+1}{2} = \frac{z-1}{2}$$

Thus $\alpha = 1,\ \beta = 2,\ \gamma = 2$

$\therefore$ direction $\quad \bar{u} = \hat{i}+2\,\hat{j}+2\,\hat{k}$

Step 5 : Find $\hat{u}$

$$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{\hat{i}+2\,\hat{j}+2\,\hat{k}}{\sqrt{1=4+4}} = \frac{\hat{i}+2\,\hat{j}+2\,\hat{k}}{3}$$

Step 6 :

$$DD = (\nabla\phi)_P \cdot \hat{u}$$
$$= (i-3\,j-3\,k)\cdot\frac{\hat{i}+2\,\hat{j}+2\,\hat{k}}{3}$$
$$= \frac{1-6-6}{3}$$
$$= -\frac{11}{3}$$

Example 12.43 : *Find the directional derivative of $\phi = x^2yz^2$ along the curve $x = e^{-u}$, $y = 2\sin u+1$, $z = u-\cos u$ at the point P where $u = 0$.*

Solution : Step 1 : Here to get point p put u = 0 in x, y, z.

$\therefore$ P is $(1,1,-1)$

$$(\nabla\phi)_p = (2a+c)\,\hat{i}+(a+2b)\,\hat{j}+(b+2c)\,\hat{k}$$

Step 4 : The direction of given line is $\bar{u} = 2i - 2j + k$.

Unit vector in this direction $\hat{u} = \dfrac{2\hat{i}-2\hat{j}+\hat{k}}{\sqrt{4+4+1}} = \dfrac{2\hat{i}-2\hat{j}+\hat{k}}{3}$

We know that the directional derivative is max in the direction of $\nabla\phi$. But here it is given that DD is max in the direction of $\bar{u}$.

$\therefore$ The direction of $\nabla\phi$ and $\bar{u}$ must be same.

$\therefore$ Their unit vectors must be equal.

$\therefore \quad \dfrac{\nabla\phi}{|\nabla\phi|} = \hat{u}$

$\therefore \quad \nabla\phi = |\nabla\phi|\,\hat{u}$

given $\quad |\nabla\phi| = 15$

Thus $\quad \nabla\phi = 15\,\hat{u}$

Substituting $\nabla\phi$ and $\hat{u}$ we get

$$(2a+c)\,\hat{i}+(a+2b)\,\hat{j}+(b+2c)\,\hat{k} = 15\left(\frac{2\hat{i}-2\hat{j}+\hat{k}}{3}\right) = 5\left(2\hat{i}-2\hat{j}+\hat{k}\right)$$

Step 5 : Equating we get

$$2a + c = 10$$

$$a + 2b = -10$$

$$b + 2c = 5$$

Step 6 : Solving we get

$$a = \frac{20}{9} \qquad b = -\frac{55}{9} \qquad c = \frac{50}{9}$$

Example 12.46 : *If the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has maximum magnitude 64 in the direction parallel to z axis. Find the values of a, b, c.*

Solution : **Step 1 :** Consider ϕ

$$\phi = axy^2 + byz + cz^2x^3$$

Step 2 : Find $\nabla\phi$

$$\nabla\phi = \left(a\,y^2 + 3\,c\,z^2\,x^2\right)\hat{i} + \left(ax + 2ybz\right)\hat{j} + \left(by + 2\,c\,z\,x^3\right)\hat{k}$$

Step 3 : Find $\nabla\phi$ at point P $(1, 2, -1)$

$$(\nabla\phi)_p = (4a+3c)\,\hat{i}+(4a-b)\,\hat{j}+(2b-2c)\,\hat{k}$$

Find all the directions say $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$.

Let
$$\bar{u}_1 = \overline{AB} = \bar{B} - \bar{A} = \hat{i} + \hat{j}$$
$$\bar{u}_2 = \overline{AC} = \bar{C} - \bar{A} = 0\,\hat{i} - 2\,\hat{j}$$
$$\bar{u}_3 = \overline{AD} = \bar{D} - \bar{A} = 3\,\hat{i} + 4\,\hat{j}$$

Step 2 : Find their unit vectors

$$\hat{u}_1 = \frac{\hat{i}+\hat{j}}{\sqrt{2}}, \quad \hat{u}_2 = \frac{0\,\hat{i}-2\,\hat{j}}{\sqrt{4}}, \quad \hat{u}_3 = \frac{3\,\hat{i}+4\,\hat{j}}{\sqrt{9+16}}$$
$$\hat{u}_1 = \frac{\hat{i}+\hat{j}}{\sqrt{2}} \qquad \hat{u}_2 = -\,\hat{j} \qquad \hat{u}_3 = \frac{3\,\hat{i}+4\,\hat{j}}{5}$$

Step 3 : Here $f(x, y)$ is not given $\therefore$ Assume ∇f.

Let
$$(\nabla f)_A = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j}$$

Step 4 : To find $(\nabla f)_A$ we need $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$

These are two unknowns. To find two unknowns we need two conditions.

Given
$$(\nabla f)_A \cdot \hat{u}_1 = 2\sqrt{2} \quad \text{and} \quad (\nabla f)_A \cdot \hat{u}_2 = -3$$

Step 5 : Substitute ∇f and $\hat{u}_1$ and $\hat{u}_2$

$$\therefore \left(\frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j}\right) \cdot \left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right) = 2\sqrt{2}, \quad \left(\frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j}\right) \cdot \left(-\,\hat{j}\right) = -3$$

i.e.
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 4 \quad , \quad 0\,i - \frac{\partial f}{\partial y} = -3$$
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 4 \quad \text{.... (1)} \qquad \frac{\partial f}{\partial y} = 3 \quad \text{.... (2)}$$

Thus solving (1) and (2) we get
$$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = 3$$

Thus
$$(\nabla f)_A = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j}$$
$$(\nabla f)_A = \hat{i} + 3\,\hat{j}$$

Step 5 : Let $\bar{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ be the direction in which DD = 0.

$$\therefore \quad (\nabla T)_p \cdot \hat{a} = 0$$

$$\therefore \quad (\nabla T)_p \cdot \frac{\bar{a}}{|\bar{a}|} = 0$$

$$\therefore \quad (\nabla T)_p \cdot \bar{a} = 0$$

$$\therefore \left(2\hat{i} + 3\hat{j} + 4\hat{k}\right) \cdot \left(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}\right) = 0$$

$$\therefore 2a_1 + 3a_2 + 4a_3 = 0$$

If $a_1 = 1, \quad a_2 = 0 \quad \Rightarrow \quad a_3 = -1/2$

$a_1 = 0 \quad a_2 = 1 \quad \Rightarrow \quad a_3 = -3/4$

$a_1 = 1 \quad a_2 = 1 \quad \Rightarrow \quad a_3 = -5/4$

$$\therefore \quad \hat{i} + 0\hat{j} - \frac{1}{2}\hat{k}, \quad 0\hat{i} + \hat{j} - \frac{3}{4}\hat{k}, \quad \hat{i} + \hat{j} - \frac{5}{4}\hat{k}$$

are the three directions in which the DD is zero.

Exercise 12.7

1. *Find the directional derivative of* $f = 4xz^3 - 3x^2y^2z$ *at*

   i) $(2, -1, 2)$ *in the direction towards the point* $\hat{i} + \hat{j} + \hat{k}$.

   Hint : $\bar{u} = \hat{i} + \hat{j} + \hat{k}$ **[Ans. :** $\frac{140}{\sqrt{3}}$**]**

   ii) $(2, -1, 1)$ *along the line which makes equal angles with co-ordinate axes.*

   Hint : $\bar{u} = \hat{i} + \hat{j} + \hat{k}$ **[Ans. :** $\frac{28}{\sqrt{3}}$**]**

2. *Find the directional derivative of* $\phi = xy^2 + yz^3$ *at* $(2, -1, 1)$ *in the direction normal to the surface* $x \log z - y^2 = -4$ *at* $(-1, 2, 1)$ **[Ans. :** $\frac{15}{\sqrt{17}}$**]**

3. *Find the rate of change of* $\phi = r e^{\sqrt{r}}$ *at A* $(1, 2, 5)$ *in the direction joining this point to the point B* $(2, 3, 7)$.

   Hint : $\nabla\phi = \frac{\partial \phi}{\partial r}\hat{r}$

   $= e^{\sqrt{r}}\left(1 + \frac{\sqrt{r}}{2}\right)\hat{r}$ **[Ans. :** $\frac{13 \cdot e^{(30)^{1/4}}}{6\sqrt{5}}\left[1 + \frac{(30)^{1/4}}{2}\right]$**]**

4. *Find the directional derivative of* $\phi = x^2 + 2y^2 - 3z^2$ *at* $(1, 2, 1)$ *in the direction.*

   i) normal to $xy^2 + yz^3 = 4$ *at* $(1, -1, 1)$ **[Ans. :** $\frac{12}{\sqrt{11}}$**]**

   ii) tangent to $x = t^2 + t,\ y = 2t,\ z = 2 - t$ *at* $t = 1$ **[Ans. :** $2\sqrt{14}$**]**

iii) *Along* $x - 1 = -2y = 2(z + 1)$ **[Ans. :** $-\frac{10}{\sqrt{6}}$**]**

5. *Find the directional derivative of* $\phi = e^{2x-y-z}$ *at (1, 1, 1) in the direction along* $x = e^{-t}$, $y = 2\sin t + 1$, $y = t - \cos t$ *at* $t = 0$. **[Ans. :** $-\frac{5}{\sqrt{6}}$**]**

6. *Find the D.D. of* $\phi = xy^2 + yz^2$ *at (2, – 1, 1) in the direction* $2i + j + 3k$. **[Ans. :** $-\sqrt{\frac{7}{2}}$**]**

7. *Find the directional derivative of* $\phi = 4xz^3 - 3x^2y^2z$ *at (2, – 1, 2) in the directional from this point towards the point (4, – 4, 8).* **[Ans. :** $\frac{376}{7}$**]**

8. *Find the directional derivative of* $x^2y + y^3z$ *at (2, – 1, 1) along the direction which make equal angles with co-ordinate axes.* **[Ans. :** $\frac{-2}{\sqrt{3}}$**]**

9. *Find D.D of* $\phi = div\left(x^5 i + y^5 j + z^5 k\right)$ *at (2, 2, 1) in the direction of the outward normal to the surface* $x^2 + y^2 + z^2 = 9$.

   Hint : div $\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

   $\therefore \qquad \phi = 5x^4 + 5y^4 + 5z^4$ **[Ans. : 220]**

10. *The directional derivative of* $\phi = ax^2y + by^2z + cz^2x$ *at (1, 1, 1) has maximum magnitude 4 in the direction parallel to x axis find a, b, c.* **[Ans. :** $a = \frac{16}{9}$, $b = -\frac{8}{9}$, $c = \frac{4}{9}$**]**

11. *The directional derivative of* $\phi = axy + byz + czx$ *at (1, 1, 1) has maximum magnitude 4 parallel to x axis find a, b, c.* **[Ans. : 2, – 2, 2]**

12. *Find the rate of change of* $\phi = 4x^2 + y^2 - 16z$ *at (2, 4, 2) in the direction normal to the plane* $x + 2y + 2z = 9$. **[Ans. : 0]**

13. *If the directional derivative of* $\phi = a(x + y) + b(y + z) + c(z + x)$ *has maximum value 12 in the direction parallel to the line* $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-1}{3}$. *Find a, b, c.* **[Ans. :** $0, \pm\frac{24}{\sqrt{14}}, \pm\frac{12}{\sqrt{14}}$**]**

14. *In what direction from the point* $(1, 2, 3)$ *is the D.D. of* $\phi = 2xy - y^2$ *a maximum ? What is its magnitude ?* **[Ans. :** $4\,\hat{i} - 2\,\hat{j} + 0\,\hat{k}$, **magnitude =** $\sqrt{20}$**]**

12.18 Divergence of a Vector Point Function

Let $\overline{F} = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$ be any V.P.F. defined in a certain field.

The expression

$$\nabla \cdot \overline{F} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \cdot \left(F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

which is a scalar point function, called the divergence of $\overline{F}$.

Thus

$$\boxed{\text{div } \overline{F} = \nabla \cdot \overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

If $\nabla \cdot \overline{F} = 0$ then $\overline{F}$ is said to be **solenoidal.**

12.19 Curl of a Vector Point Function

Let $\overline{F} = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$ be any V.P.F. defined in a certain field.

The expression

$$\nabla \times \overline{F} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \times \left(F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y} F_3 - \frac{\partial}{\partial z} F_2\right) - j\left(\frac{\partial}{\partial x} F_3 - \frac{\partial}{\partial z} F_1\right) + k\left(\frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1\right)$$

which is a vector point function called the curl of $\overline{F}$.

If $\nabla \times \overline{F} = 0$ then E is said to be irrotational or conservative.

In this case there exists a scalar point function ϕ such that $\overline{F} = \nabla\phi$, where ϕ is called as scalar potential of $\overline{F}$.

The formula for obtaining scalar potential ϕ for an irrotational field $\overline{F}$ is

$$d\phi = F_1\,dx + F_2\,dy + F_3\,dz$$

Integration gives ϕ

$$\phi = \int\limits_{\substack{y,\,z \\ \text{constant}}} F_1\,dx + \int\limits_{\substack{z \text{ constant} \\ \text{free } x}} F_2\,dy + \int\limits_{\substack{\text{free} \\ x,\,y}} F_3\,dz$$

Note : The work done in moving an object from P to Q in an irrotational field is $[\phi]_P^Q$ where ϕ is scalar potential of $\overline{F}$.

a) Physical meaning of divergence :

Consider the motion of a fluid with velocity $\overline{q} = u\hat{i} + v\hat{j} + w\hat{k}$ at a point P (x, y, z) consider a small parallopiped with edges δx, δy, δz with one of its corner at P (x, y, z).

As $\qquad \overline{q} = u\hat{i} + v\hat{j} + w\hat{k}$

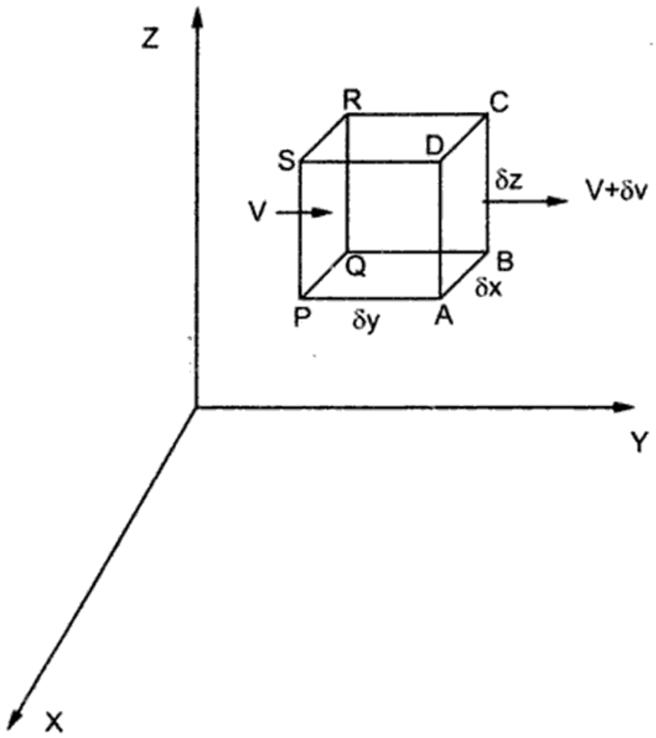


Fig 12.20

$\therefore$ v is the rate flow of the fluid in the direction of Y axis, hence the quantity of fluid entering through face PQRS per unit time is v. (Area of PQRS)

$$= v \cdot \delta x \, \delta z$$

and quantity of fluid leaving the face ABCD is $(v + \delta v)$ (Area of ABCD)

$$= (v + \delta v) \; (\delta x \, \delta z)$$

$$= v \, \delta x \, \delta z + \delta v \, \delta x \, \delta z$$

$$= \left[v \, \delta x \, \delta z + \frac{\delta v}{\delta y} \delta x \, \delta y \, \delta z \right]$$

Thus the actual quntity of fluid flowing out in the direction of y axis is

$$= \left[v \, \delta x \, \delta z + \frac{\delta v}{\delta y} \delta x \, \delta y \, \delta z \right] - [v \, \delta x \, \delta z]$$

$$= \frac{\delta v}{\delta y} \, \delta x \; \delta y \; \delta z \qquad \text{... (i)}$$

Similarly, the actual quantity of fluid flowing out in the direction of X axis is

$$= \frac{\delta u}{\delta x} \, \delta x \; \delta y \; \delta z \qquad \text{... (ii)}$$

We know from above that

$$\text{work done} = \int \bar{F}\cdot d\bar{r}$$

$$= \int F_1\, dx + F_2\, dy + F_3\, dz$$

$$= \int_c d\phi \qquad \{\text{in case of irrotational } \bar{F}$$

$$= [\phi]_c$$

$$= 0$$

As the curve 'c' is closed value of ϕ at upper limit = value of ϕ at lower limit.

$\therefore$ work done $= 0 \Rightarrow \bar{F}$ conservative.

g) The group operator $\bar{a}\cdot\nabla$:

If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ be a constant vector and

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \quad \text{then}$$

$$\bar{a}\cdot\nabla = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z} \text{ is a scalar differential operator.}$$

Note (i) $(\bar{a}\cdot\nabla)\,\bar{r} = \bar{a}$

$$\text{L.H.S} = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= a_1\left(\hat{i} + 0 + 0\right) + a_2\left(0 + \hat{j} + 0\right) + a_3\left(0 + 0 + \hat{k}\right)$$

$$= \bar{a}$$

ii) $(\bar{a}\cdot\nabla) \neq (\nabla\cdot\bar{a})$

iii) $(\bar{a}\cdot\nabla)\phi\bar{F} = \phi(\bar{a}\cdot\nabla)\,\bar{F} + \bar{F}(\bar{a}\cdot\nabla)\phi$

i.e. product rule for the operator $(\bar{a}\cdot\nabla)$

h) Standard Results

If $\bar{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$ (constant vector) and $\bar{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ (position vector) then

i) $\nabla\cdot\bar{a} = 0$ ii) $\nabla\times\bar{a} = 0$

iii) $\nabla\cdot\bar{r} = 3$ iv) $\nabla\times\bar{r} = 0$

v) $\nabla\cdot r^n\,\bar{r} = (n+3)\,r^n$ iv) $\nabla\times r^n\,\bar{r} = 0$

Proofs :

Put $\phi = r^n \quad \bar{F} = \bar{r}$

$\therefore \quad \nabla \times r^n \bar{r} = \nabla r^n \times \bar{r} + r^n (\nabla \times \bar{r})$

Step 2 : use $\nabla f(r) = f'(r) \dfrac{\bar{r}}{r}, \quad \nabla \times \bar{r} = 0$

$$\nabla \times r^n \bar{r} = n\, r^{n-1} \frac{\bar{r}}{r} \times \bar{r} + 0$$

$$= n\, r^{n-2} (\bar{r} \times \bar{r})$$

Step 3 : $\bar{r} \times \bar{r} = 0$

$\therefore \quad \nabla \times r^n \bar{r} = 0$

Example 12.50 : *Prove that* $F = (2xy + z^3)\hat{i} + x^2 \hat{j} + 3xz^2 \hat{k}$ *is conservative force field. Find its scalar potential and the work done in moving an object in this field from* $(1, -2, 1)$ *to* $(3, 1, 4)$.

Solution : Step 1 : Consider $\nabla \times \bar{F}$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= (0-0)\,\hat{i} - (3z^2 - 3z^2)\,\hat{j} + (2x - 2x)\,\hat{k}$$

$$= 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$$

Step 2 : As $\nabla \times \bar{F} = 0 \quad \therefore \bar{F}$ is irrotational there exists a scalar point function and such that

$$\bar{F} = \nabla \phi$$

The scalar potential is given by

$$\phi = \int\limits_{\substack{y,\,z \\ \text{const}}} f_1\, dx + \int\limits_{\substack{z \text{ const} \\ \text{free } x}} f_2\, dy + \int\limits_{\substack{\text{free} \\ x,\,y}} f_3\, dz$$

Step 3 : Substituting the values of f_1, f_2, f_3

$$\phi = \int (2xy + z^3)\, dx + \int 0\, dy + \int 0\, dz$$

Step 4 : Integrate

$$\phi = x^2 y + x z^3$$

Step 5 : The work done in moving an object in an irrotational field is

$$= [\phi]_P^Q \qquad (\phi = \text{scalar potential of } \bar{F})$$

$\therefore$ Workdone $= [x^2 y + x z^3]_{(1,-2,1)}^{(3,1,4)}$

$= 202$ units

Example 12.51 : *Prove that* $\overline{F} = 2xyz^2\,\hat{i} + (x^2z^2 + z\cos yz)\hat{j} + (2x^2yz + y\cos yz)\hat{k}$ *is irrotational. Find* $\int_c 2xyz^2dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz$ *on any path c from* $(0,0,1)$ *to* $(1, \pi/4, 2)$.

Solution : Step 1 : Consider $\nabla \times \overline{F}$

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= \hat{i}(2x^2z + \cos yz - yz\sin yz - 2x^2z - \cos yz + yz\sin yz)$$

$$- \hat{j}(4xyz - 4xyz) + \hat{k}(2xz^2 - 2xz^2)$$

$$= 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$$

Step 2 : As $\nabla \times \overline{F} = 0 \therefore \overline{F}$ is irrotational.

$\therefore$ There exists a s.p.f. ϕ such that $\overline{F} = \nabla\phi$

The scalar potential ϕ is given by

$$\phi = \int_{\substack{y,z\\ \text{constant}}} f_1\,dx + \int_{\substack{z \text{ constant}\\ \text{free } x}} f_2\,dy + \int_{\substack{\text{free}\\ x,y}} f_3\,dz$$

Step 3 : Substitute f_1, f_2, f_3

$$\phi = \int 2xyz^2\,dx + \int z\cos yz\,dy + \int 0\,dz$$

Step 4 : Integrate

$$\phi = x^2yz^2 + z\,\frac{\sin yz}{z}$$

$$= x^2yz^2 + \sin yz$$

Step 5 : To find work done

$$\int_c \overline{F}\cdot d\overline{r} = [\text{work done}] = [\phi]_P^Q$$

$$\therefore \int_c \overline{F}\cdot d\overline{r} = \int_P^Q \overline{F}\cdot d\overline{r}$$

$$= [x^2yz^2 + \sin yz]_{(0,0,1)}^{(1,\pi/4,2)}$$

$$= \left[1 \cdot \frac{\pi}{4} \cdot 4 + \sin \frac{\pi}{2}\right] - [0-0]$$

$$= \pi + 1 \quad \text{units}$$

Example 12.52 : *If the vector field* $\overline{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ *is irrotational, find the values of a, b, c and the work done in moving a particle from* $(1,2,-4)$ *to* $(3,2,2)$ *along the straight line joining these points.*

Solution : Step 1 : Given $\nabla \times \overline{F} = \theta$ $\therefore$ Consider $\nabla \times \overline{F}$

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ x+2y+az & bx-3y-z & 4x+xy+2z \end{vmatrix}$$

$$0 = \hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2)$$

Compare

$$\therefore \quad c+1=0 \qquad 4-a=0 \qquad b-2=0$$

$$c=-1 \qquad a=4 \qquad b=2$$

Step 2 : Substitute a, b, c in $\overline{F}$

$$\overline{F} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}$$

Step 3 : As $\overline{F}$ is irrotational there exists a scalar potential ϕ such that $\overline{F} = \nabla \phi$. The scalar potential ϕ is given by

$$\phi = \int\limits_{\substack{y,\,z \\ \text{const}}} f_1\, dx + \int\limits_{\substack{z \text{ const} \\ \text{free } x}} f_2\, dy + \int\limits_{\substack{\text{free} \\ x,\,y}} f_3\, dz$$

Step 4 : Substitute F_1, F_2, F_3

$$\phi = \int (x+2y+4z)\,dx + \int (-3y-z)\,dy + 2z\,dz$$

Step 5 : Integrate

$$\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - yz + z^2$$

Step 6 : Work done $= [\phi]_P^Q$

$$= \left[\frac{x^2}{2} + 2xy + 4xz - \frac{3}{2}y^2 - yz + z^2\right]_{(1,2,-4)}^{(3,2,2)}$$

$$= 28 \text{ units}$$

Example 12.53 : *Show that*

$\bar{F} = \dfrac{x\hat{i} + y\hat{j}}{x^2 + y^2}$ *is solenoidal as well as irrotational evaluate* $\int_c \bar{F} \cdot d\bar{r}$.

Solution : **Step 1 :** Consider $\bar{F}$

$$\bar{F} = \frac{x\,\hat{i} + y\,\hat{j}}{x^2 + y^2}$$

Step 2 : To show $\bar{F}$ solenoidal consider $\nabla \cdot \bar{F}$

$$\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{z}$$

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial x}\frac{x}{x^2 + y^2} + \frac{\partial}{\partial y}\frac{y}{x^2 + y^2}$$

Step 3 : Find partial derivatives

$$\nabla \cdot \bar{F} = \frac{(x^2 + y^2)\,1 - x(2x)}{(x^2 + y^2)^2} + \frac{(x^2 + y^2)\,1 - y(2y)}{(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$= 0$$

$\therefore$ $\bar{F}$ is solenoidal.

Step 4 : To show $\bar{F}$ irrotational consider $\nabla \times \bar{F}$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{x^2 + y^2} & \dfrac{y}{x^2 + y^2} & 0 \end{vmatrix}$$

Step 5 : Solve the determinant

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}\left(\frac{\partial}{\partial x}\frac{y}{x^2 + y^2} - \frac{\partial}{\partial y}\frac{x}{x^2 + y^2}\right)$$

$$= \hat{k}\left[y\frac{-2x}{(x^2 + y^2)^2} - x\frac{-2y}{(x^2 + y^2)^2}\right]$$

$$= \hat{k}\,[0]$$

$$= 0$$

Thus $\bar{F}$ is irrotational.

Step 8 : Use formula : $\nabla \times \bar{r} = 0$

$$= \nabla\left(\frac{\bar{a}\cdot\bar{r}}{r}\right)\times \bar{r}+0$$

Step 9 : Use formula $\nabla\left(\frac{U}{V}\right) = \frac{v\nabla\, u - u\nabla\, v}{v^2}$

$$= \left[\frac{r\nabla(\bar{a}\cdot\bar{r})-(\bar{a}\cdot\bar{r})\nabla\, r}{r^2}\right]\times \bar{r}$$

Step 10 : Use formula : $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$

$$\nabla r = \frac{\bar{r}}{r}$$

$$= \left[\frac{r\,\bar{a}-(\bar{a}\cdot\bar{r})\,\bar{r}/r}{r^2}\right]\times \bar{r}$$

Step 11 : Open the bracket and take x with $\bar{r}$

$$\frac{\bar{a}\times\bar{r}}{r} = \frac{(\bar{a}\cdot\bar{r})}{r^3}(\bar{r}\times\bar{r})$$

Step 12 : Use formula $\bar{r}\times\bar{r} = 0$

$$= \frac{\bar{a}\times\bar{r}}{r} \qquad \dots (2)$$

Step 13 : Add (1) and (2)

$$\nabla\times\bar{F} = (1)+(2) = 0$$

Step 14 : To find the scalar potential consider $\bar{F}$.

$$\bar{F} = r\,\bar{a} + (\bar{a}\cdot\bar{r})\frac{\bar{r}}{r}$$

Step 15 : $\bar{a} = \nabla(\bar{a}\cdot\bar{r})$ and $\frac{\bar{r}}{r} = \nabla\, r$

Use formula :

$$\bar{F} = r\nabla(\bar{a}\cdot\bar{r}) + (\bar{a}\cdot\bar{r})\nabla\, r$$

Step 16 : We know that

$$\nabla(u\,v) = u\nabla\, v + v\nabla\, u$$

Thus $\bar{F} = \nabla[r(\bar{a}\cdot\bar{r})]$

$\therefore$ $\bar{F} = \nabla\phi$

$\therefore$ $\phi = r(\bar{a}\cdot\bar{r})$ is the required scalar potential.

Step 17 : To find the work done

$$\text{W.D.} = \int_c \overline{F}\cdot d\,\overline{r} = [\phi]_P^Q$$

Step 18 : Substituting ϕ we get

$$= [r(\overline{a}\cdot\overline{r})]_{(1,0,0)}^{(1,1,1)}$$

Step 19 : Put $r = \sqrt{x^2+y^2+z^2}$, $\overline{r} = x\hat{i}+y\hat{j}+z\hat{k}$, $\overline{a} = \hat{i}+\hat{j}+\hat{k}$

$$= \sqrt{x^2+y^2+z^2}\left[\left(\hat{i}+\hat{j}+\hat{k}\right)\cdot\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\right]$$

Step 20 : Take the dot product.

$$= \left\{\sqrt{x^2+y^2+z^2}\,(x+y+z)\right\}_{(1,0,0)}^{(1,1,1)}$$

Step 21 : Substitute the limits.

$$\left\{\sqrt{1+1+1}\,(3)-\sqrt{1+0+0}\,[1+0+0]\right\}$$

$$\text{W.D.} = 3\sqrt{3}-1$$

Example 12.57 : *Show that $\overline{F}=(\overline{a}\cdot\overline{r})\overline{a}$ is irrotational. Also find scalar potential functional ϕ such that $\overline{F}=\nabla\phi$ also find $\int_c \overline{F}\cdot d\overline{r}$ where c is curve joining $(1,1,2)$; $(2,2,3)$ where $\overline{a}=\hat{i}+2\hat{j}+\hat{k}$.*

Solution : Step 1 : Consider $\nabla\times\overline{F}$

$$\nabla\times\overline{F} = \nabla\times(\overline{a}\cdot\overline{r})\,\overline{a}$$

Step 2 : Use formula :

$$\nabla\times\phi\,\overline{F} = \nabla\phi\times\overline{F}+\phi\left(\nabla\times\overline{F}_1\right)$$

Put $\phi = (\overline{a}\cdot\overline{r})$, $\overline{F} = \overline{a}$

$$= \nabla(\overline{a}\cdot\overline{r})\times\overline{a}+(\overline{a}\cdot\overline{r})(\nabla\times\overline{a})$$

Step 3 : Use formula :

$$\nabla(\overline{a}\cdot\overline{r}) = \overline{a}, \quad \nabla\times\overline{a} = 0$$

$$= \overline{a}\times\overline{a}+0$$

Step 4 : $\overline{a}\times\overline{a} = 0$

$\therefore$ $\nabla\times\overline{F} = 0$

$\therefore$ Irrotational.

Step 5 : To find scalar potential consider $\bar{F}$.

$$\bar{F} = (\bar{a} \cdot \bar{r})\,\bar{a}$$

Step 6 : Use

$$\bar{a} = \nabla(\bar{a} \cdot \bar{r})$$

$$= (\bar{a} \cdot \bar{r})\nabla(\bar{a} \cdot \bar{r})$$

Step 7 : Let

$$\bar{a} \cdot \bar{r} = u$$

Use

$$\nabla f(u) = f'(u)\nabla u$$

$$\nabla\left(\frac{u^2}{2}\right) = \left(\frac{2u}{2}\right)\nabla u$$

$$\nabla\left(\frac{u^2}{2}\right) = u\nabla u$$

$\therefore$

$$\bar{F} = u\nabla u$$

$$= \nabla\left(\frac{u^2}{2}\right)$$

$$= \nabla\left[\frac{(\bar{a} \cdot \bar{r})^2}{2}\right]$$

Thus

$$\bar{F} = \nabla\phi$$

$\therefore$

$$\phi = \frac{(\bar{a} \cdot \bar{r})^2}{2} \quad \text{is the required scalar potential.}$$

Step 8 : Put the values of $\bar{a}$ and $\bar{r}$

$$\phi = \frac{\left[\left(\hat{i}+2\hat{j}+\hat{k}\right)\cdot\left(r\,\hat{i}+y\,\hat{j}+2\,\hat{k}\right)\right]^2}{2}$$

Step 9 : Take the dot product

$$\phi = \frac{(x+2y+z)^2}{2}$$

Step 10 : To find the work done

$$\text{W.D.} = [\phi]_P^Q$$

$$= \left[\frac{(x+2y+z)^2}{2}\right]_{(1,1,2)}^{(2,2,3)}$$

Step 11 : Put the limits

$$= \left[\frac{(2+4+3)^2}{2} - \frac{(1+2+2)^2}{2}\right]^2 = 28$$

Example 12.58 : *Show that $\bar{F} = f(r)\bar{r}$ is irrotational. Find $f(r)$ such that $\bar{F}$ is solenoidal.*

Solution : Step 1 : Consider $\nabla \times \bar{F}$

$$\nabla \times \bar{F} = \nabla \times f(r)\,\bar{r}$$

Step 2 : Use formula :

$$\nabla \times \phi\,\bar{F} = \nabla\,\phi \times \bar{F} + \phi(\nabla \times \bar{F})$$

Put $\phi = f(r)$ and $\bar{F} = \bar{r}$

$$= \nabla\, f(r) \times \bar{r} + f(r)\,(\nabla \times \bar{r})$$

Step 3 : Use formula

$$\nabla f(r) = \frac{f'(r)}{r}\,\bar{r}, \quad \nabla \times \bar{r} = 0$$

$$= \frac{f'(r)}{r}\,\bar{r} \times \bar{r} + 0$$

Step 4 : use $\bar{r} \times \bar{r} = 0$

$$\therefore \quad \nabla \times \bar{F} = 0$$

$\therefore$ $\bar{F}$ irrotational.

Step 5 : Given $\nabla \cdot \bar{F} = 0$

$$\therefore \quad 0 = \nabla \cdot f(r)\,\bar{r}$$

Step 6 : Use $\nabla \cdot \phi\,\bar{F} = \nabla\,\phi \cdot \bar{F} + \phi(\nabla \cdot \bar{F})$ formula

put $\phi = f(r)$ and $\bar{F} = \bar{r}$

$$\therefore \quad 0 = \nabla\, f(r) \cdot \bar{r} + f(r)\,(\nabla \cdot \bar{r})$$

Step 7 : Use $\nabla f(r) = \frac{f'(r)}{r}\,\bar{r}, \quad \nabla \cdot \bar{r} = 3$ formula

$$0 = \frac{f'(r)}{r}\,\bar{r} \cdot \bar{r} + 3\,f(r)$$

Step 8 : Use $\bar{r} \cdot \bar{r} = r^2$ formula

$$0 = f'\frac{(r)}{r}\,(r^2) + 3\,f(r)$$

$$0 = f'(r)\ r + 3\,f(r)$$

$$\therefore \quad -f'(r)\,r = 3\,f(r)$$

$$\therefore \quad \frac{f'(r)}{f(r)} = -\frac{3}{r} \qquad \text{Multiply by } (-)$$

Step 9 : Integrate w.r.t. r

$$\log f(r) = -3\log r + \log c$$

$$\log f(r) = \log \frac{c}{r^3}$$

$$f(r) = \frac{c}{r^3}$$

Exercise 12.8

1. *Compute curl and divergence of* $\overline{F} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$ **[Ans. : $\nabla \times \overline{F} = 0,\ \nabla \cdot \overline{F} = \frac{2}{r}$]**

2. *Verify whether the following vectors are irrotational if so find the corresponding sclar potential* ϕ.

   i) $\overline{F} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$ **[Ans. : $\phi = xy\sin z + \cos x + y^2 z + c$]**

   ii) $\overline{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ **[Ans. : $\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c$]**

   iii) $\overline{F} = (4xy + z^3)\hat{i} + (2x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ **[Ans. : $\phi = 2x^2y + xz^3 - yz + c$]**

   iv) $\overline{F} = (xyz)^m (x^n\hat{i} + y^n\hat{j} + z^n\hat{k})$ **[Ans. : $\overline{F}$ is irrotational if $m = 0$, $n = -1$ $\phi = \log(xyz)$]**

   v) $\overline{F} = 2xye^z\,\hat{i} + x^2e^z\,\hat{j} + x^2ye^z\,\hat{k}$ **[Ans. : $\phi = x^2y\,e^z + c$]**

   vi) $\overline{F} = (y^2\cos x + z^2)\hat{i} + (2y\sin x)\hat{j} + 2xz\hat{k}$ **[Ans. : $\phi = y^2\sin x + xz^2 + c$]**

3. *Find the work done in moving a particle from (0, 1, – 1) to* $\left(\frac{\pi}{2}, -1, 2\right)$ *in a force field*

   $\overline{F} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2 + 2)\hat{k}$ **[Ans. : $4\pi + 15$]**

4. *Show that* $\overline{F} = \dfrac{-(y\hat{i} - x\hat{j})}{x^2 + y^2}$ *is irrotational evaluate* $\int_c \overline{F} \cdot d\overline{r}$ *where 'c' is a circle containing origin. Is* $\overline{F}$ *solenoidal ?*

5. *Prove that the following vectors are irrotational also find the work done in moving a particle in an irrotational field for the given two points.*

   i) $\overline{F} = (2xz^3 + 6y)\hat{i} + (6x - 2yz)\hat{j} + (3x^2z^2 - y^2)\hat{k}$. *Given P (1, – 2, 1), Q (3, 1, 4)*

   [Ans. : Work done = 605]

   ii) $\overline{F} = (2xyz^3)\hat{i} + (x^2z^3 + 2y)\hat{j} + (3x^2yz^2)\hat{k}$, *P (1, – 2, 1), Q (3, 1, 4).*

   [Ans. : Work done = 575]

Step 3 : Dropping the suffixes and use $\nabla \cdot \nabla = \nabla^2$

$$= \left[\nabla\left(\nabla \cdot \overline{F}\right) - \nabla^2 \overline{F}\right]$$

Thus $$\boxed{\nabla \times \left(\nabla \times \overline{F}\right) = \nabla\left(\nabla \cdot \overline{F}\right) - \nabla^2 \overline{F}}$$

Note that : $d\,(xy) = y\,dx + x\,dy$

$$= d_y\,(xy) + d_x\,(xy)$$

where suffixes of d are to be treated as a constant in each expression.

5) Divergence $\left(\phi\overline{F}\right)$:

It is convenient to employ the symbolic method as follows

$$\nabla \cdot \left(\phi\overline{F}\right) = \nabla_\phi \cdot \left(\phi\overline{F}\right) + \nabla_F \cdot \left(\phi\overline{F}\right)$$

where the suffix of del (∇) is to be treated as a constant in each expression.

$$= \phi\left(\nabla_\phi \cdot \overline{F}\right) + \nabla_F\, \phi \cdot \overline{F}$$

dropping the suffixes

$$= \phi\left(\nabla \cdot \overline{F}\right) + \nabla\phi \cdot \overline{F}$$

$$\therefore \quad \nabla \cdot \left(\phi\overline{F}\right) = \nabla\phi \cdot \overline{F} + \phi\left(\nabla \cdot \overline{F}\right)$$

6) Curl $\left(\phi\overline{F}\right)$: It is convenient to employ the symbolic method as follows.

$$\nabla \times \left(\phi\overline{F}\right) = \nabla_\phi \times \left(\phi\overline{F}\right) + \nabla_F \times \left(\phi\overline{F}\right)$$

where the suffix of del (∇) is to be treated as a constant in each expression.

$$= \phi\left(\nabla_\phi \times \overline{F}\right) + \nabla_F\, \phi \times \overline{F}$$

dropping the suffixes

$$= \phi\left(\nabla \times \overline{F}\right) + \nabla\phi \times \overline{F}$$

$$\therefore \quad \boxed{\nabla \times \left(\phi\overline{F}\right) = \nabla\phi \times \overline{F} + \phi\left(\nabla \times \overline{F}\right)}$$

7) Divergence $\left(\overline{A} \times \overline{B}\right)$:

$$\nabla \cdot \left(\overline{A} \times \overline{B}\right) = \nabla_A \cdot \left(\overline{A} \times \overline{B}\right) + \nabla_B \cdot \left(\overline{A} \times \overline{B}\right)$$

Now, $$\nabla_A \cdot \left(\overline{A} \times \overline{B}\right) = \overline{A} \cdot \left(\overline{B} \times \nabla_A\right)$$

$$= -\overline{A} \cdot \left(\nabla_A \times \overline{B}\right)$$

and $$\nabla_B \cdot \left(\overline{A} \times \overline{B}\right) = \overline{B} \cdot \left(\nabla_B \times \overline{A}\right)$$

$$\therefore \quad \nabla \cdot \left(\overline{A} \times \overline{B}\right) = -A \cdot \left(\nabla_A \times \overline{B}\right) + \overline{B} \cdot \left(\nabla_B \times \overline{A}\right)$$

$$= \overline{B} \cdot \left(\nabla \times \overline{A}\right) - \overline{A} \cdot \left(\nabla \times \overline{B}\right)$$

$$\text{Put} \quad \phi = \frac{f'(r)}{r} \qquad \bar{F} = \bar{r}$$

$$= \left(\nabla \frac{f'(r)}{r}\right)\cdot \bar{r} + \frac{F'(r)}{r}(\nabla\cdot\bar{r})$$

Step 4 : Use $\nabla \dfrac{u}{v} = \dfrac{v\nabla u - u\nabla v}{v^2}$, $\nabla\cdot\bar{r} = 3$

$$= \left[\frac{r\nabla f'(r) - f'(r)\nabla r}{r^2}\right]\cdot\bar{r} + \frac{f'(r)}{r}(3)$$

Step 5 : Use $\nabla f'(r) = \dfrac{f''(r)}{r}\bar{r}$

$$\nabla r = \frac{\bar{r}}{r}$$

$$= \frac{1}{r^2}\left[r\,\frac{f''(r)}{r}\,\bar{r} - f'(r)\,\frac{\bar{r}}{r}\right]\cdot\bar{r} + 3\,\frac{f'(r)}{r}$$

Step 6 : Take $\bar{r}$ common

$$= \frac{1}{r^2}\left[f''(r) - \frac{f'(r)}{r}\right](\bar{r}\cdot\bar{r}) + 3\,\frac{f'(r)}{r}$$

Step 7 : Use $\bar{r}\cdot\bar{r} = r^2$

$$= f''(r) - \frac{1}{r}\,f'(r) + \frac{3}{r}\,f'(r)$$

Thus $\quad \nabla^2 f(r) = f''(r) + \dfrac{2}{r}\,f'(r)$

Example 12.60 : $\nabla^2\left[\nabla\cdot\dfrac{\bar{r}}{r^2}\right] = \dfrac{2}{r^4}$

Solution :

Step 1 : Use $\nabla\cdot r^n\,\bar{r} = (n+3)\,r^n$ put $n = -2$

$$\therefore \quad \nabla\cdot\frac{\bar{r}}{r^2} = (-2+3)\,\bar{r}^2 = \frac{1}{r^2}$$

Step 2 : Consider L.H.S.

$$\therefore \quad \nabla^2\left[\nabla\cdot\frac{\bar{r}}{r^2}\right] = \nabla^2\left[\frac{1}{r^2}\right]$$

where $G(r) = \left(1+\frac{2}{r}\right) e^r$

Step 5 : Again use $\nabla^2 G(r) = G''(r) + \frac{2}{r} G'(r)$ ... (i)

$\therefore$ Find $G'(r)$ and $G''(r)$

and
$$G'(r) = \left(1+\frac{2}{r}-\frac{2}{r^2}\right) e^r$$

$$G''(r) = \left(1+\frac{2}{r}-\frac{4}{r^2}+\frac{4}{r^3}\right) e^r$$

Step 6 : Substituting in

$$\nabla^2 G(r) = G''(r) + \frac{2}{r} G'(r)$$

$$= \left(1+\frac{2}{r}-\frac{4}{r^2}+\frac{4}{r^3}\right) e^r + \frac{2}{r}\left(1+\frac{2}{r}-\frac{2}{r^2}\right) e^r$$

$$= \left(1+\frac{4}{r}\right) e^r$$

$$= \text{R.H.S.}$$

Example 12.62 : $\nabla^2 \dfrac{(\bar{a}\cdot\bar{b})}{r} = 0$

Solution :

Step 1 : As $\bar{a}\cdot\bar{b}$ is a constant therefore we can take $\bar{a}\cdot\bar{b}$ outside $= \left(\bar{a}\cdot\bar{b}\right)\left(\nabla^2 \frac{1}{r}\right)$

Step 2 : Use $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

$$= \left(\bar{a}\cdot\bar{b}\right)\left[\left(\frac{1}{r}\right)'' + \frac{2}{r}\left(\frac{1}{r}\right)'\right]$$

Step 3 : Find the derivatives

$$= \left(\bar{a}\cdot\bar{b}\right)\left[\frac{2}{r^3} + \frac{2}{r}\left(\frac{-1}{r^2}\right)\right]$$

Step 4 : Simplify

$$= \left(\bar{a}\cdot\bar{b}\right)\left[\frac{2}{r^3} - \frac{2}{r^3}\right]$$

$$= \left(\bar{a}\cdot\bar{b}\right)\cdot 0 = 0$$

$$= -\frac{6}{r^2}+\frac{12}{r^2}$$

$$= \frac{6}{r^2}$$

Example 12.65 : *Prove that* $\nabla \times \left(\frac{\bar{a}\times \bar{r}}{r^n}\right)=\frac{(2-n)\,\bar{a}}{r^n}+\frac{n(\bar{a}\cdot\bar{r})\,\bar{r}}{r^{n+2}}$.

Solution :

Step 1 : Use formula : $\bar{1}\times\left(\bar{2}\times\bar{3}\right)=\left(\bar{1}\cdot\bar{3}\right)\bar{2}-\left(\bar{2}\cdot\bar{1}\right)\bar{3}$

$$\therefore \quad \nabla\times\left(\bar{a}\times\frac{\bar{r}}{r^n}\right) = \left(\nabla\cdot\frac{\bar{r}}{r^n}\right)\bar{a}-(\bar{a}\cdot\nabla)\frac{\bar{r}}{r^n}$$

Step 2 : Use formula $\nabla\cdot r^n\,\bar{r}=(n+3)\,r^n$

$$= (-n+3)\,r^{-n}\,\bar{a}-(\bar{a}\cdot\nabla)\frac{1}{r^n}\,\bar{r}$$

Step 3 : Use $(\bar{a}\cdot\nabla)\phi\,\bar{F}=\phi(\bar{a}\cdot\nabla)\,\bar{F}+\bar{F}(\bar{a}\cdot\nabla)\phi$ product rule for $\bar{a}\cdot\nabla$

$$= (-n+3)\,r^{-n}\,\bar{a}-\frac{1}{r^n}(\bar{a}\cdot\nabla)\,\bar{r}-\bar{r}(\bar{a}\cdot\nabla)\frac{1}{r^n}$$

Step 4 : Use $(\bar{a}\cdot\nabla)\,\bar{r} = \bar{a}$

$$= \frac{(3-n)\,\bar{a}}{r^n}-\frac{\bar{a}}{r^n}-\bar{r}\left(\bar{a}\cdot\nabla\frac{1}{r^n}\right)$$

Step 5 : Use $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$ i.e. $\nabla\frac{1}{r^n}=\frac{-n}{r^{n+1}}\frac{\bar{r}}{r}$

$$= \frac{(2-n)\,\bar{a}}{r^n}-\bar{r}\left(\bar{a}\cdot\frac{-n}{r^{n+1}}\frac{\bar{r}}{r}\right)$$

Step 6 : Simplify

$$= \frac{(2-n)\,\bar{a}}{r^n}+\frac{n\,\bar{r}(\bar{a}\cdot\bar{r})}{r^{n+2}}$$

Alternative method

Solution : Step 1 : Consider

$$\nabla\times\left(\frac{\bar{a}\times\bar{r}}{r^n}\right) = \nabla\times\left[\left(\frac{1}{r^n}\right)(\bar{a}\times\bar{r})\right]$$

Step 2 : Use $\nabla\times\phi\bar{F} = \nabla\phi\times\bar{F}+\phi\left(\nabla\times\bar{F}\right)$

where $\phi=\frac{1}{r^n}$ and $\bar{F}=(\bar{a}\times\bar{r})$

$$\therefore \quad \nabla \times \left(\frac{\bar{a}\times\bar{r}}{r^n}\right) = \left[\nabla\left(\frac{1}{r^n}\right)\right]\times(\bar{a}\times\bar{r})+\frac{1}{r^n}\nabla\times(\bar{a}\times\bar{r})$$

Step 3 : Use $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$

$$\therefore \quad \nabla\frac{1}{r^n} = \frac{-n}{r^{n+1}}\,\frac{\bar{r}}{r} = \frac{-n\bar{r}}{r^{n+2}}$$

$$\therefore \quad = \frac{-n\,\bar{r}}{r^{n+2}}\times(\bar{a}\times\bar{r})+\frac{1}{r^n}\nabla\times(\bar{a}\times\bar{r})$$

$$= \frac{-n}{r^{n+2}}[\bar{r}\times(\bar{a}\times\bar{r})]+\frac{1}{r^n}[\nabla\times(\bar{a}\times\bar{r})]$$

Step 4 : Use

$$\bar{1}\times\left(\bar{2}\times\bar{3}\right) = \left(\bar{1}\cdot\bar{3}\right)\bar{2}-\left(\bar{2}\cdot\bar{1}\right)\bar{3}$$

$$= \frac{-n}{r^{n+2}}[(\bar{r}\cdot\bar{r})\,\bar{a}-(\bar{a}\cdot\bar{r})\,\bar{r}]+\frac{1}{r^n}[(\nabla\cdot\bar{r})\,\bar{a}-(\bar{a}\cdot\nabla)\,\bar{r}]$$

Step 5 : Use $\bar{r}\cdot\bar{r} = r^2$, $\nabla\cdot\bar{r} = 3$, $(\bar{a}\cdot\nabla)\,\bar{r} = \bar{a}$

$$= \frac{-n}{r^{n+2}}[r^2\,\bar{a}-(\bar{a}\cdot\bar{r})\bar{r}]+\frac{1}{r^n}[3\,\bar{a}-\bar{a}]$$

Step 6 : Simplify.

$$= \frac{-n\bar{a}}{r^n}+\frac{n(\bar{a}\cdot\bar{r})\,\bar{r}}{r^{n+2}}+\frac{2\bar{a}}{r^n}$$

$$= \frac{(2-n)\,\bar{a}}{r^n}+\frac{n(\bar{a}\cdot\bar{r})\,\bar{r}}{r^{n+2}} \quad \therefore \text{ Proved}$$

Example 12.66 : *Prove that* $\nabla\times(\bar{a}\times\bar{r}) = 2\bar{a}$.

Solution :

Step 1 : Use $\bar{1}\times\left(\bar{2}\times\bar{3}\right) = \left(\bar{1}\cdot\bar{3}\right)\bar{2}-\left(\bar{2}\cdot\bar{1}\right)\bar{3}$

$$\nabla\times(\bar{a}\times\bar{r}) = (\nabla\cdot\bar{r})\;\bar{a}-(\bar{a}\cdot\nabla)\,\bar{r}$$

Step 2 :

$$\nabla\cdot\bar{r} = 3$$

$$(\bar{a}\cdot\nabla)\,\bar{r} = \bar{a}$$

$$= 3\,\bar{a}-\bar{a}$$

$$= 2\,\bar{a}$$

Example 12.67 : *Prove that* $\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right)=\frac{\bar{a}}{r^n}-\frac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}$

Solution :

Step 1 : Use $\nabla\left(\frac{u}{v}\right) = \frac{v\nabla u - u\nabla v}{v^2}$

$$\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right) = \frac{r^n\,\nabla(\bar{a}\cdot\bar{r})-(\bar{a}\cdot\bar{r})\nabla r^n}{r^{2n}}$$

Step 2 : Use $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$ and $\nabla f(r)=f'(r)\frac{\bar{r}}{r}$

$$\nabla r^n = n\,r^{n-1}\frac{\bar{r}}{r}$$

$$= n\,r^{n-2}\,\bar{r}$$

$$\therefore \quad \nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right) = \frac{r^n\,\bar{a}-(\bar{a}\cdot\bar{r})\,n\,r^{n-2}\,\bar{r}}{r^{2n}}$$

$$= \frac{\bar{a}}{r^n}-\frac{n(\bar{a}\cdot\bar{r})\,\bar{r}}{r^{n+2}}$$

Exercise 12.68 : *Prove that* $\nabla\times\left[(\bar{r}\times\bar{a})\times\left(\bar{r}\times\bar{b}\right)\right]=\left(\bar{a}\times\bar{b}\right)\times\bar{r}$

Solution : Step 1 : Use $\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)=\left[\bar{a}\ \bar{b}\ \bar{d}\right]\bar{c}-\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}$

$$(\bar{r}\times\bar{a})\times\left(\bar{r}\times\bar{b}\right) = \left[\bar{r}\ \bar{a}\ \bar{b}\right]\bar{r}-\left[\bar{r}\ \bar{a}\ \bar{r}\right]\bar{b}$$

Step 2 : If two rows in a determinant are same then its value is $\therefore [\bar{r}\,\bar{a}\,\bar{r}]=0$

$$= \left(\bar{r}\,\bar{a}\,\bar{b}\right)\bar{r}-0$$

$$= \left\{\bar{r}\left(\bar{a}\times\bar{b}\right)\right\}\bar{r}$$

Step 3 : Let $\bar{a}\times\bar{b} = \bar{d}$

$$\therefore \quad (\bar{r}\times\bar{a})\times\left(\bar{r}\times\bar{b}\right) = \left(\bar{r}\cdot\bar{d}\right)\bar{r}$$

Step 4 : Consider $\nabla\times\left[(\bar{r}\times\bar{a})\times\left(\bar{r}\times\bar{b}\right)\right]$

$$\nabla\times\left[(\bar{r}\times\bar{a})\times\left(\bar{r}\times\bar{b}\right)\right] = \nabla\times\left(\bar{r}\cdot\bar{d}\right)\bar{r}$$

Step 5 : Use $\nabla\times\phi\,\bar{F} = \nabla\phi\times\bar{F}+\phi\left(\nabla\times\bar{F}\right)$

Put $\phi = \left(\bar{r}\cdot\bar{d}\right),\quad \bar{F}=\bar{r}$

$$= \nabla\left(\bar{r}\cdot\bar{d}\right)\times\bar{r}+\left(\bar{r}\cdot\bar{d}\right)(\nabla\times\bar{r})$$

Step 2 : Consider

$$\bar{a}\cdot\nabla \log r = \frac{\bar{a}\cdot\bar{r}}{r^2}$$

Step 3 : Operate ∇ on both sides

$$\nabla\left[\bar{a}\cdot\nabla \log r\right] = \nabla\left[\frac{(\bar{a}\cdot\bar{r})}{r^2}\right]$$

Step 4 : Use formula

$$\nabla\left(\frac{u}{v}\right) = \frac{v\nabla u - u\nabla v}{v^2}$$

$$= \frac{r^2\nabla(\bar{a}\cdot\bar{r}) - (\bar{a}\cdot\bar{r})\nabla r^2}{r^4}$$

Step 5 : Use $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$, $\nabla r^2 = 2r\,\frac{\bar{r}}{r} = 2\,\bar{r}$

$$= \frac{r^2\,\bar{a} - (\bar{a}\cdot\bar{r})\,2\,\bar{r}}{r^4}$$

Step 6 : Simplify

$$\therefore \quad \nabla\left[\bar{a}\cdot\nabla \log r\right] = \frac{\bar{a}}{r^2} - \frac{2(\bar{a}\cdot\bar{r})}{r^4}\,\bar{r}$$

Step 7 : As cross product exists only for vectors.

Take x with $\bar{b}$ on both sides.

$$\bar{b}\times\left[\bar{a}\cdot\nabla \log r\right] = \frac{b\times\bar{a}}{r^2} - \frac{2(\bar{a}\cdot\bar{r})}{r^4}\left(\bar{b}\times\bar{r}\right)$$

Example 12.75 : *If $\bar{v}_1$ and $\bar{v}_2$ are vectors which join the fixed points $p_1(x_1, y_1, z_1)$ and $p_2(x_2, y_2, z_2)$ to a variable point $p(x, y, z)$ prove that*

1) $\nabla(\bar{v}_1\cdot\bar{v}_2) = \bar{v}_1 + \bar{v}_2$

2) $\nabla\times(\bar{v}_1\times\bar{v}_2) = 2(\bar{v}_1 - \bar{v}_2)$

3) $\nabla\cdot(\bar{v}_1\times\bar{v}_2) = 0$

Solution : Step 1 : Let $p_1(x_1, y_1, z_1) = (0, 0, 0)$

As p_1 and p_2 are fixed point.

$$\therefore \quad \overline{p_1 p_2} = \bar{a} \text{ constant vector}$$

$$\overline{p_1 p} = \overline{op} = \bar{r}$$

$$\therefore \quad \overline{p_2 p} = \overline{p_1 p} - \overline{p_1 p_2}$$

$$= \bar{r} - \bar{a}$$

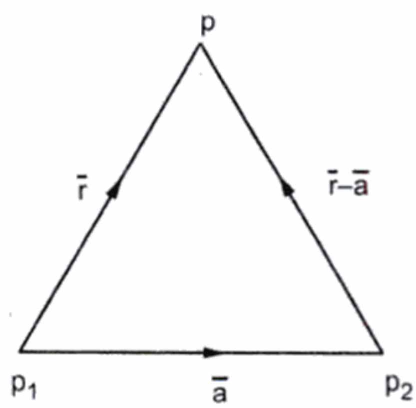


Fig. 12.22

Step 2 : Thus

$$\bar{v}_1 = \bar{r}$$

$$\bar{v}_2 = \bar{r} - \bar{a}$$

Step 3 : Consider $\bar{v}_1 \cdot \bar{v}_2$

$$\therefore \quad \bar{v}_1 \cdot \bar{v}_2 = \bar{r} \cdot (\bar{r} - \bar{a})$$

Step 4 : Open the ( ) and use $\bar{r} \cdot \bar{r} = r^2$

$$= r^2 - \bar{r} \cdot \bar{a}$$

Step 5 : Consider $\bar{v}_1 \times \bar{v}_2$

$$\bar{v}_1 \times \bar{v}_2 = \bar{r} \times (\bar{r} - \bar{a})$$

Step 6 : Open the ( ) and use $\bar{r} \times \bar{r}$

$$= \bar{0} - \bar{r} \times \bar{a}$$

$$\therefore \quad \bar{v}_1 \times \bar{v}_2 = \bar{a} \times \bar{r}$$

Step 7 : Consider $\nabla (\bar{v}_1 \cdot \bar{v}_2)$

$$\nabla (\bar{v}_1 \cdot \bar{v}_2) = \nabla \left[r^2 - (\bar{r} \cdot \bar{a}) \right]$$

Step 8 : Distribute ∇ over –

$$= \nabla r^2 - \nabla (\bar{r} \cdot \bar{a})$$

Step 9 : Use $\nabla r^2 = 2r \dfrac{\bar{r}}{r} = 2\bar{r}$ and $\nabla (\bar{a} \cdot \bar{r}) = \bar{a}$

$$= 2\bar{r} - \bar{a}$$

$$= \bar{r} + \bar{r} - \bar{a}$$

$$= \bar{v}_1 + \bar{v}_2$$

which proves (i).

Step 10 : Consider $\nabla \times (\bar{v}_1 \times \bar{v}_2)$

ii) $\nabla \times (\bar{v}_1 \times \bar{v}_2)$

$\nabla \times (\bar{a} \times \bar{r})$

Step 11 : $\left[\bar{1} \times (\bar{2} \times \bar{3}) = (\bar{1} \cdot \bar{3})\bar{2} - (\bar{2} \cdot \bar{1})\bar{3} \right]$

$$= \left[(\nabla \cdot \bar{r})\bar{a} - (\bar{a} \cdot \nabla)\bar{r} \right]$$

Step 12 : Use $\nabla \cdot \bar{r} = 3$ and $(\bar{a} \cdot \nabla)\bar{r} = \bar{a}$

$$\therefore \quad = 3\bar{a} - \bar{a} = 2\bar{a}$$

$$= 2(\bar{v}_1 - \bar{v}_2) \text{ which proves (ii)}$$

iii) Consider $\nabla \cdot (\overline{v}_1 \times \overline{v}_2)$

Step 1 : $\nabla (\overline{a} \times \overline{r})$

Step 2 : Use $\nabla\left(\overline{A} \times \overline{B}\right) = \overline{B}\cdot\left(\nabla \times \overline{A}\right) - \overline{A}\cdot\left(\nabla \times \overline{B}\right)$

$$= \overline{r}\cdot(\nabla \times \overline{a}) - \overline{a}\cdot(\nabla \times \overline{r})$$

Step 3 : Use $\nabla \times \overline{a} = 0$ and $\nabla \times \overline{r} = 0$

$$= 0 - 0 \text{ which proves (iii)}$$

Example 12.76 : *For solenoidal. Vector $\overline{E}$ show that $\nabla \times \nabla \times \nabla \times \nabla \times \overline{E} = \nabla^4 \overline{E}$.*

Solution : Step 1 : Consider $\nabla \times \left(\nabla \times \overline{E}\right)$

Step 2 : Use $\nabla \times \nabla \times \overline{F} = \nabla\left(\nabla \cdot \overline{F}\right) - \nabla^2 \overline{F}$

$$= \nabla\left(\nabla \cdot \overline{E}\right) - \nabla^2 \overline{E}$$

Step 3 : Given $\nabla \cdot \overline{E} = 0$ As $\overline{E}$ solenoidal.

$$= 0 - \nabla^2 \overline{E}$$

Step 4 : Let $\overline{F} = \left(-\nabla^2 \overline{E}\right)$

Step 5 : Consider LHS.

$\nabla \times \nabla \times \nabla \times \nabla \times \overline{E}$

Step 6 : Put $\nabla \times \nabla \times \overline{E} = \overline{F}$.

$$= \nabla \times \nabla \times \overline{F}$$

Step 7 : Use vector identity.

$$= \nabla\left(\nabla \cdot \overline{F}\right) - \nabla^2 \overline{F}$$

Step 8 : As $\nabla \cdot \overline{E} = 0$ Thus $\nabla \cdot \overline{F} = 0$

Step 9 : Put $\overline{F} = -\nabla^2 \overline{E}$

$$= -\nabla^2\left(-\nabla^2 \overline{E}\right)$$

Step 10 : Simplify.

$$= \nabla^4 \overline{E}$$

Example 12.77 : *If $\overline{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$ then show that $\nabla \times \nabla \times \nabla \times \nabla \times \overline{F} = \nabla^4 \overline{F}$.*

Solution : Step 1 : Consider $\nabla \cdot \overline{F}$

$$\nabla \cdot \overline{F} = \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(z+x) + \frac{\partial}{\partial y}(x+y)$$

$$= 0 + 0 + 0$$

14. *Prove that :*

i) $\bar{b} \times \nabla[\bar{a} \cdot \nabla \log r] = \frac{\bar{b} \times \bar{a}}{r^2} - \frac{2(\bar{a} \cdot \bar{r})(\bar{b} \times \bar{r})}{r^4}$

ii) $\bar{a} \cdot \nabla\left[\bar{b} \cdot \nabla \frac{1}{r}\right] = \frac{3(\bar{a} \cdot \bar{r})(\bar{b} \cdot \bar{r})}{r^5} - \frac{(\bar{a} \cdot \bar{b})}{r^3}$

iii) $r^5\left[\bar{b} \cdot \nabla\left(\bar{a} \cdot \nabla \frac{1}{r}\right)\right] = 3(\bar{a} \cdot \bar{r})(\bar{b} \cdot \bar{r})(\bar{a} \cdot \bar{b})r^2$

15. *If $\bar{r}$ is a position vector and $\overline{U}$ is a vector field prove that :*

i) $\nabla \cdot (\overline{U} \times \bar{r}) = \bar{r} \cdot \nabla \times \overline{U}$

ii) $\nabla \times (\overline{U} \times \bar{r}) = 2\overline{U} + (\bar{r} \cdot \nabla)\overline{U} - \bar{r}(\nabla \cdot \overline{U})$

iii) $\nabla(\overline{U} \cdot \bar{r}) = \overline{U} + \bar{r} \times (\nabla \times \overline{U}) + (\bar{r} \cdot \nabla)\overline{U}$

iv) $\nabla \int r^n \, dr = r^{n-1}\bar{r}$

16. *If* $\overline{E} = \frac{1}{\rho}\nabla\phi$ *then prove that* $\overline{E} \cdot curl\, \overline{E} = 0$

University Questions

Dec. - 98

1. *A particle is moving along the curve*

$x = t^3 + 1, \quad y = t^2, \quad z = t.$

Find the magnitude of the tangential and normal components of acceleration at $t = 1$. **[7 Marks]**

2. *Find the directional derivative of*

$\phi = 4xz^3 - 3x^2y^2z$ *at* $(2, -1, 2)$

along a line equally inclined to the co-ordinate axes. **[5 Marks]**

3. *Show that*

i) $Div\,(Curl\, \bar{v}) = 0$

ii) $\nabla(\bar{a} \cdot \bar{r}) = \bar{a}$ **[4 Marks]**

4. *Show that the vector field given by*

$\overline{F} = (y^2 \cos x + z^2)\hat{i} + (2y \sin x)\hat{j} + (2xz)\hat{k}$

is conservative and find the scalar-field ϕ such that $\overline{F} = \nabla\phi$ **[6 Marks]**

5. *With usual notations prove that (any two) :*

i) $\nabla^4(r^2 \log r) = \frac{6}{r^2}$ ii) $Curl\left[\bar{a} \times (\bar{b} \times \bar{r})\right] = \bar{a} \times \bar{b}$

iii) $\nabla \cdot \left[r\, \nabla(\bar{r}^n)\right] = \frac{n(n-2)}{r^{n+1}}$ **[6 Marks]**

6. *Find the function $f(r)$ so that $f(r)\bar{r}$ is a Solenoidal-field, $\bar{r}$ being space vector $x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$.* **[5 Marks]**

May - 99

1. *A particle P moves in a plane with constant angular velocity ω about 'O'. If the rate of increase of acceleration is parallel to PO, prove that :*

$$\frac{d^2 r}{dt^2} = \frac{1}{3} r\,\omega^2$$

[6 Marks]

2. *The directional derivative of $f(x, y)$ at the point $A(3,2)$ towards the point $B(2,3)$ is $3\sqrt{2}$ and towards the point $C(1,0)$ is $\sqrt{8}$. Find the directional derivative at the point A towards the point $D(2, 4)$.* **[6 Marks]**

3. *Show that :* $\bar{F} = \frac{1}{r}\left[r^2\,\bar{b} + \left(\bar{b}\cdot\bar{r}\right)\bar{r}\right]$

is irrotational. Hence find the scalar potential φ, such that $\bar{F} = \nabla\phi$. **[6 Marks]**

4. *Establish the following :*

i) $\bar{a}\cdot\nabla\left[\bar{b}\cdot\nabla\left(\frac{1}{r}\right)\right] = \frac{3\left(\bar{a}\cdot\bar{r}\right)\left(\bar{b}\cdot\bar{r}\right)}{r^5} - \frac{\left(\bar{a}\cdot\bar{b}\right)}{r^3}$

ii) $\text{curl}\left(\bar{a}\times \text{grad}\,\frac{1}{r}\right) + \text{grad}\left(\bar{a}\cdot \text{grad}\,\frac{1}{r}\right) = 0$

iii) $\nabla\times\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = -\frac{\bar{a}}{r^3} + \frac{3\left(\bar{a}\cdot\bar{r}\right)}{r^5}\bar{r}$

[9 Marks]

Dec. - 99

1. *If* $\bar{r}\times\frac{dr}{dt} = 0$*, show that* $\bar{r}$ *has a constant direction.* **[4 Marks]**

2. *A particle described an ellipse* $\frac{l}{r} + 1 + e\cos\theta$

with uniform angular velocity ω. Show that when particle is at one end of latus rectum through the pole, the component of acceleration towards the pole is $\left(1-2e^2\right)\omega^2 l$. **[6 Marks]**

3. *If directional derivative of* $\phi = ax^2y + by^2z + cz^2x$ *at (1, 1, 1) has maximum magnitude 15 in the direction parallel to* $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$.

Find the values of a, b, c. **[6 Marks]**

4. *Show that the vector field* $\bar{F} = \frac{\bar{r}}{r^3}$ *is irrotational and find the scalar point function φ such that* $\bar{F} = \nabla\phi$. **[6 Marks]**

5. *With usual notations prove that :*

i) $\nabla\cdot\left(\frac{\bar{a}\times\bar{r}}{r}\right) = 0$

ii) $\nabla\times\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = -\frac{\bar{a}}{r^3} + \frac{3\left(\bar{a}\cdot\bar{r}\right)}{r^5}\bar{r}$

iii) $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$.

[9 Marks]

4. *With usual notations prove that (any two) :*

   i) $\nabla\left[\dfrac{\bar{a}\times\bar{r}}{r}\right]=0$

   ii) $\nabla\times\left[\dfrac{\bar{a}\times\bar{r}}{r^3}\right]=\dfrac{-\bar{a}}{r^3}+\dfrac{3(\bar{a}\,\bar{r})}{r^5}\bar{r}$

   iii) $\nabla^2 f(r)=f''(r)+\dfrac{2}{r}f'(r).$ **[8 Marks]**

5. *A particle P moves in plane with constant angular velocity ω about 'O'. If the rate of increase of acceleration is parallel to PO, prove that :*

   $\dfrac{d^2r}{dt^2}=\dfrac{1}{3}r\,\omega^2$ **[6 Marks]**

6. *If the vector field*

   $\bar{F}=(x+2y+az)\,\hat{i}+(bx-3y-z)\,\hat{j}+(4x+cy+2z)\,\hat{k}$

   is irrotational, find a, b, c and determine scalar potential ϕ such that $\bar{F}=\nabla\phi$. **[5 Marks]**

Dec. - 2002

1. *The position vector of a particle at time 't' is*

   $\bar{r}=\cos(t-1)\,\hat{i}+\sinh(t-1)\,\hat{j}+mt^3\,\hat{k}$

   Find the condition imposed on m by requiring that at $t=1$, *the acceleration is normal to the position vector.* **[6 Marks]**

2. *For the curve,* $x=\cos t+t\sin t$, $y=\sin t-t\cos t$. *Find the tangential and normal components of acceleration at any time 't'.* **[6 Marks]**

3. *Find the directional derivative of the function* $\phi=x^2-y^2+2z^2$ *at the point* $P(2,-1,3)$ *along the normal to the surface* $x^2+y^2+z^2=9$ *at* $(1,2,2)$. **[6 Marks]**

4. *Prove that :* $\nabla\times\left[(\bar{r}\times\bar{a})\times\bar{b}\right]=\bar{b}\times\bar{a}$ **[4 Marks]**

5. *Prove the following :*

   i) $\nabla^4 e^r=e^r+\dfrac{4}{r}e^r$ ii) $\bar{F}=(y^2\cos x+z^2)\,\hat{i}+(2y\sin x)\,\hat{j}+2xz\,\hat{k}$

   is conservative and hence find the corresponding scalar field ϕ such that $\bar{F}=\nabla\phi$. **[8 Marks]**

May - 2003

1. *For the curve* $x=t^3+1$, $y=t^2$, $z=t$, *find the magnitude of tangential and normal components of acceleration for a particle moving on the curve at* $t=1$. **[6 Marks]**

2. *At any point of the curve* $x=3\cos t$, $y=3\sin t$, $z=4t$.

   Find :

   i) *Tangent vector*

   ii) *Unit tangent vector*

   iii) *Normal vector*

   iv) *Unit normal vector.* **[6 Marks]**

3. *For the function*

$$\phi(x, y) = \frac{x}{x^2 + y^2}$$

find the magnitude of the directional derivative along a line making an angle 30° with the positive x-axis at $(0,2)$. **[6 Marks]**

4. *Let* $f(x,y,z)$ *and* $g(x,y,z)$ *be two scalar functions. Find an expression for* $\nabla^2(fg)$ *in terms of* $\nabla^2 f$, $\nabla^2 g$, ∇f *and* ∇g. **[4 Marks]**

5. *Prove the following :*

i) $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$, *where* $r = |\bar{r}|$,

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

ii) *If* $F = \nabla\phi$, *show that the work done in moving a particle in the force field* $\bar{F}$ *from* $A(x_1, y_1, z_1)$ *to* $B(x_2, y_2, z_2)$ *is independent of the path joining the two points.* **[4 Marks]**

Dec. - 2003

1. *If* $\bar{F} = (x - 4y + az)\hat{i} + (bx + 2y + z)\hat{j} + (2x - cy + 3z)\hat{k}$ *is conservative, find a, b, c and work done in moving a particle in this field from the point* $(1,1,1)$ *to the point* $(2,1,3)$. **[6 Marks]**

2. *Find the tangential and normal component of acceleration at any time t for the curve* $\bar{r}(t) = at\cos t\,\hat{i} + at\sin t\,\hat{j}$. **[6 Marks]**

3. *Find the directional derivative of* $\phi = 4e^{2x-y+z}$ *at A* $(1,1,-1)$ *in the direction towards the point* $B(-3,5,6)$. **[6 Marks]**

4. *Prove that* **[5 Marks]**

$$\nabla^2\left[\nabla\cdot\frac{\bar{r}}{r^3}\right] = \frac{2}{r^4}$$

5. *Prove that* **[4 Marks]**

$$\bar{b}\times\nabla[\bar{a}\cdot\nabla\log r] = \frac{\bar{b}\times\bar{a}}{r^2} - \frac{2(\bar{a}\cdot\bar{r})}{r^4}(\bar{b}\times\bar{r}).$$

May - 2004

1. *The position vector of a particle at time t is* **[6 Marks]**

$\bar{r} = \cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + \alpha t^3\hat{k}$

If the acceleration at $t = 1$ *is normal to its radius vector at that time, then find value of* α.

2. *If* $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = xy + yz + zx$, *then prove that* ∇u, ∇v *and* ∇w *represent co-planar vectors.* **[5 Marks]**

3. *Find the values of* λ *and* μ *so that the surfaces* $\lambda x^2 - \mu yz = (\lambda + 2)x$ *and* $3x^2 - y^2 = 1 - 2z$ *are orthogonal to each other at (1, – 2, 1).* **[6 Marks]**

4. *Prove that :*

$$\nabla\cdot\left[r\nabla\left(\frac{1}{r^3}\right)\right] + \nabla^2\left(\frac{1}{r}\right) = \frac{3}{r^4}.$$

[5 Marks]

13

Vector Integration

13.1 Line Integral

Any integral which is evaluated along a curve is called a line integral.

Let $\quad \overline{F}(x, y, z) = F_1(x, y, z)\,\hat{i} + F_2(x, y, z)\,\hat{j} + F_3(x, y, z)\,\hat{k}$

be a continuos vector function defined at every point of a curve C in space.

Divide the curve C into n-parts by the points.

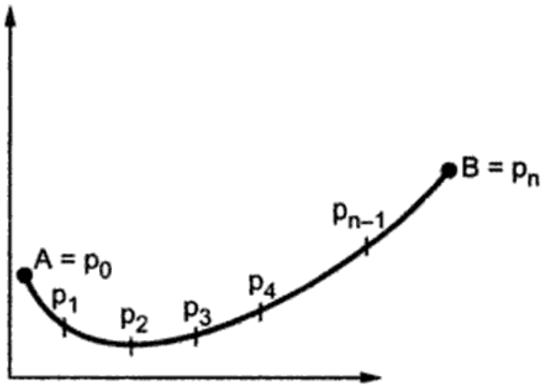


Fig. 13.1

$A = p_0, p_1, p_2, \ldots, p_n = B$. Let $\bar{r}_0, \bar{r}_1, \ldots, \bar{r}_n$ be the position vector of these points. Let Q_i be any point between p_{i-1} and p_i. Then limit of sum

$\sum_{i=1}^{n} \overline{F}(Q_i) \cdot \delta\,\overline{p}_i$ where $\delta\overline{p}_i = \overline{p}_i - \overline{p}_{i-1}$ as $n \to \infty$ and every $\left|\delta\overline{R}_i\right| \to 0$, if it exists is called a line integral of $\overline{F}$ along C and is denoted by

$$\int_C \overline{F} \cdot d\bar{r} = \int_C \overline{F} \cdot \frac{d\bar{r}}{ds}\, ds$$

But $\frac{d\bar{r}}{ds} = T$ = unit tangent $d\bar{s}$ the curve ds = elementary arc length of C.

$\therefore$ Line integral may be termed as integral of tangential component of $\overline{F}$ along C.

If $\quad \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

then $\quad d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$$\therefore \quad \int_C \overline{F} \cdot d\bar{r} = \int_C F_1 dx + F_2 dy + F_3 dz$$

If the parametric equation of the curve C are x = x (t), y = y (t), z = z (t) and $t = t_1$ at A and $t = t_2$ at B, then

$$\int_C \overline{F}\cdot d\overline{r} = \int_{t_1}^{t_2} \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt$$

If C is a closed curve then line integral is denoted by $\oint_C \overline{F}\cdot d\overline{r}$.

13.2 Physical Interpretation of Line Integration

Work done by $\overline{F}$

If $\overline{F}$ represents the force field, then $\overline{F}\cdot\hat{T}$ represents a force acting along the tangent to the curve C, under this force say the particle is displaced by distance say ds along C.

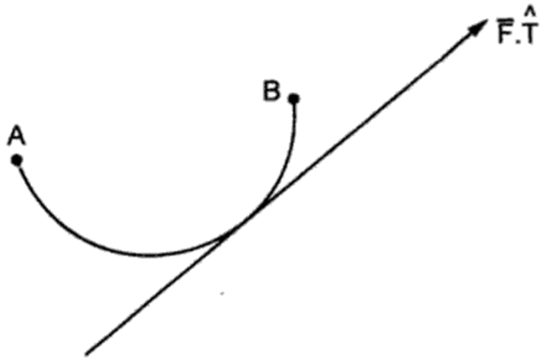


Fig. 13.2

$$\therefore \quad (\overline{F}\cdot\hat{T})\, ds = \text{force} \times \text{displacement}$$

$$= \text{work done}$$

$\therefore$ Total work done in moving a particle along C from A to B is given by

$$\text{work done} = \int_A^B \overline{F}\cdot\hat{T}\, ds = \int_A^B \overline{F}\cdot d\overline{r}$$

Note : Circulation

In fluid dynamics, if $\overline{V}$ represents the velocity of the fluid particle and C is the closed curve, then the integral $\oint_C \overline{V}\cdot d\overline{R}$ is called circulation of $\overline{V}$ round the curve C.

Note :

In evaluation of line integrals we have to express the line integral $\int_C \overline{F}\cdot d\overline{r}$ in terms of one variable either x or y or z or t or θ and then integrate within corresponding limits.

13.3 Type 1 : Evaluation of Line Integral

Example 13.1 : *If $\overline{F} = 3xy\,\hat{i} - y^2\hat{j}$, evaluate $\int_C \overline{F}\cdot d\overline{r}$ where C is the arc of the parabola $y = 2x^2$ from (0, 0) to (1, 2).*

Solution : Step 1 : Given

$$y = 2x^2$$

$$\because \quad dy = 4x\,dx$$

and x-varies from 0 to 1.

Step 2 : Consider

$$\therefore \quad \bar{F}\cdot d\bar{r} = 3xy\,dx - y^2 dy$$

Step 3 : Substituting we get

$$= 3x(2x^2)\,dx - (2x^2)^2 \cdot 4x\,dx$$

$$= (6x^3 - 16x^5)\,dx$$

Step 4 : Integrate both sides

$$\therefore \quad \int_C \bar{F}\cdot d\bar{r} = \int_0^1 (6x^3 - 16x^5)\,dx$$

$$= \left[\frac{3}{2}x^4 - \frac{16x^6}{x^6}\right]_0^1$$

$$= \frac{3}{2} - \frac{8}{3}$$

$$= -\frac{7}{6}$$

Example 13.2 : *Evaluate* $\int_C \bar{F}\cdot d\bar{r}$ *where* $\bar{F} = (2xy + z^3)\,\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ *along the straight line joining (1, – 2, 1) and (3, 1, 4).*

Solution : Step 1 : Equation of straight line joining (1, – 2, 1) and (3, 1, 4) is

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-1}{3} = t \text{ (say)}$$

Step 2 : $\therefore$ Its parametric equation

$$x = 2t + 1,\ y = 3t - 2,\ z = 3t + 1$$

$$dx = 2dt,\ dy = 3dt,\ dz = 3dt$$

x	1	3
t	0	1

$\therefore$ t varies from 0 to 1.

Step 3 : Consider

$$\bar{F}\cdot d\bar{r} = (2xy + z^3)\,dx + x^2 dy + 3xz^2 dz$$

$$= [2(2t+1)(3t-2)\,2dt + (2t+1)^2\,3dt + 3(2t+1)(3t+1)^2\,3dt]$$

$$= 4[6t^2 - t - 2]\,dt + 3[4t^2 + 4t + 1]\,dt$$

$$\bar{F}\cdot d\bar{r} = (216t^3 + 279t^2 + 98t + 6)\,dt$$

Step 4 : $\therefore \int_C \overline{F}\cdot d\overline{r} = \int_0^1 (216t^3+279t^2+98t+6)\,dt$

$$= \left[216\frac{t^4}{4}+279\frac{t^3}{3}+98\frac{t^2}{2}+6t\right]_0^1$$

$$= 202$$

Example 13.3 : *If $\overline{F}=(2y+3)\hat{i}+(xz)\hat{j}+(yz-x)\hat{k}$, evaluate $\int_C \overline{F}\cdot d\overline{r}$ along the following path C.*

i) The straight line joining (0, 0, 0) and (2, 1, 1)

ii) $x=2t^2$, $y=t$, $z=t^3$ from $t=0$ to $t=1$.

iii) The polygonal path joining A to B, B to C and C to D where A (0, 0, 0), B = (0, 0, 1), C (0, 1, 1) and D (2, 1, 1).

Solution : i) Equation of straight line joining (0, 0, 0) and (2, 1, 1) is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0}=\frac{z-0}{1-0}=t \ \text{(say)}$$

i.e. $x = 2t,\ y = t,\ z = t$

$dx = 2dt,\ dy = dt,\ dz = dt, \quad 0 \le t \le 1$

Substituting we get

$$\int_C \overline{F}\cdot d\overline{r} = \int_0^1 (3t^2+2t+6)\ dt$$

$$= [t^3+t^2+6t]_0^1 = 8$$

ii) $x = 2t^2,\ y = t,\ z = t^3$

$dx = 4t\ dt,\ dy = dt,\ dz = 3t^2dt$

Substituting we get

$$\overline{F}\cdot d\overline{r} = (3t^6+2t^5-6t^4+8t^2+12t)\ dt$$

$$\int_C \overline{F}\cdot d\overline{r} = \int_0^1 (3t^6+2t^5-6t^4+8t^2+2t)\ dt$$

$$= \frac{288}{85}$$

iii) $\int_C \overline{F}\cdot d\overline{r} = \int_{AB} \overline{F}\cdot d\overline{r}+\int_{BC} \overline{F}\cdot d\overline{r}+\int_{CD} \overline{F}\cdot d\overline{r}$

Along AB : $\dfrac{x-0}{0-0}=\dfrac{y-0}{0-0}=\dfrac{z-0}{1-0}=t$ (say)

$\Rightarrow x = 0,\ y = 0,\ z = t$

$dx = 0,\ dy = 0,\ dz = dt$

$$\overline{F}\cdot d\overline{r} = 0$$

$$\int_{AB} \overline{F}\cdot d\overline{r} = 0$$

Along BC : $\dfrac{x-0}{0-0}=\dfrac{y-0}{1-0}=\dfrac{z-0}{1-1}=t$ (say)

$x = 0,\ y = t,\ z = 1$

$dx = 0,\ dy = dt,\ dz = 0$

$$\therefore \quad \overline{F}\cdot d\overline{r} = 0$$

$$\int_{BC} \overline{F}\cdot d\overline{r} = 0$$

Along CD : $\dfrac{x-0}{2-0}=\dfrac{y-1}{1-1}=\dfrac{z-1}{1-1}=t$ (say)

$x = 2t,\ y = 1,\ z = 1$

$dx = 2dt,\ dy = 0,\ dz = 0$

Substituting we get

$$\overline{F}\cdot d\overline{r} = 10\ dt$$

$$\int_{CD} \overline{F}\cdot d\overline{r} = \int_0^1 10\ dt = 10$$

$$\therefore \quad \int_{C} \overline{F}\cdot d\overline{r} = 10$$

Example 13.4 : *Find work done in moving a particle along $y^2 = x$ from (0, 0) to (1, 1) in force field.*

Solution : $$\overline{F} = \left(x^2 - y^2\right)\hat{i} + 2xy\hat{j}$$

Step 1 : Consider $\overline{F} = \left(x^2 - y^2\right)\hat{i} + 2xy\hat{j}$

Step 2 : Consider $\int \overline{F}\cdot d\overline{r}$

$$= \int \left(x^2 - y^2\right)dx + 2xy\ dy$$

Step 3 : Put $x = y^2$

Now along OA (0, 0, 0) to (1, 0, 0)

$$\frac{x-0}{1-0} = \frac{y-0}{0-0} = \frac{z-0}{0-0}$$

x = 0 to 1 y = 0, z = 0, dy = 0, dz = 0

Substituting in (i) we get

$$\int_{OA} \bar{F} \cdot d\bar{r} = \int_0^1 2y dx = \int_0^1 0 dx = 0$$

Step 5 : Along AB (1, 0, 0) and (1, 1, 0)

$$\frac{x-1}{1-1} = \frac{y-0}{1-0} = \frac{z-0}{0-0}$$

y = 0 to 1 x = 1, z = 0, dx = 0, dz = 0

Substituting in (i) we get

$$\int_{AB} \bar{F} \cdot d\bar{r} = \int_0^1 x dy = \int_0^1 1 dy = 1$$

Step 6 : Along BC (1, 1, 0) and (1, 1, 1)

$$\frac{x-1}{1-1} = \frac{y-1}{1-1} = \frac{z-0}{1-0}$$

z = 0 to 1 x = 1, y = 1, dx = 0, dy = 0

Substituting in (i) we get

$$\int_{BC} \bar{F} \cdot d\bar{r} = \int_0^1 z^2 \, dz = \frac{1}{3}$$

Step 7 : From (ii)

$$\int \bar{F} \cdot d\bar{r} = \int_{OA} \bar{F} \cdot d\bar{r} + \int_{AB} \bar{F} \cdot d\bar{r} + \int_{BC} \bar{F} \cdot d\bar{r}$$

Substituting ali values of integrals we get

$$\int_C \bar{F} \cdot d\bar{r} = 0 + 1 + \frac{1}{3}$$

$$\int_C \bar{F} \cdot d\bar{r} = \frac{4}{3}$$

Step 2 : $\bar{r} = t\hat{i}+t^3\hat{j}+t^2\hat{k} = x\hat{i}+y\hat{j}+z\hat{k}$

$x = t, y = t^3, z = t^2$

$dx = dt, dy = 3t^2dt, dz = 2tdt$

Step 3 : Substituting in (i) we get

As the limits of t are given from t = 0 to t = 1.

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 \left[2(t)(t)^3+3(t^2)^2\right]dt+(t^2+4t^5)\,3t^2dt+(2t^6+6t^3)\,2tdt$$

Step 4 : Simplify

$$= \int_0^1 (2t^4+3t^4)\,dt+(3t^4+12t^7)\,dt+(4t^7+12t^4)\,dt$$

Step 5 : Integrate

$$= \left(5\frac{t^5}{5}+3\frac{t^5}{5}+12\frac{t^8}{8}+4\frac{t^8}{8}+12\frac{t^5}{5}\right)_0^1$$

Step 6 : Substitute the limits of t.

$$=\left(\frac{5}{5}+\frac{3}{5}+\frac{12}{8}+\frac{4}{8}+\frac{12}{5}\right)-101$$

$$= \frac{20}{5}+\frac{16}{8} = 4+2 = 6$$

ii) x = y = z from (0, 0, 0) to (1, 1, 1)

Step 1 :

$$\int_C \bar{F}\cdot d\bar{r} = \int (2xy+3z^2)\,dx+(x^2+4yz)\,dy+(2y^2+6xz)\,dz$$

Step 2 : Given x = y = z

$\therefore$ dx = dy = dz

limits are given from (0, 0, 0) to (1, 1, 1)

$$\int_c \bar{F}\cdot d\bar{r} = \int_0^1 (2x^2+3x^2)\,dx+(x^2+4x^2)\,dx+(2x^2+6x^2)\,dx$$

Step 3 : Integrate

$$= \left(18\frac{x^3}{3}\right)_0^1$$

Step 4 : Substitute the limits of x.

$$= 6$$

Use conversion formula

$$= \int_0^{2\pi} 0 - 12 \cdot 4 \int_0^{\pi/2} \sin^2\theta \, d\theta - 0 - 12 \cdot 4 \int_0^{\pi/2} \cos^2\theta \, d\theta + 0 - 0 - 0$$

Step 5 : Using reduction formula

$$= -12 \times 4 \times \frac{1}{2} \cdot \frac{\pi}{2} - 12 \times 4 \times \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= -12\pi - 12\pi$$

$$= -24\pi$$

Example 13.15 : *If* $\int_C \overline{F} \cdot d\overline{r} = 0$ *where C is closed curve. Then prove that the line integral* $\overline{F} \cdot d\overline{r}$ *is independent of the path joining the points A and B where A and B are the point on the curve.*

Solution : Step 1 :

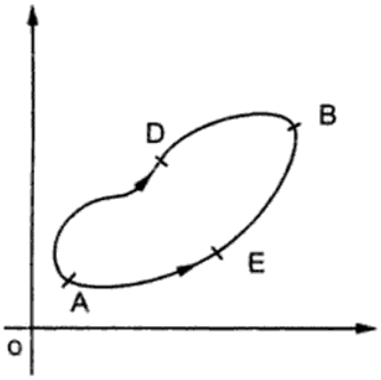


Fig. 13.6

Step 2 :

$$= \int_C \overline{F} \cdot d\overline{r} = 0$$

$$= \int_{AEBDA} \overline{F} \cdot d\overline{r} = 0$$

Step 3 :

$$\int_{AEB} \overline{F} \cdot d\overline{r} + \int_{BDA} \overline{F} \cdot d\overline{r} = 0$$

Step 4 :

$$\int_{AEB} \overline{F} \cdot d\overline{r} - \int_{ADB} \overline{F} \cdot d\overline{r} = 0$$

Step 5 :

$$\therefore \quad \int_{AEB} \overline{F} \cdot d\overline{r} = \int_{ADB} \overline{F} \cdot d\overline{r}$$

Type 3

Example 13.16 : *Find work done in moving a particle once around circle* $x^2 + y^2 = a^2$, $Z = 0$ *in the force field.*

Solution : Step 1 :

Consider $\quad \overline{F} = \sin y i + x(1 + \cos y) j$

Step 2 :

Consider $\int \overline{F}\cdot d\overline{r} = \int \sin y dx + x \cos y\, dy + x dy$

Step 3 : Integrate

$$= \int d\,(x \sin y) + \int x dy$$

$$= [x \sin y]_C + \int x dy$$

$$= 0 + \int x dy$$

Over the circle $x^2 + y^2 = a^2$ $\therefore$ Put $x = a \cos \theta$, $y = a \sin \theta$

$$= a^2 \int_0^{2\pi} \cos\theta \cdot \cos\theta\, d\theta$$

$$= a^2 4 \int_0^{\pi/2} \cos^2 \theta d\theta$$

Step 4 : Using reduction formula.

$$= a^2 4\left[\frac{1}{2}\cdot\frac{\pi}{2}\right] = \pi a^2$$

Example 13.17 : *If* $\overline{v} = \frac{a}{x}\hat{i} + \frac{b}{y}\hat{j}$, *find* $\int_C \overline{v}\cdot d\overline{r}$ *along the curve joining (1, 2) to (3, 3).*

Solution : Step 1 : Consider

$$\overline{v} = \frac{a}{x}\hat{i} + \frac{b}{y}\hat{j}$$

Step 2 : Consider

$$\int \overline{v}\cdot d\overline{r} = \int_C \frac{a}{x} dx + \frac{b}{y} dy$$

$$= (a \log x + b \log y)_{(1,2)}^{(3,3)}$$

Step 3 : Substitute the limits

$$= (a \log 3 - a \log 1 + b \log 3 - b \log 2)$$

$$= a \cdot \log \frac{3}{1} + b \log \frac{3}{2}$$

13.5 Simply Connected Regions

A region of the plane is called simply connected if it contains all the points enclosed by any closed curve in the region.

Example

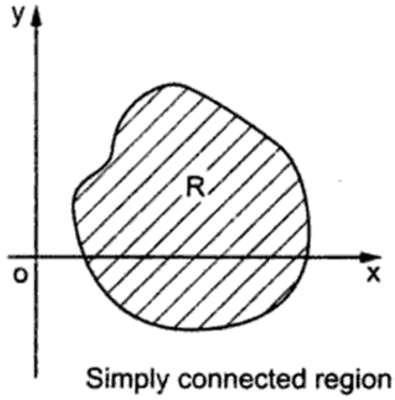


Fig. 13.7

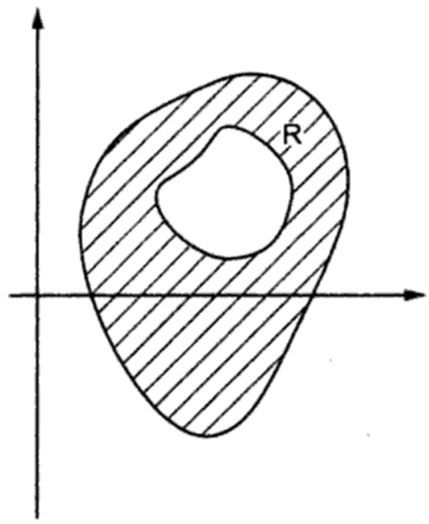


Fig. 13.8

Theorem

If domain of $\overline{F}$ is simply connected and $\nabla \times \overline{F} = \overline{0}$, then $\overline{F}$ is conservative i.e. the line integral over 'c' is zero.

Theorem

If $\overline{F}$ is conservative, then these exists scalar function ϕ (called as potential of $\overline{F}$) such that $\overline{F} = \nabla\phi$

Method of finding ϕ

$$\nabla\phi \cdot d\overline{r} = \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= d\phi \text{ (Total differential or Exact differential)}$$

$$\therefore \quad \phi = \int d\phi + C$$

$$= \int \overline{F} \cdot d\overline{r} + C$$

$$= \int\limits_{y,\, z \text{ constant}} F_1 dx + \int\limits_{z \text{ constant}} (\text{Terms in } F_2 \text{ independent of } x)\, dy$$

$$+ \int (\text{Terms in } F_3 \text{ independent of } x, y) + C$$

Thus work done in an irrotational field

i.e. $$\int_C \overline{F} \cdot d\overline{r} = [\phi]_P^Q$$

Exercise 13.1

1. *Evaluate* $\int_{(1,1)}^{(4,2)} \overline{F} \cdot d\overline{r}$ *where* $F = (x+y)\hat{i} + (y-x)\hat{j}$ *along i) the parabola* $y^2 = x$ *ii) the straight line joining (1, 1) and (4, 2).* **[Ans. : $\frac{20}{3}$]**

2. *Evaluate* $\int_{(0,1)}^{(2,3)} (2xy-1)\,dx + (x^2+1)\,dy$ *along i)* $y = x + 1$ *and ii)* $y = \frac{x^2}{2} + 1$. **[Ans. : i) 12, ii) 12]**

3. *Evaluate the line integral* $\int_C xdx - yzdy + e^z dz$ *if C is given by* $x = t^3,\ y = -t,\ z = t^2;\ 1 \le t \le 2$

   [Ans. : $\frac{111}{4} + e^4 - e$]

4. *Evaluate* $\int_C xyz\,dx - \cos(yz)\,dy + xzdz$ *over the straight line segment from (1, 1, 1) to (– 2, 1, 3)*

   [Ans. : $\frac{3}{2}$]

5. *What is the value of the line integral* $\int_C 3x^2y^2dx + 2x^3ydy$ *in the positive direction around C, where C is the ellipse* $x^2 + 4y^2 = 4$ *?*

6. *The force on a particle p is given by the vector field* $\overline{F}(x,y) = xy\hat{i} + (x^2+y^2)\hat{j}$. *Find the work done by the force as it moves p from (0, 0) to (4, 2) along* $x = y^2$.

7. *If* $\overline{F} = (3x-2y)\hat{i} + (y+2z)\hat{j} - x^2\hat{k}$ *evaluate* $\int_C \overline{F} \cdot d\overline{r}$ *from (0, 0, 0) to (1, 1, 1) where C is a path consisting of*

   i) the curve $x = t,\ y = t^2,\ z = t^3$;

   ii) the straight line joining (0, 0, 0) to (1, 1, 1).

   iii) the straight lines from (0, 0, 0) to (0, 1, 0) then to (0, 1, 1) and then to (1, 1, 1).

8. *Find the work done by the force* $\overline{F} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$ *when it moves a particle from the point (0, 0, 0) to (2, 1, 1) along the curve* $x = 2t^2,\ y = t,\ z = t^3$.

9. *Find the work done in moving a particle once around a circle C in the x-y plane if the circle has centre at the origin and radius 3 when the force field is given by* $\overline{F} = (2x-y+z)\hat{i} + (x+y-z^2)\hat{j} + (3x-2y+4z)\hat{k}$.

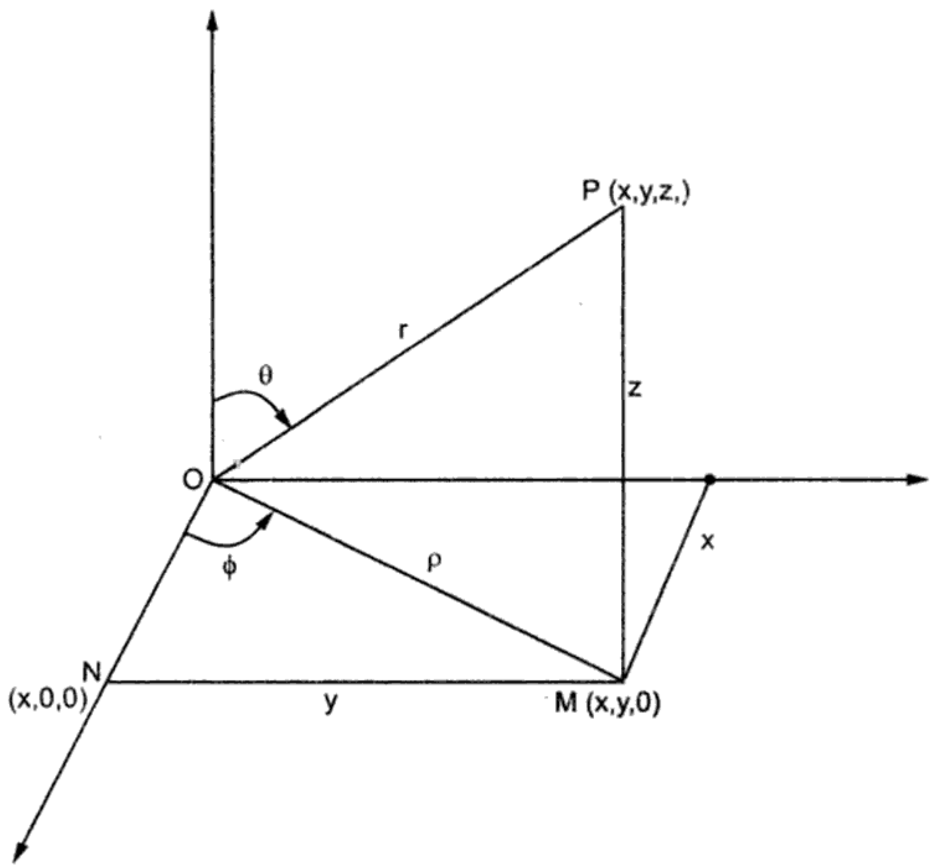


Fig. 13.10

In case of cylinder of radius a (i.e. $x^2+y^2=a^2$)

surface element = ds = a dϕdz

(r, θ, ϕ) spherical polar co-rodinates.

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

$$r = \sqrt{x^2+y^2+z^2},\ \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\text{Volume element} = dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$$

In case of sphere of radius a ($x^2+y^2+z^2=a^2$)

$$\text{surface element} = ds = a^2\sin\theta\,d\theta\,d\phi$$

13.9 Green's Theorem in the XOY Plane

If R is a closed region of the XY-plane bounded by a simple closed curve C and if M and N are continuous function of x and y having continuous derivation in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is traversed in the positive (counter clockwise) direction.

Proof : Let C be a closed curve such that any straight line parallel to co-ordinate axis cuts C in at most two points.

Let the equation of curve AEB and AFB be $y = \phi_1(x)$ and $y = \phi_2(x)$ respectively. If R is the region bounded by C we have

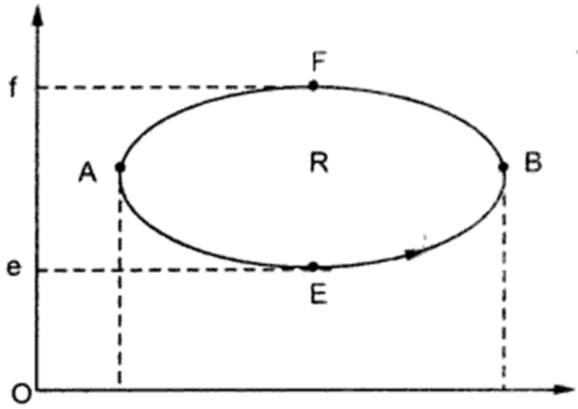


Fig. 13.11

$$\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial M}{\partial y}\,dy\,dx$$

$$= \int_a^b [M(x, \phi_2) - M(x, \phi_1)]\,dx$$

$$= -\int_a^b M(x, \phi_1)\,dx - \int_b^a M(x, \phi_2)\,dx$$

$$= -\oint_C M\,dx$$

Then $$\oint_C M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dx\,dy \qquad \text{... (1)}$$

Similarly let the equations of the curves EAF and EBF be $x = \psi_1(y)$ and $x = \psi_2(y)$ respectively.

$$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_e^f \int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial N}{\partial x}\,dx\,dy$$

$$= \int_e^f [N(\psi_2, y) - N(\psi_2, y)]\,dy$$

$$= \int_f^e N(\psi_1, y)\,dy + \int_e^f N(\psi_2, y)\,dy$$

$$= \oint_C N\,dy$$

$$\therefore \quad \oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy \quad \text{... (2)}$$

Adding (1) and (2),

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

13.10 Green's Theorem in Plane in Vector Notation

We have

$$M\,dx + N\,dy = (M\,\hat{i} + N\,\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j})$$

$$= \bar{F} \cdot d\bar{r}$$

where $\quad \bar{F} = M\,\hat{i} + N\,\hat{j},\ \bar{r} = x\,\hat{i} + y\,\hat{j}$

$$d\bar{r} = dx\,\hat{i} + dy\,\hat{j}$$

Also, if $\quad \bar{F} = M\,\hat{i} + N\,\hat{j}$ then

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = -\frac{\partial N}{\partial z}\,\hat{i} + \frac{\partial M}{\partial z}\,\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

$$\therefore \quad (\nabla \times \bar{F}) \cdot \hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

($\frac{\partial N}{\partial z} = \frac{\partial M}{\partial z} = 0$, since M and N are function of x and y above)

Then, Green's theorem in the XOY plane can be written as

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_R (\nabla \times \bar{F}) \cdot \hat{k}\,dx\,dy$$

Another Method

Let $\quad M\,dx + N\,dy$

$$= \bar{F} \cdot d\bar{r}$$

$$= \bar{F} \cdot \frac{d\bar{r}}{ds}\,ds$$

$$= \bar{F} \cdot \hat{T}\,ds$$

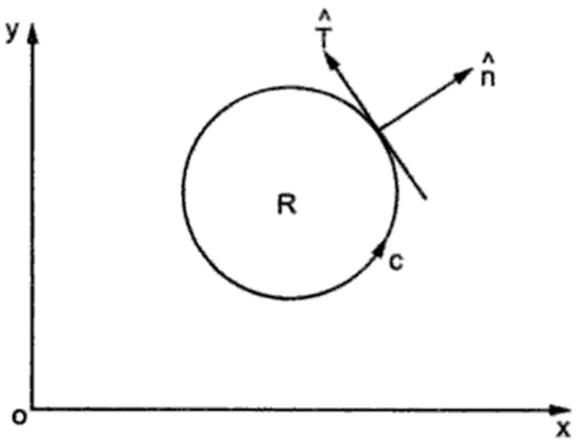


Fig. 13.12

where $\frac{d\bar{r}}{ds} = \hat{T}$ = unit tangent vector to C.

If $\hat{n}$ is the outward drawn normal to C, then $\hat{T} = \hat{k}\times\hat{n}$, so that

$$\begin{aligned} M\,dx + N\,dy & \\ &= \bar{F}\cdot\hat{T}\,ds \\ &= \bar{F}\cdot(\hat{k}\times\hat{n})\,ds \\ &= (\bar{F}\times\hat{k})\cdot\hat{n}\,ds \end{aligned}$$

Since
$$\begin{aligned} \bar{F} &= M\,\hat{i} + N\,\hat{j} \\ \bar{G} &= \bar{F}\times\hat{k} = M\,(\hat{i}\times\hat{k}) + N(\hat{j}\times\hat{k}) \\ &= N\,\hat{i} - M\,\hat{j} \text{ and} \\ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} &= \nabla\cdot\bar{G} \end{aligned}$$

then Green's theorem in plane becomes

$$\oint_C \bar{G}\cdot\hat{n}\,ds = \iint_R \nabla\cdot\bar{G}\,dx\,dy$$

13.11 Illustrations

Example 13.23 : *Verify Green's theorem in plane for* $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ *where C is the boundary of the region defined by* $y = \sqrt{x}$, $y = x^2$.

Solution : $y^2 = x$, $y = x^2$, intersects at O (0, 0), A (1, 1) we have $M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16\,y$$

$$\frac{\partial N}{\partial x} = -6y$$

$$\therefore \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10\,y$$

Let R be the region bounded by two parabolas.

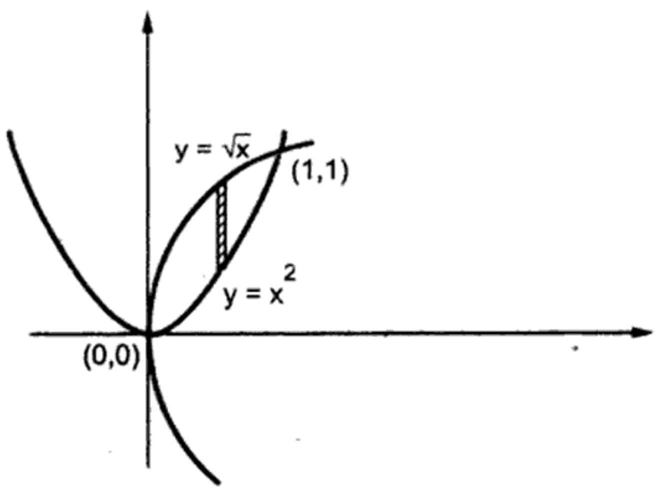


Fig. 13.13

$$\therefore \quad \oint_C M\,dx + N\,dy = -1+\frac{5}{2}=\frac{3}{2} \qquad \ldots (2)$$

From (1) and (2), Green's theorem is verified.

Example 13.24 : *Show that area bounded by the closed curve C is given by* $\frac{1}{2}\oint_C (x\,dy - y\,dx)$, *hence obtained the area of ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution : Green's theorem state that

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy$$

Let $\quad M = -y,\ N = x$

$$\therefore \quad \frac{\partial M}{\partial y} = -1,\ \frac{\partial N}{\partial x} = 1$$

$$\therefore \quad \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y} = 2$$

$$\therefore \quad \oint_C -y\,dx + x\,dy = 2\iint_R dx\,dy$$

$$= 2\,[\text{Area of the region R}] \qquad \ldots (1)$$

$\therefore$ Area bounded by the curve C

$$= \frac{1}{2}\oint_C x\,dy - y\,dx$$

To find the area of the ellipse

Let $\quad x = a\cos\theta,\ y = b\sin\theta$

$\therefore \quad dx = -a\sin\theta\,d\theta,\ dy = b\cos\theta\,d\theta$

From (1) $\quad$ Area $= \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} ab\left(\cos^2\theta+\sin^2\theta\right)d\theta = \frac{ab}{2}\int_0^{2\pi} d\theta$

$$= \pi ab$$

Example 13.25 : *Using Green's theorem evaluate* $\oint_C (xy-x^2)\,dx + x^2y\,dy$, *along the closed curve C formed by* $y = 0$, $x = 1$ *and* $y = x$.

Solution : By Green's Lemma

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy \qquad \ldots (1)$$

$$= \int_{-1}^{1} (1+y^2)\,dy + \int_{-1}^{1} (x^2+x)\,dx$$

$$+ \int_{-1}^{1} (1+y^2)\,dy + \int_{-1}^{1} (x^2-x)\,dx$$

Step 4 : Integrate

$$= \left[y+\frac{y^3}{3}\right]_{-1}^{1} + \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{1}^{-1} \left[y+\frac{y^3}{3}\right]_{1}^{-1} + \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-1}^{1}$$

Step 5 : Substitute the limits

$$= 2+\frac{2}{3}-\frac{2}{3}+0-2-\frac{2}{3}+\frac{2}{3}-0$$

$$= 0 \qquad \text{... (2)}$$

From (1) and (2). Green's Theorem is verified.

Example 13.29 : *Verify Green's Theorem in the plane for* $\oint_C (3x^2-8y^2)dx+(4y-6xy)dy$ *where 'C' is the boundary of the region defined by* $y^2 = x,\ y = x^2$.

Solution : Step 1 : Green's Theorem is

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Here $M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$$\therefore \quad \frac{\partial M}{\partial y} = -16\,y$$

$$\frac{\partial N}{\partial x} = -6y$$

$$\therefore \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y + 16\,y = 10\,y$$

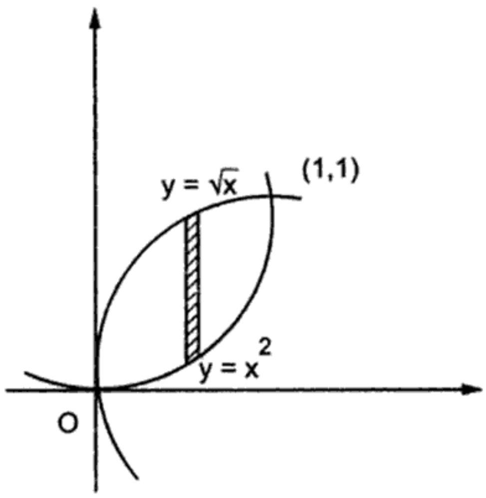


Fig. 13.18

Step 2 :

Now consider

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1 \int_{x^2}^{\sqrt{x}} 10\,y dxdy$$

Step 3 : Integrate

$$= 10\int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx = 5\int_0^1 \left(x - x^4\right) dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left[\frac{1}{2} - \frac{1}{5}\right] = \frac{3}{2} \qquad \text{... (1)}$$

Step 4 : Now consider the line integral

$$\int_C Mdx + Ndy = \int_{OA} \left(3x^2 + 8y^2\right) dx + (4y - 6xy)\, dy$$

$$+ \int_{AO} \left(3x^2 - 8y^2\right) dx + (4y - 6xy)\, dy$$

Along OA $x^2 = y$, $2xdx = dy$

Along AO $y^2 = x$, $2ydy = dx$

Step 5 :

$$= \int_0^1 \left(3x^2 - 8x^4\right) dx + \left(4x^2 - 6x^3\right) 2\,xdx$$

$$+ \int_1^0 \left(3y^4 - 8y^2\right) 2\,ydy + \left(4y - 6y^3\right) dy$$

$$= \int_0^1 \left(3x^2 + 8x^3 - 20x^4\right) dx + \int_1^0 \left(4y - 22\,y^3 + 6y^5\right) dy$$

Step 6 :

$$= \left[x^3 + 2x^4 - 4x^5\right]_0^1 + \left[2y^2 - \frac{11}{2}y^4 + y^6\right]_1^0$$

$$= -1 + \frac{5}{2} = \frac{3}{2} \qquad \text{... (2)}$$

From (1) and (2), Green's Theorem is verified.

Example 13.30 : *Verify Green's Theorem for* $\int_C (y - \sin x)\, dx + \cos x dy$ *where 'C' is a triangle whose vertices are (0, 0),* $\left(\frac{\pi}{2}, 0\right)$ *and* $\left(\frac{\pi}{2}, 1\right)$

Solution : Step 1 : Here M = (y – sin x); N = cos x

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -\sin x$$

13.12 Stokes Theorem

If S be an open surface bounded by a closed curve C and $\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be any vector point function having continuous first order partial derivatives, then

$$\oint_C \overline{F}\cdot d\overline{r} = \iint_S \nabla\times\overline{F}\cdot\hat{n}\,ds, \text{ where}$$

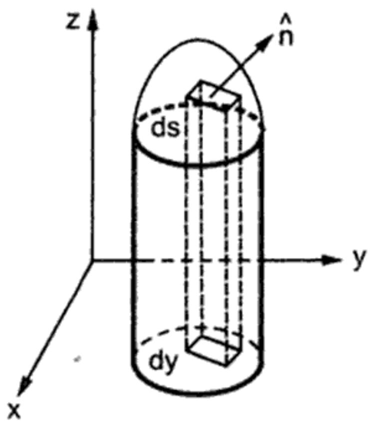


Fig. 13.20

$\hat{n}$ is a unit normal vector at any point of S drawn in the sense in which a right handed screw would advance when rotated in the sense of description of C.

Proof : Let S be a surface which is such that its projection on XY, YZ and ZX-plane are regions bounded by simple closed curves. Assume that S to have representation Z = f (x, y) or x = g (y, z) or y = h (x, z), where f, g, h are single value continuous and differentiable functions,

Consider,

$$\iint_S (\nabla\times F_1\hat{i})\cdot\hat{n}\,ds$$

$$\nabla\times F_1\,\hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix}$$

$$= \hat{j}\frac{\partial F_1}{\partial z} - \hat{k}\frac{\partial F_1}{\partial y}$$

$$\therefore \quad (\nabla\times F_1\hat{i})\cdot\hat{n}\,ds = \left[\frac{\partial F_1}{\partial z}(\hat{j}\cdot\hat{n}) - \frac{\partial F_1}{\partial y}(\hat{k}\cdot\hat{n})\right] ds \qquad \text{... (1)}$$

If Z = F (x, y) is taken as equation of S, then position vector of any point of S is,

$$\overline{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

$$= x\,\hat{i} + y\,\hat{j} + f(x, y)\,\hat{k}$$

So that

$$\frac{d\overline{r}}{dy} = \hat{j} + \frac{\partial f}{\partial y}\hat{k}$$

is a vector tangent to s, thus perpendicular to $\hat{n}$, so that

$$\hat{n}\cdot\frac{d\overline{r}}{dy} = \hat{n}\cdot\hat{j} + \frac{\partial f}{\partial y}\hat{n}\cdot\hat{k} = 0$$

$$\Rightarrow \qquad \hat{n}\cdot\hat{j} = -\frac{\partial f}{\partial y}(\hat{n}\cdot\hat{k})$$

Substitute in (1), we get

$$= \left(\frac{\partial F_1}{\partial z}\hat{n}\cdot\hat{j} - \frac{\partial F_1}{\partial y}\hat{n}\cdot k\right) ds$$

$$= \left(-\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} - \frac{\partial F_1}{\partial y}\right)\hat{n}\cdot\hat{k}\, ds$$

i.e. $$[\nabla \times \hat{i}]\cdot\hat{n}\, ds = -i\left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y}\right)\hat{n}\cdot\hat{k}\, ds \qquad \text{... (2)}$$

Now on S $F_1(x, y, z) = F_1(x, y, f(x, y) = F(x, y)$

Hence $\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial F}{\partial y}$ and (2) becomes

$$[\nabla \times (F_1\ \hat{i})]\cdot\hat{n}\, ds = -\frac{\partial F}{\partial y}\hat{n}\cdot\hat{k}\, ds = -\frac{\partial F}{\partial y}dx\, dy$$

Thus $$\iint_S [\nabla \times (F_1\ \hat{i})\cdot\hat{n}\, ds = \iint_R -\frac{\partial F}{\partial y}dx\, dy$$

where R is the projection of S on XY-plane.

$\therefore$ By green's lemma, the last integral equals $\oint_\Gamma \overline{F}\, dx$ where 'Γ' is a boundary of R.

Since at each point (x, y) of Γ the value of F is same the value of F_1 at each point (x, y, z) of C, and dx is same for both curves

$$\oint_\Gamma F\, dx = \oint_C F_1\ dx$$

or $$\iint_S [\nabla \times F_1\ \hat{i}]\cdot\hat{n}\, ds = \oint_C F_1\ dx$$

Similarly, by projection on the other co-ordinate planes,

$$\iint_S [\nabla \times (F_2\hat{j})\cdot\hat{n}\, ds = \oint_C F_2\ dy$$

$$\iint_S [\nabla \times (F_3\ \hat{k})]\cdot\hat{n}\, ds = \oint_C F_3\ dz$$

Thus by adding, we get

$$\iint_S (\nabla \times \overline{F})\cdot\hat{n}\, ds = \oint_C \overline{F}\cdot d\overline{r}$$

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & +x^2 & -(x+z) \end{vmatrix}$$

$$= \hat{j} + 2(x-y)\hat{k}$$

$$\nabla \times \overline{F} \cdot \hat{n} = \nabla \times \overline{F} \cdot \hat{k} = 2(x-y)$$

$$\therefore \quad \iint_R \nabla \times \overline{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^x 2(x-y)\, dy\, dx$$

$$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx = 2\int_0^1 \frac{x^2}{2}\, dx = \frac{1}{3} \qquad \text{... (1)}$$

Next, to evaluate $\oint_C \overline{F} \cdot d\overline{r}$

where $C = OA + AB + BO$

$$\therefore \quad \oint_C \overline{F} \cdot d\overline{r} = \int_{OA} \overline{F} \cdot d\overline{r} + \int_{AB} \overline{F} \cdot d\overline{r} + \int_{BO} \overline{F} \cdot d\overline{r} \qquad \text{... (2)}$$

Along OA : $y = 0,\ dy = 0,\ z = 0,\ dz = 0$

$$\therefore \quad \overline{F} \cdot d\overline{r} = y^2 dx + x^2 dy - (x+z)\, dz$$

$$= 0$$

$$\therefore \quad \int_{OA} \overline{F} \cdot d\overline{r} = 0$$

Along AB : $x = 1,\ z = 0 \Rightarrow dx = 0,\ dz = 0$ and y-varies from $y = 0$ to $y = 1$

$$\int_{AB} \overline{F} \cdot d\overline{r} = \int_0^1 dy = 1$$

Along BO : $x = y,\ z = 0\ dx = dy,\ dz = 0$, x varies from 1 to 0.

$$\overline{F} \cdot d\overline{r} = 2x^2 dx$$

$$\therefore \quad \int_{BO} \overline{F} \cdot d\overline{r} = 2\int_{1}^{0} x^2 dx = -\frac{2}{3}$$

Substitute in (2), we get

$$\oint_{C} \overline{F} \cdot d\overline{r} = 1 - \frac{2}{3} = \frac{1}{3} \quad \text{... (3)}$$

Hence from (1) and (3), stokes theorem is verified.

Example 13.32 : *Verify stokes theorem for* $\overline{F} = (2x - y)\,\hat{i} - yz^2\hat{j} - y^2 z\,\hat{k}$ *taken over the rectangular parallelopiped* $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, *above the XOY-plane.*

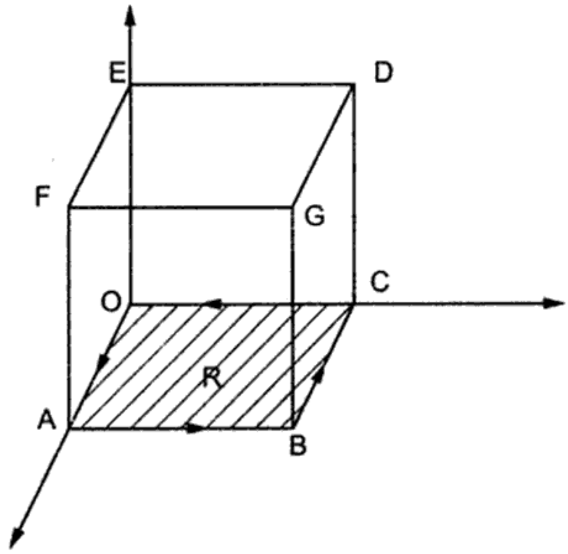


Fig. 13.23

Solution : Stokes theorem states that

$$\oint_{C} \overline{F} \cdot d\overline{r} = \iint_{S} \nabla \times \overline{F} \cdot \hat{n}\, ds$$

To evaluate $\iint_{S} \nabla \times \overline{F} \cdot \hat{n}\, ds$

where $\quad S = BCDG + OCDE + OAEF + ABGF + EFGD$

But the bounding curve C' is source for the surface S and the region R, shown in the figure.

$$\therefore \quad \iint_{S} \nabla \times \overline{F} \cdot \hat{n}\, ds = \iint_{R} \nabla \times \overline{F} \cdot \hat{k}\, dx\, dy \quad \text{... (1)}$$

$$= \frac{-a^2}{2}\left[\int_0^a dy\right]$$

$$= \frac{-a^2}{2}[y]_0^a$$

$$= \frac{-a^2}{2}[a] = \frac{-a^3}{2}$$

Step 9 : Consider the expression $\bar{F} \cdot d\bar{r}$

$$\bar{F} \cdot d\bar{r} = (xy)\,dx + (yz)\,dy + (z^2)\,dz$$

Step 10 : As the bounding curve lie in x-o-y plane put $z = 0$.

$$= xy\,dx + 0 + 0$$

Step 11 : Write the data for line segments OA, AB, BC, CO.

Along OA	$Y = 0$ $dy = 0$	$z = 0$ $dz = 0$	$x = 0$ to a	$\bar{F} \cdot d\bar{r} = 0\,dx$
Along AB	$x = a$ $dx = 0$	$z = 0$ $dz = 0$	$y = 0$ to a	$\bar{F} \cdot d\bar{r} = ax\,dx$
Along BC	$Y = a$ $dy = 0$	$z = 0$ $dz = 0$	$x = a$ to 0	$\bar{F} \cdot d\bar{r} = 0$
Along CO	$x = 0$ $dx = 0$	$z = 0$ $dz = 0$	$y = a$ to 0	$\bar{F} \cdot d\bar{r} = 0$

Step 12 : Consider the line integral.

$$\int_C \bar{F} \cdot d\bar{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Step 13 : Substitute all values.

$$= 0 + 0 + \int_a^0 x \cdot a\,dx + 0$$

Step 14 : Integrate

$$= a \cdot \left(\frac{x^2}{2}\right)_a^0$$

$$= \frac{-a^3}{2}$$

$\therefore$ L.H.S = R.H.S

Thus stoke's theorem is verified.

Example 13.34 : *Verify Stoke's theorem for* $\overline{F} = (y-z-2)\hat{i} + (yz+4)\hat{j} - xz\hat{k}$ *over the cube whose side is 2 and open at bottom.*

Solution : Step 1 : As the cube is open at bottom i.e. in x-o-y plane it is enough to verify stoke's theorem over the square in x-o-y plane.

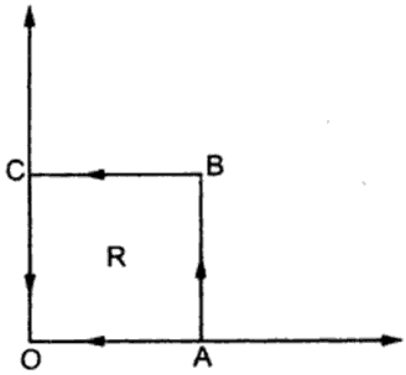


Fig. 13.25

Step 2 : Consider

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y-z-2 & yz+4 & -xz \end{vmatrix}$$

$$= \hat{i}\,(0 - y) - \hat{j}\,(-z-1) + \hat{k}\,(-1)$$

$$= -y\hat{i} + (z+1)\,\hat{j} - k$$

Step 3 : As the surface is in x-o-y plane $\hat{N} = \hat{k}$, ds = dx dy

Consider $\iint (\nabla \times \overline{F}) \cdot \hat{N}\, ds$

Step 4 : Substitute the values

$\iint \left[-y\hat{i} + (z+1)\,\hat{j} - \hat{k}\right] \cdot \hat{k}\, dx\, dy$

Step 5 : We know that $\hat{i} \cdot \hat{k} = 0$, $\hat{k} \cdot \hat{k} = 1$, $\hat{j} \cdot \hat{k} = 0$

Step 6 : To integrate over the square put the limits

$\int_0^2 \int_0^2 -dx\, dy$

Step 7 : Integrate

$$= \int_0^2 [-y]_0^2\, dx$$

Step 8 : Substitute the limits

$$= -2[x]_0^2 = -4$$

evaluation of the line integral $\int_C \overline{F} \cdot d\overline{r}$ is kept for student's exercise.

Example 13.35 : *Evaluate* $\int xy\, dx + xy^2\, dy$ *over the square with vertices* $(1, 0), (-1, 0), (0, 1), (0, -1)$.

Solution : Step 1 : Consider

$$\overline{F} = xy\,\hat{i} + xy^2\,\hat{i}$$

Step 2 : Consider

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(y^2 - x)$$

$$= 0\hat{i} + 0\hat{j} + (y^2 - x)\hat{k}$$

Step 3 : As the surface is in x-o-y plane $\hat{N} = \hat{k}$, $ds = dx\, dy$

Step 4 : Consider $\iint (\nabla \times \overline{F}) \cdot \hat{N}\, ds$

Step 5 : Substitute the values

$\iint (y^2 - x)\, \hat{k} \cdot \hat{k}\, ds$

Step 6 : We know that $\hat{i} \cdot \hat{k} = 0$, $\hat{k} \cdot \hat{k} = 1$, $\hat{j} \cdot \hat{k} = 0$

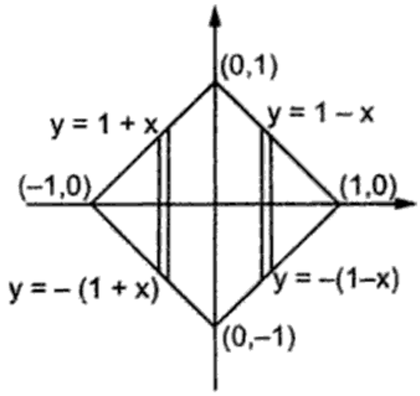


Fig. 13.26

Step 7 :

To integrate over the square put the limits

$$= \int_{-1}^{0} \int_{-(1+x)}^{1+x} (y^2 - x)\, dx\, dy + \int_{0}^{1} \int_{-(1-x)}^{1-x} (y^2 - x)\, dx\, dy$$

Step 8 : As the function is an even function of y thus

$$= 2\int_{-1}^{0} \int_{0}^{1+x} (y^2 - x)\, dx\, dy + 2\int_{0}^{1} \int_{0}^{1-x} (y^2 - x)\, dx\, dy$$

$$= 2\int_{-1}^{0} \left[\frac{y^3}{3} - xy\right]_0^{1+x} dx + 2\int_{0}^{1} \left[\frac{y^3}{3} - xy\right]_0^{1-x} dx$$

$$= 2\int_{-1}^{0} \left[\frac{(1+x)^3}{3} - x(1+x)\right] dx + 2\int_{0}^{1} \left[\frac{(1-x)^3}{3} - x(1-x)\right] dx$$

$$= 2\left[\frac{(1+x)^4}{3.4} - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{0} + 2\left[\frac{(1-x)^4}{-3.4} - \frac{x^2}{2} + \frac{x^3}{3}\right]_0^1$$

Step 9 : Substitute the limits.

$$= 2\left[\left(\frac{1}{12} - 0\right) - \left(0 - \frac{1}{2} + \frac{1}{3}\right)\right] + 2\left[\left(0 - \frac{1}{2} + \frac{1}{3}\right) - \left(\frac{1}{-12} - 0\right)\right]$$

$$= 2\left[\frac{1}{12} + \frac{1}{12}\right] = \frac{1}{3}$$

$$+ \left[3a^3 \cos\theta \, d\theta + 6a^3 \cos^2\theta \, d\theta + 3a^3 \cos^3\theta \, d\theta - a^3 \sin^2\theta \cos\theta \, d\theta\right]$$

Step 5 : Simplify

$$= a^3 \int_0^{2\pi} \left[-2\sin^3\theta \, d\theta + \sin\theta \, d\theta + 2\sin\theta\cos\theta \, d\theta + \sin\theta\cos^2\theta \, d\theta\right]$$

$$+ \left[3\cos\theta \, d\theta + 6\cos^2\theta \, d\theta + 3\cos^3\theta - \sin^2\theta\cos\theta \, d\theta\right]$$

Using conversion formula, we get

$$= a^3 \int_0^{\pi/2} \left[0+0+0+0+0+6\cos^2\theta \, d\theta + 0 - 0\right]$$

$$= a^3 6 \cdot 4 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

Step 6 : Using reduction formula

$$= a^3 \cdot 6 \cdot 4 \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= 6a^3\pi$$

Example 13.38 : *$\bar{F} = (2x-y+z)\hat{i} + (x+y-z^2)\hat{j} + (3x-2y+4z)\hat{k}$ over the surface of cylinder $x^2+y^2=4$ bounded by Z = 9 open at Z = 0.*

Solution : Step 1 :

$$\int_C \bar{F} \cdot d\bar{r} = \int_C (2x-y+z)\,dx + (x+y-z^2)\,dy + (3x-2y+4z)\,dz$$

Step 2 : The equation of surface over cylinder $x^2+y^2=4$, Z = 0, thus the bounding curve of this surface will be x-o-y plane.

∴ Put Z = 0 in equation of surface over cylinder to get the equation of bounding curve 'C'.

Thus 'C' is $x^2+y^2=4$.

Step 3 : Put $x = 2\cos\theta$, $y = 2\sin\theta$

$dx = -2\sin\theta \, d\theta$, $dy = 2\cos\theta \, d\theta$

Step 4 : Substitute the values

$$= \int_0^{2\pi} [2(2\cos\theta) - (2\sin\theta) + (0)](-2\sin\theta \, d\theta)$$

$$+ [2\cos\theta + 2\sin\theta - 0]\,(2\cos\theta \, d\theta)$$

Step 5 : Integrate

$$= \int_0^{2\pi} \left[-8 \sin\theta \cos\theta \, d\theta + 4 \sin^2\theta \, d\theta + 4 \cos^2\theta \, d\theta + 4 \sin\theta \cos\theta \, d\theta\right]$$

Using conversion formula we get

$$= \int_0^{\pi/2} \left[0 + 4 \sin^2\theta \, d\theta + 4 \cos^2\theta \, d\theta + 0\right]$$

$$= 4 \cdot 4 \int_0^{\pi/2} \sin^2\theta \, d\theta + 4 \cdot 4 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

Step 6 : Simplify using reduction formula.

$$= 4 \cdot 4 \cdot \frac{1}{2} \cdot \frac{\pi}{2} + 4 \cdot 4 \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= 4\pi + 4\pi = 8\pi$$

Example 13.39 : *$\bar{F} = y\hat{i} + z\hat{j} + x\hat{k}$ over surface of parabolic $z = 1 - x^2 - y^2$ for which $z \geq 0$*

Solution : Step 1 :

$$\int_C \bar{F} \cdot d\bar{r} = \int_C y dx + z dy + x a z$$

Step 2 : The equation parabolic surface $z = 1 - x^2 y^2$ for which $z \geq 0$.

Thus the bounding curve of this surface will be in x-o-y plane.

$\therefore$ Put $Z = 0$ in equation of surface over parabolic to get the equation of curve 'C'.

Thus 'C' is $1 - x^2 y^2 = z$.

$\therefore$ $x^2 + y^2 = 1$ and $Z = 0$

Step 3 : Put $x = \cos\theta$, $y = \sin\theta$

$dx = -\sin\theta \, d\theta$, $dy = \cos\theta \, d\theta$

Step 4 : Substitute the values.

$$= 4 \int_0^{\pi/2} -\sin^2\theta \, d\theta$$

Step 5 : Integrate using reduction formula

$$= 4\left[-\left(\frac{1}{2} \cdot \frac{\pi}{2}\right)\right] = -\pi$$

$dx = -2 \sin \theta \, d\theta, \; dy = 2 \cos \theta \, d\theta$

Step 4 : Substitute the values.

$$= \int_0^{2\pi} (2\cos\theta - 0)(-2\sin d\theta) + (8\cos^3\theta - 0)(2\cos\theta \, d\theta)$$

Step 5 : Simplify.

$$= \int_0^{2\pi} -4\sin\theta\cos\theta \, d\theta + 16\cos^4\theta \, d\theta$$

$$= \int_0^{\pi/2} 0 + 16 \cdot 4 \int_0^{\pi/2} \cos^4\theta \, d\theta$$

Step 6 : Using reduction formula.

$$= 16 \cdot 4 \cdot \frac{3}{4} \cdot \frac{1}{2} \times \frac{\pi}{2}$$

$$= 12\pi$$

Note :

1) For double integral over the circle $x^2 + y^2 = a^2$

Put $\left.\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}\right\}$ $dx\,dy = r\,dr\,d\theta$

The limits are $\int_0^{2\pi} \int_0^{a} \ldots r\,dr\,d\theta$

2) For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Put $\therefore x = a\,r\cos\theta, \; y = br\sin\theta$

$\therefore dr\,dy = ab\,r\,dr\,d\theta$

The limits are

$$= \int_0^{2\pi} \int_0^{1} a\,b\,r\,dr\,d\theta$$

Example 13.42 : *Verify Stoke's theorem* $\overline{F} = -y^3\hat{i} + x^3\hat{j}$ *over ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution : Step 1 : Consider

Step 9 : Thus $\iint\limits_S (\nabla \times \overline{F}) \cdot \hat{N}\, ds$

$$= 3\left[a^3 b \frac{\pi}{4} + ab^3 \frac{\pi}{4}\right]$$

$$= 3\, ab \frac{\pi}{4}\left[a^2 + b^2\right] \qquad \text{... (1)}$$

Step 10 : To find $\int\limits_C \overline{F} \cdot d\overline{r}$

Consider $\int\limits_C \overline{F} \cdot d\overline{r} = \int\limits_C -y^3 dx + x^3\, dy$

over the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Step 11 : The parametric equations of ellipse are

$x = a \cos \theta,\ y = b \sin \theta,\ z = 0$

$\therefore\ dx = -a \sin \theta\, d\theta,\ dy = b \cos \theta\, d\theta,\ dz = 0$

Step 12 : Substituting we get

$$\int\limits_C \overline{F} \cdot d\overline{r} = \int\limits_0^{2\pi} -b^3 \sin^3 \theta(-a \sin\theta)\, d\theta + a^3 \cos^3 \theta\, b \cos\theta\, d\theta$$

$$= \int\limits_0^{2\pi} \left(ab^3 \sin^4 \theta + a^3 b \cos^4 \theta\right) d\theta$$

Step 13 : Using conversion formula

$$= 4\int\limits_0^{\pi/2} \left(ab^3 \sin^4 \theta + a^3 b \cos^4 \theta\right) d\theta$$

Step 14 : Using reduction formula

$$= 4\left[ab^3\left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right) + a^3 b\left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right)\right]$$

$$= 3ab \frac{\pi}{4}\left(a^2 + b^2\right) \qquad \text{... (2)}$$

From (1) and (2) Stoke's theorem is verified.

Example 13.43 : *Verify Stoke's theorem* $\overline{F} = y^3 \hat{i}$ *over* $x^2 + y^2 = a^2$, $z = 0$.

Solution : Step 1 : Consider

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^3 & 0 & 0 \end{vmatrix}$$

$$= \hat{i}\,(0-0) - \hat{j}\,(0-0) + \hat{k}\left(0-3y^2\right)$$

Step 2 : As the surface is in x-o-y plane.

$\because \hat{N} = k,\ \partial s = \partial x \partial y$

Step 3 : Consider $\iint \left(\nabla \times \overline{F}\right) \cdot \hat{N}\, \partial s$

Step 4 : Substitute the values

$$= \iint -3y^2\, dx\, dy$$

Step 5 : To integrate over $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Put $x = r\cos\theta$, $y = r\sin\theta$, $dx\, dy = r\, dr\, d\theta$

Step 6 : Substitute the values

$$= \int_0^{2\pi} \int_0^1 -3\left(r^2 \sin^2\theta\right) \cdot (r\, dr\, d\theta)$$

Step 7 : Integrate

$$= -3\int_0^{2\pi} \sin^2\theta\, d\theta \int_0^1 r^3\, dr - 3\int_0^{2\pi} \sin^2\theta \left[\frac{r^4}{4}\right]_0^1$$

$$= -3a^4 \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= \frac{-3a^4}{4}(4)\int_0^{\pi/2} \sin^2\theta\, d\theta$$

Step 8 : Using reduction formula

$$= \frac{-3a^4}{4}(4) \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= -3a^4 \frac{\pi}{4} \qquad \ldots (1)$$

Step 9 : To find $\int_C \overline{F} \cdot d\overline{r}$

Consider $\int_C \overline{F} \cdot d\overline{r} = \int_C + y^3\, dx$

over $x^2 + y^2 = a^2$

Step 10 : The parametric equations are x =a cos θ, y = a sin θ.

$\therefore$ dx = – a sin θ dθ, dy = a cos θ dθ

Substituting we get

$$\int_C \overline{F} \cdot d\overline{r} = \int_0^{2\pi} + a^3 \sin^3 \theta \cdot - a \cdot \sin\theta \, d\theta$$

$$= -a^4 \int_0^{2\pi} \sin^4 \theta \cdot d\theta$$

Step 11 : Using conversion formula

$$= -4a^4 \int_0^{\pi/2} \sin^4 \theta \, d\theta$$

Step 12 : Using reduction formula

$$= -4a^4 \left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right)$$

$$= -3a^4 \cdot \frac{\pi}{4} \qquad \text{... (2)}$$

From (1) and (2) Stoke's theorem is verified.

Example 13.44 : *Verify Stoke's theorem* $\overline{F} = (x^3 - y^3)\hat{i} - xyz\hat{j} + y^3\hat{k}$ *over* $x^2 + 4y^2 + z^2 - 2z = 4$ *above* $z = 0$.

Solution : Step 1 :

$$\int_C \overline{F} \cdot d\overline{r} = \int_C (x^3 - y^3)\, dx - (xyz)\, dy + y^3 dz$$

Step 2 : $x^2 + 4y^2 = 0$ and z = 0

x = 2 cos θ, 2y = 2 sin θ

dx = – 2 sin θ dθ, y = sin θ, dy = cos θ dθ

Step 3 : Substitute the values.

$$= \int_0^{2\pi} -16\cos^3 \theta \sin\theta \, d\theta + 2\sin^4 \theta \, d\theta$$

Step 4 : Integrate

$$= 2 \cdot 4 \int_0^{\pi/2} \sin^4 \theta \, d\theta$$

Step 5 : Simplify using reduction formula.

$$= 2\cdot 4\cdot \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= 3\frac{\pi}{2}$$

Step 6 :

$$\nabla\times\overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3-y^3 & -xyz & y^3 \end{vmatrix}$$

$$= i(3y^2+xy)-j(0-0)+k(-yz+3y^2)$$

Step 7 : As the surface is in xoy plane $\hat{N} = k$, $ds = dx\,dy$

$$\therefore \quad (\nabla\times\overline{F})\cdot\hat{N}\,ds = (\nabla\times\overline{F})\cdot k\,dx\,dy$$

$$= (-yz+3y^2)\,dx\,dy$$

$$= 3y^2\,dx\,dy \qquad \{\text{As } z = 0\}$$

Step 8 : Thus $\iint (\nabla\times\overline{F})\cdot\hat{N}\,ds$

$$= \iint 3y^2\,dx\,dy$$

over the ellipse $x^2+4y^2 = 4$, $z = 0$

i.e. $\frac{x^2}{4}+\frac{y^2}{1} = 1$, $z = 0$

Step 9 : For double integral over the ellipse $\frac{x^2}{4}+\frac{y^2}{1} = 1$

Put $x = 2r\cos\theta$, $y = 1\cdot r\cdot\sin\theta$, $dx\,dy = 2r\,dr\,d\theta$

$$\therefore \quad \iint (\nabla\times\overline{F})\cdot\hat{N}\,ds = \int_0^{2\pi}\int_0^1 3\,r^2\sin^2\theta\cdot 2r\,dr\,d\theta$$

$$= 6\int_0^{2\pi} \sin^2\theta\left[\frac{r^4}{4}\right]_0^1 d\theta$$

Using conversion formula

$$= 6\cdot 4\cdot\int_0^{\pi/2} \sin^4\theta\,d\theta\left(\frac{1}{4}\right)$$

Using reduction formula

$$= 6 \cdot 4 \cdot \left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right)\left(\frac{1}{4}\right)$$

$$= \frac{3\pi}{2} \qquad \text{... (2)}$$

From (1) and (2) Stoke's theorem is verified.

Example 13.45 : *Verify Stoke's theorem for* $\bar{F} = (x^2 + y - 4)\hat{i} + 3xy\hat{j} + (2xz + z^2)\hat{k}$ *over the surface of the paraboloid.* $z = 9 - (x^2 + y^2)$, *above the xoy-plane.*

Solution : Stoke's theorem state that

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F} \cdot \hat{n})\, ds$$

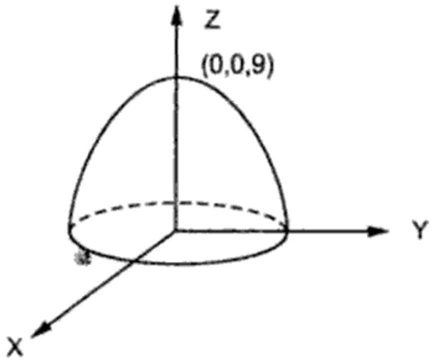


Fig. 13.27

To evaluate

$$\iint_S \nabla \times \bar{F} \cdot \hat{n}\, ds$$

Since the bounding curve C for the surface S and region R (shown in Fig.) is same.

$$\therefore \quad \iint_S \nabla \times \bar{F} \cdot \hat{n}\, ds = \iint_R \nabla \times \bar{F} \cdot \hat{k}\ dx\, dy \qquad \text{... (1)}$$

where $\hat{k}$ = Unit normal to xy-plane.

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y - 4 & 3xy & 2xz + z^2 \end{vmatrix}$$

$$= \hat{i}(0-0) + \hat{j}\,(0 - 2z) + \hat{k}(3y - 1)$$

$$\therefore \quad \nabla \times \bar{F} \cdot \hat{k} = 3y - 1$$

$$\therefore \quad \iint_R \nabla \times \bar{F} \cdot \hat{k}\, dx\, dy = \iint_R (3y - 1)\, dx\, dy$$

Changing to polar's

$x = r\cos\theta$, $\quad y = r\sin\theta$

$$dx\, dy = r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^3 (3 \cdot r\sin\theta - 1)\, r\, dr\, d\theta$$

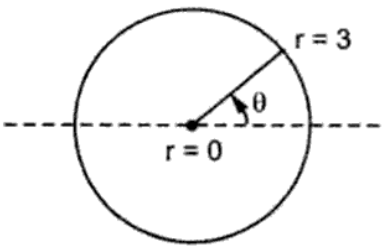


Fig. 13.28

$$\cos 45^\circ = \frac{1(0)+0(0)+\lambda(1)}{\sqrt{1^2+\lambda^2}\sqrt{0^2+0^2+1^2}}$$

$1+\lambda^2 = 2\lambda^2$, $\lambda^2 = 1$

Giving $\lambda = \pm 1$

Step 2 : Substituting in (1) the planes $x \pm z = 0$.

Let us take the equation of the wedge as $x - z = 0$. This will cut the cylinder $x^2+y^2 = a^2$ in the curve 'C' whose parametric equations are $x = a \cos\theta$, $y = \sin\theta$ so that

$z = a\cos\theta$, $dx = -a\sin\theta\, d\theta$, $dy = a\cos\theta\, d\theta$

$dz = -\sin\theta\, d\theta$ and θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

Step 3 : By Stoke's theorem

$$\iint_S \nabla\times\bar{F}\cdot d\bar{s} = \int_C \bar{F}\cdot d\bar{r} = \int_C 4x\,dx - 2y^2\,dy + z^2\,y dz$$

Step 4 : Substitute the values.

$$= \int_{-\pi/2}^{\pi/2} 4a\cos\theta(-a\sin\theta\,d\theta) - 2\cdot a^2\sin^2\theta(a\cos\theta\,d\theta)$$

$$+\ a^2\cos^2\theta\ a\sin\theta(-a\sin\theta\,d\theta)$$

Step 5 : Integrate

$$= -2a^2\int_{-\pi/2}^{\pi/2}\sin 2\theta\,d\theta - 2a^3\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos d\theta - a^4\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos^2 d\theta$$

$$= -2a^2\left(-\frac{\cos 2\theta}{2}\right)_{-\pi/2}^{\pi/2} - 2a^3\left(-\frac{\sin^3\theta}{3}\right)_{-\pi/2}^{\pi/2} - 2a^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

$$= 0 - \frac{2a^3}{3}(1+1) - 2a^4\frac{1\times 1}{4\times 2}\left(\frac{\pi}{2}\right)$$

$$= -\frac{4}{3}a^3 - \frac{\pi}{8}a^4$$

If we take the equation of the wedge as $x + z = 0$, the result will be $\frac{\pi}{8}a^4 - \frac{4}{3}a^3$.

Type 3

Example 13.47 : *Use Stoke's theorem* $\int_C ydx + zdy + xdzC$ ≡ *intersection of* $x^2+y^2+z^2 = a^2$ *and* $x+z=a$, $\bar{F} = yi+zj+xk$

Solution : Step 1 : Consider

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \hat{i}(0-1) - \hat{j}(1-0) + \hat{k}(0-1)$$

$$= -\hat{i} - \hat{j} - \hat{k}$$

Step 2 : The sphere cut by the plane gives circle. This particular circle lies in cutting plane $\therefore$ Normal to the circle is normal to the cutting plane here the plane is $x + z = a$.

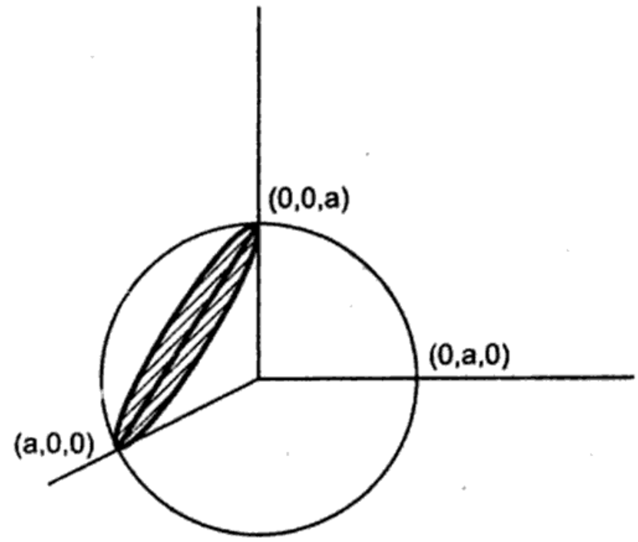


Fig. 13.29

Sphere $x^2 + y^2 + z^2 = a^2$, Centre 0, 0, 0

Radius = a $\therefore$ Diameter = $a\sqrt{2}$

$\therefore$ Radius $\frac{a}{\sqrt{2}}$

Step 3 : Let $\phi = x + z - a$

$$\nabla\phi = \hat{i} + 0\hat{j} + \hat{k}$$

$$|\nabla\phi| = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \hat{i} + 0\hat{j} + \hat{k} = \hat{i} + \hat{k}$$

$$|\nabla\phi| = \sqrt{1+1} = \sqrt{2}$$

Step 4 :

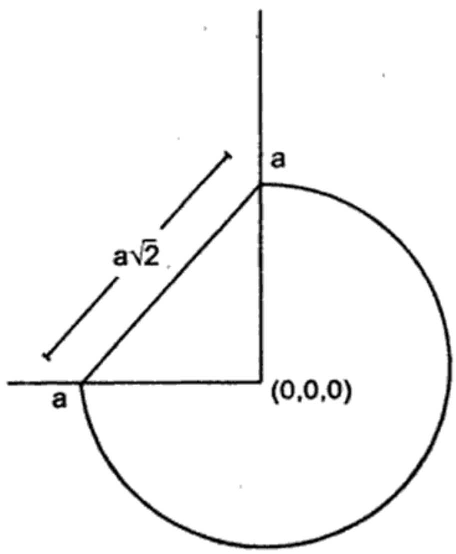


Fig. 13.30

$$\hat{N} = \frac{\nabla\phi}{|\nabla\partial|} = \frac{\hat{i} + \hat{k}}{\sqrt{2}}$$

Note : Here $\hat{N} \neq k$

$$\therefore \quad (\nabla \times \bar{F}) \cdot \hat{N} = (-\hat{i} - \hat{j} - \hat{k}) \cdot \frac{\hat{i} + \hat{k}}{\sqrt{2}}$$

$$= \frac{-1-1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

Step 5 :

Now, $$\int_C \bar{F} \cdot d\bar{r} = \iint (\nabla \times \bar{F}) \cdot \hat{N}\, \partial s$$

Step 6 : Substitute all the values.

$$\int_C \bar{F} \cdot d\bar{r} = \iint -\sqrt{2}\,\partial s$$

$$= -\sqrt{2} \text{ Core of circle}$$

$$= -\sqrt{2} \times \pi r^2$$

$$= -\sqrt{2} \times \pi \cdot \left(\frac{a}{\sqrt{2}}\right)^2$$

$$= -\frac{\pi a^2}{\sqrt{2}}$$

Example 13.48 : $\int_c ydx + zdy + xdz$ *where* $C \equiv x^2 + y^2 + z^2 - 2ax - 2ay = 0,\ x + y = 2a.$ *Use Stoke's theorem* $\bar{F} = y\hat{i} + z\hat{j} + x\hat{k}.$

Solution : Step 1 : Consider

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \hat{i}\,(0 - 1) - \hat{j}\,(1 - 0) + \hat{k}\,(0 - 1)$$

Step 2 : Let $\phi = x + z - 2a$

$$\nabla\phi = \hat{i} + \hat{j} - 0\hat{k}$$

$$|\nabla\phi| = \sqrt{1 + 1 + 0}$$

Step 3 :

$$\hat{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\hat{i} + \hat{j}}{\sqrt{2}}$$

$$(\nabla \times \bar{F}) \cdot \hat{N} = (-\hat{i} - \hat{j} - \hat{k}) \cdot \frac{(\hat{i} + \hat{j})}{\sqrt{2}} = -\sqrt{2}$$

Step 4 :

Now $$\int_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \hat{N}\,\partial s$$

Step 5 : Substitute the values.

$$= \iint -\sqrt{2}\,ds$$

$$= -\sqrt{2} \text{ (Area of surface)}$$

$$= -\sqrt{2} \text{ (Area of circle)}$$

Step 5 : Consider $\iint\limits_S \nabla\times\overline{F}\cdot\hat{N}\,ds$

Over circle $X^2+Y^2=1$

Step 6 : Substitute the values.

$\iint 3aX^2\ 3bY^2\ dX\,dY$

Step 7 : Using polar co-ordinates

$X = r\cos\theta$, $Y = r\sin\theta$, $dX\,dY = r\,d\theta\,dr$, $X^2+Y^2=1$, given circle is transformed into $r^2=1$ or $r=1$.

$$\therefore \quad \iint\limits_S \nabla\times\overline{F}\cdot\hat{N}\,ds = 9ab\int_0^{2\pi}\int_0^1 r^2\cos^2\theta\, r^2\sin^2\theta\, r\,d\theta\,dr$$

Step 8 : Integrate

$$= 9ab\,4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta\int_0^1 r^5\,dr$$

$$= 36\,ab\frac{1\times1}{4\times2}\left(\frac{\pi}{2}\right)\cdot\left(\frac{r^6}{6}\right)_0^1 = \frac{36\,ab\pi}{4\times2\times2\times6}$$

$$= \frac{3\pi\,ab}{8} \qquad \text{... (1)}$$

Step 9 :

Now $\quad \overline{F}\cdot d\overline{r} = 2ydx+3xdy-z^2\,dz$

Where 'C' is the boundary of $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$ in $z=0$ plane.

Step 10 : Take parametric equations

$x = a\cos^3\theta$, $y = b\sin^3\theta$, $z=0$

$dx = -3a\cos^2\theta\sin\theta\,d\theta$

$dy = 3b\sin^2\theta\cos\theta\,d\theta$, $dz=0$

Step 11 :

$$\int\limits_C \overline{F}\cdot d\overline{r} = \int_0^{2\pi} 2b\sin^3\theta\left(-3a\cos^2\theta\sin\theta\,d\theta\right)+3a\cos^3\theta\left(3b\sin^2\theta\cos\theta\,d\theta\right)$$

$$= -6ab\times4\int_0^{\pi/2}\sin^4\theta\ \cos^2\theta\,d\theta+9ab\times4\int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta$$

Step 12 : Using reduction formula

$$= -24\,ab\frac{1\times 3\times 1}{6\times 4\times 2}\left(\frac{\pi}{2}\right)+36\,ab\frac{3\times 1\times 1}{6\times 4\times 2}\left(\frac{\pi}{2}\right)$$

$$= \pi ab\left(-\frac{3}{4}+\frac{9}{8}\right)=\pi ab\left(\frac{-6+9}{8}\right)$$

$$= \frac{3\pi ab}{8} \qquad \text{... (2)}$$

Type 4

Example 13.54 : *Verify Stoke's theorem $\overline{F}=(x+y)\hat{i}+(2x-z)\hat{j}+(y+z)\hat{k}$ over Δ ABC. Cut off plane $3x+2y+z=6$ by co-ordinate planes.*

Solution : Step 1 : We have

$$\overline{F} = (x+y)\,\hat{i}+(2x-z)\,\hat{j}+(y+z)\,\hat{k}$$

Step 2 : Consider

$$\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$\nabla\times\overline{F} = (1-(-1))\,\hat{i}-\hat{j}\,(0-0)+\hat{k}(2-1)$$

$$\nabla\times\overline{F} = 2\hat{i}-0\hat{j}+\hat{k}$$

Step 3 : L.. $\phi = 3x + 2y + z - 6$

$$\nabla\phi = 3\hat{i}+2\hat{j}+\hat{k}$$

$$|\nabla\phi| = \sqrt{9+4+1}$$

$$= \sqrt{14}$$

Step 4 :

$$\hat{N} = \frac{\nabla\phi}{|\nabla\phi|}=\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{4}}$$

Step 5 : $(\nabla\times\overline{F})\cdot\hat{N} = 2\hat{i}-0\hat{j}+\hat{k}=\dfrac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$

$$= \frac{6+1}{\sqrt{14}}=\frac{7}{\sqrt{14}}$$

Step 6 :

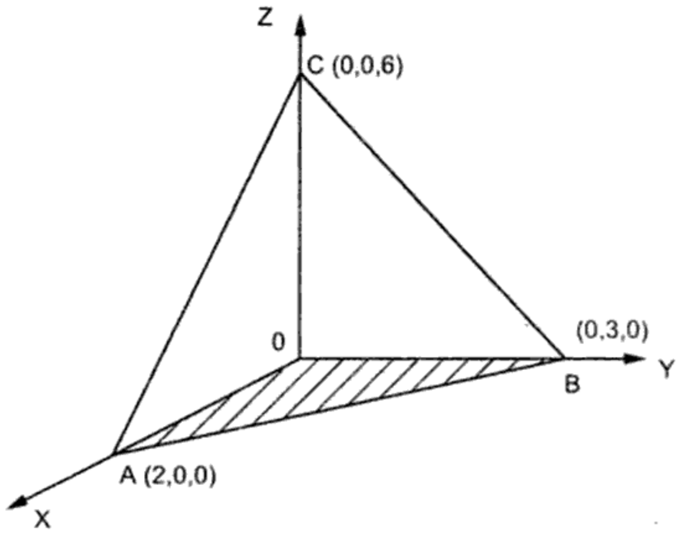


Fig. 13.35

Step 7 : The projection ABC on x-o-y plane is Δ in OAB projection formula

$$ds = \frac{dx\,dy}{|\hat{N}\cdot\hat{k}|} = \frac{dx\,dy}{|1/\sqrt{14}|}$$

$$= \iint (\nabla \times \overline{F})\cdot\hat{N}\,ds$$

$$= \iint \frac{7}{\sqrt{14}}\cdot\frac{dx\,dy}{\frac{1}{\sqrt{14}}}$$

$$= \iint 7\,dx\,dy$$

$$= 7 \text{ (area of triangle)}$$

$$= 7\cdot\frac{1}{2}\cdot 3\cdot 2 = 21$$

Step 8 :

$$\overline{F}\cdot d\overline{r} = (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$$

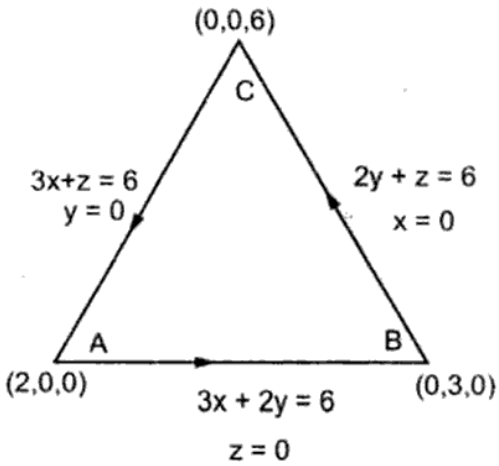


Fig. 13.36

Step 2 :

$$\iint_{S_1} \nabla \times \overline{F} \cdot \hat{N}\, ds = \int_C \overline{F} \cdot d\,\overline{r} \quad \text{... (1)}$$

and $$\iint_{S_2} \nabla \times \overline{F} \cdot \hat{N}\, ds = -\int_C \overline{F} \cdot d\,\overline{r} \quad \text{... (2)}$$

Step 3 : (As the surfaces S_1 and S_2 are oppositely oriented, curves C_1 and C_2 enclosing open surfaces S_1 and S_2 will have opposite orientation).

Adding (1) and (2), we have

$$\iint_{S_1} \nabla \times \overline{F} \cdot \hat{N}\, ds + \iint_{S_2} \nabla \times \overline{F} \cdot \hat{N}\, ds = 0$$

$$\iint_{S_1 \cup S_2} \nabla \times \overline{F} \cdot \hat{N}\, ds = 0$$

$$\iint_{S} \nabla \times \overline{F} \cdot \hat{N}\, ds = 0 \quad (\because\ S = S_1 \cup S_2)$$

Similar argument holds for any other closed surface.

Example 13.59 : *If* ψ *is scalar point function of x and z and* $\overline{F} = \frac{\partial \psi}{\partial z}\hat{i} - \frac{\partial \psi}{\partial x}\hat{k}$, *and* $\nabla^2 \psi = 0$, *then show that* $\int_C \overline{F} \cdot d\overline{r} = 0$.

Solution : Step 1 : We have

$$\nabla^2 \psi = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0$$

Step 2 :

Since ψ is independent of y, giving $\frac{\partial^2 \psi}{\partial y^2} = 0$

$$\therefore \text{ We have } \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \quad \text{... (1)}$$

Step 3 : Now by Stoke's theorem

$$\int_C \overline{F} \cdot d\overline{r} = \iint_S \left(\nabla \times \overline{F}\right) d\overline{s} \quad \text{... (2)}$$

Example 13.61 : *Show by using Stoke's theorem* $\oint_C y\,dx + z\,dy + x\,dz = -\iint_S (\cos\alpha + \cos\beta + \cos\gamma)\,ds$ *where* α, β, γ *are the angles made by the normal to the surface S with X, Y, Z axes respectively.*

Solution : Step 1 : Given

$$\oint_C y\,dx + z\,dy + x\,dz = \oint_C \overline{F}\cdot d\,\overline{r} \qquad \text{... (1)}$$

$$\therefore \qquad \overline{F} = y\,\hat{i} + z\,\hat{j} + x\,\hat{k}$$

Step 2 : Now, by Stoke's theorem we have,

$$\oint_C \overline{F}\cdot d\,\overline{r} = \iint_S \nabla\times\overline{F}\cdot\hat{N}\,ds \qquad \text{... (2)}$$

where S is the open surface bounded by curve C and $\hat{N}$ is unit outward normal vector to S.

Step 3 : Now

$$\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= i(0-1) - j(1-0) + k(0-1)$$

$$\therefore \qquad \nabla\times\overline{F} = -i - j - k \qquad \text{... (3)}$$

Step 4 : Since the normal to the surface S makes angles of α, β and γ with the X, Y, Z axes respectively.

$\therefore$ dcs of the normal are : $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$

$\therefore$ Unit vector in the direction of the normal is

$$\hat{N} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k} \qquad \text{... standard result}$$

$$= (\cos\alpha)\,\hat{i} + (\cos\beta)\,\hat{j} + (\cos\gamma)\,\hat{k}$$

Step 5 :

$$\therefore \qquad (\nabla\times\overline{F})\cdot\hat{N} = \left(-\hat{i} - \hat{j} - \hat{k}\right)\cdot\left[(\cos\alpha)\,\hat{i} + (\cos\beta)\,\hat{j} + (\cos\gamma)\,\hat{k}\right] \qquad \text{... using (3)}$$

$$= -(\cos\alpha + \cos\beta + \cos\gamma)$$

Step 6 : Substituting in (2)

$$\oint_C \overline{F}\cdot d\,\overline{r} = \iint_S -(\cos\alpha + \cos\beta + \cos\gamma)\,ds$$

i.e. $$\oint_C y\,dx + z\,dy + x\,dz = -\iint_S (\cos\alpha + \cos\beta + \cos\gamma)\,ds$$

Example 13.62 : *Verify Stoke's theorem for the function* $\bar{f}=(x-y-z)\hat{i}+(y-z-x)\hat{j}+(z-x-y)\hat{k}$ *over the unclosed surface of the cylinder,* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *bounded by the plane z = h and open at the end z = 0.* **[May-2002]**

Solution : We have to verify Stoke's theorem

i.e. $\iint_S \nabla\times\bar{F}\cdot\hat{n}\,ds = \oint_C \bar{F}\cdot d\bar{r}$.

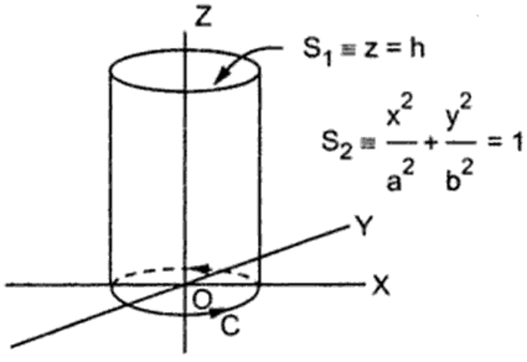


Fig. 13.37

Step 1 : It is enough to verify Stoke's theorem over the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in XOY plane.

Step 2 : Consider

$$\bar{F} = (x-y-z)\,\hat{i}+(y-z-x)\,\hat{j}+(z-x-y)\,\hat{k}$$

$$\therefore \quad \nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y-z & y-z-x & z-x-y \end{vmatrix}$$

$$= \hat{i}\,(-1+1) - \hat{j}\,(-1+1) + \hat{k}\,(-1+1)$$

$$\therefore \nabla\times\bar{F} = \bar{0}$$

$$\therefore \quad \text{LHS} = \iint_S \nabla\times\bar{F}\cdot\hat{N}\,ds = 0 \qquad \dots (1)$$

Step 3 : Now, $\int_C \bar{F}\cdot d\bar{r} = \int_C (x-y-z)\,dx+(y-z-x)\,dy+(z-x-y)\,dz$ and on the ellipse C.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$$

$$\therefore \quad dz = 0$$

$$\therefore \quad \int_C \bar{F}\cdot d\bar{r} = \int_C (x-y)\,dx+(y-x)\,dy \qquad \dots (2)$$

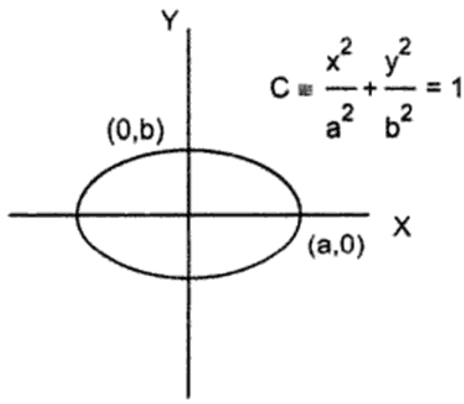


Fig. 13.38

Step 4 : The parametric equations for ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are

$$x = a\cos\theta,\ y = b\sin\theta$$

$$\therefore \quad dx = -a\sin\theta\,d\theta,\ \ dy = b\cos\theta\,d\theta$$

$\therefore$ RHS $= -16\pi$ ... (5)

$\therefore$ LHS $=$ RHS i.e. Stoke' Theorem is verified.

Exercise 13.3

1. *Verify Stoke's theorem for* $\overline{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ *over the region bounded by* $x = \pm a,\ y = 0, y = b.$

[Ans. : $4ab^2$]

2. *Verify Stoke's theorem for* $\overline{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$ *over the surface of hemisphere* $x^2 + y^2 + z^2 = a^2$ *above XOY plane.* **[Ans. = 0]**
3. *Verify Stoke's theorem for* $\overline{F} = (x^2 + y - 4)\hat{i} + 3xy\hat{j} + (2xz + z^2)\hat{k}$ *over the surface of hemisphere* $x^2 + y^2 + z^2 = a^2$ *above XOY plane.* **[Ans. : $-\pi a^2$]**
4. *Verify Stoke's theorem for* $\overline{F} = (2x - y)\hat{i} - yz\hat{j} - y^2z\hat{k}$ *over the surface of hemisphere* $x^2 + y^2 + z^2 = 1,\ z \geq 0.$ **[Ans. : π]**
5. *Verify Stoke's theorem for* $\overline{F} = (x - y - z)\hat{i} + (y - z - x)\hat{j} + (z - x - y)\hat{k}$ *over the cylinder* $x^2 + y^2 = a^2$ *bounded by* $z = h$ *open at* $z = 0.$ **[Ans. : 0]**
6. *Verify Stoke's theorem for* $\overline{F} = (2y + z)\hat{i} + (x - z)\hat{j} + (y - x)\hat{k}$ *over the triangle ABC cut off from plane* $x + y + z = 1$ *by the co-ordinate planes.* **[Ans. : $\frac{3}{2}$]**
7. *Evaluate* $\iint (\nabla \times \overline{F}) \cdot \hat{N}\, ds$ *where* $\overline{F} = (x - y)\hat{i} + (x^2 + yz)\hat{j} - 3xy^2\hat{k}$ *and S is the surface of the cone* $z = 4 - \sqrt{x^2 + y^2}$ *above XOY plane.* **[Ans. : 16π]**
8. *Evaluate* $\iint (\nabla \times \overline{F}) \cdot d\overline{s}$ *where* $\overline{F} = (x^2 + y - 4)\hat{i} + 3xy\hat{j} + (2xz + z^2)\hat{k}$ *and S is the surface* $z = 9 - x^2 - y^2,\ z \geq 0.$ **[Ans. : -9π]**
9. *Use Stoke's theorem for* $\overline{F} = y^3\hat{i}$ *to evaluate* $\iint (\nabla \times \overline{F}) \cdot d\overline{s}$ *where S is the unclosed surface of a cylinder* $x^2 + y^2 = 9$ *bounded by* $z = 2$ *and open at* $z = 0.$ **[Ans. : $-\frac{3^5\pi}{4}$]**
10. *Using Stoke's theorem prove that curl grad* $\phi = 0.$
11. *Verify Stoke's theorem for* $\overline{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$ *taken around the rectangle bounded by the lines* $x = \pm 2,\ y = 0,\ y = 1.$ **[Ans. : – 8]**
12. *Verify Stoke's theorem for* $\overline{F} = xz\hat{i} - y\hat{j} + x^2y\hat{k}$ *where S is the surface of the region bounded by* $y = 0,\ z = 0,\ 3x + y + 3z = 6$ *which is not included in YZ plane.*
13. *Verify Stoke's theorem for* $\overline{F} = x^2\hat{i} - xy\hat{j}$ *over the square bounded by* $x = 0,\ y = 0,\ x = a,\ y = a$ *in XOY plane.* **[Ans. : $-\frac{a^3}{2}$]**
14. *Using Stoke's theorem deduce that the surface integral of curl* $\overline{F}$ *taken over a closed surface is zero.*
15. *Verify stokes theorem for* $\overline{F} = (y - z + z)\hat{i} + (yz + 4)\hat{j} - xz\hat{k}$*, where S is the surface of the cube* $x = 0,\ y = 0,\ x = 2,\ y = 2,\ z = 2$ *above the xy-plane.* **[Ans. : Common value = – 4]**

16. *Verify the Stoke's theorem for* $\overline{F} = xz\hat{i} - y\hat{j} + x^2 y\hat{k}$, *where S is the surface of the region bounded by* $x = 0$, $y = 0$, $z = 0$, $2x + y + 2z = 8$ *which is not included in xz-plane.*

[Ans. : Common value = $\frac{32}{3}$**]**

17. *Verify Stoke's theorem for* $\overline{F} = \left(x^2 + y - 4\right)\hat{i} + 3yx\hat{j} + \left(2xz + z^2\right)\hat{k}$ *and S is the surface of the sphere* $x^2 + y^2 + z^2 = 16$ *above xoy-plane.* **[Ans. : Common value =** -16π**]**

18. *Verify Stoke's theorem for* $\overline{F} = (2x - y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}$ *where S is the upper half of the sphere* $x^2 + y^2 + z^2 = 1$ *and c is the circular boundary on* $z = 0$, *plane.* **[Ans. : Common value =** π**]**

19. *Verify Stoke's theorem for* $\overline{F} = \left(x^2 - y^2\right)\hat{i} + 2xy\hat{j}$ *in the rectangular lamina* $x = 0$, $y = 0$, $x = a$, $y = b$. **[Ans. : Common value =** $2ab^2$**]**

20. *Evaluate by Stoke's theorem* $\oint\limits_C \left(yz\,dx + zx\,dy + xy\,dz\right)$ *where C is the curve* $x^2 + y^2 = 1$, $z = y^2$. **[Ans. : 0]**

21. *Evaluate by Stoke's theorem* $\oint\limits_C \left(\sin z\,dz - \cos x\,dy + \sin y\,dx\right)$ *where C is the boundary of rectangle* $0 \le x \le \pi$, $0 \le y \le 1$, $z = 3$.

22. *Use Stoke's theorem to evaluate* $\oint\limits_C \left(4y\hat{i} + 2z\hat{j} + 6y\hat{k}\right)\cdot d\overline{r}$ *where C is*

    i) $x^2 + y^2 + z^2 = 2z$, $x = z - 1$ **[Ans. :** $4\pi\sqrt{2}$**]**

    ii) $x^2 + y^2 + z^2 = 6z$, $z = x + 3$ **[Ans. :** $36\pi\sqrt{2}$**]**

23. *Evaluate* $\int\limits_C y\,dx + z\,dy + x\,dz$, *C being intersection of* $x^2 + y^2 + z^2 = a^2$, $x + z = a$. **[Ans. :** $-\frac{\pi C}{\sqrt{2}}$**]**

24. *Evaluate* $\int\limits_C \left(e^z\,dz + 2y\,dy - dz\right)$ *by using Stoke's theorem, where C is the curve* $z^2 + y^2 = 4$, $z = 2$ **[Ans. : 0]**

25. *Evaluate* $\iint\limits_S \left(\nabla \times \overline{F}\right)\cdot d\overline{s}$ *where* $\overline{F} = (x - z)\hat{i} + \left(x^3 + yz\right)\hat{j} - 3xy^2\hat{k}$ *and S is the surface of the cone* $z = 2 - \sqrt{x^2 + y^2}$ *above xy-plane.*

26. *If* $\overline{F} = 2yz\hat{i} - (x + 3y - 2)\hat{j} + \left(x^2 + z\right)\hat{k}$. *Evaluate* $\iint\limits_S \left(\nabla \times \overline{F}\right)\cdot d\overline{s}$ *over the surface of intersection of cylinders.* $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$, *which is included in the first octant.* **[Ans. :** $-\frac{a^2}{12}(3\pi + 8a)$**]**

27. *Use Stoke's theorem to evaluate* $\oint\limits_C \left[(x + 2y)\,dx + (x - z)\,dy + (y - z)\,dz\right]$ *where C is the boundary of triangle with verifies* $(2, 0, 0)$, $(0, 3, 0)$ *and* $(0, 0, 6)$. **[Ans. : 15]**

Similarly, we have

$$\iiint\limits_V \frac{\partial F_2}{\partial y} dx\, dy\, dz = \iint\limits_S F_2\, \hat{n}\cdot\hat{k}\, ds \quad \text{... (5)}$$

$$\iiint\limits_V \frac{\partial F_1}{\partial x} dx\, dy\, dz = \iint\limits_S F_1\, \hat{n}\cdot\hat{k}\, ds \quad \text{... (6)}$$

Adding (4), (5) and (6), we get

$$\iiint\limits_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$$

$$= \oiint\limits_S (F_1\, \hat{i}+F_2\, \hat{j}+F_3\, \hat{k})\cdot\hat{n}\, ds$$

i.e. $$\iiint\limits_V \nabla\cdot\bar{F}\, dv = \oiint\limits_S \bar{F}\cdot\hat{n}\, ds \quad \text{... (7)}$$

Secondly, consider the general volume V. Assume that it can be split up into a finite number of sub-regions each of which is met by a line parallel to any axis is only two points applying (1) to each of these sub-regions and adding the results.

13.15 Divergence Theorem in Component

Let $\hat{n} = \cos\alpha\, \hat{i}+\cos\beta\, \hat{j}+\cos\gamma\, \hat{k}$, equation (7) taken the form

$$\iiint\limits_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$$

$$= \iint\limits_S (F_1\, \cos\alpha+F_2\, \cos\beta+F_3\, \cos\gamma)\, ds$$

the above equation may also be written as

$$\iiint\limits_V \nabla\cdot\bar{F}\, dv = \oiint\limits_S \left(F_1\, dy\, dz+F_2\, dz\, dx+F_3\, dx\, dy\right)$$

13.16 Physical Interpretation of Divergence Theorem

Let $\bar{F}$ = Velocity $\bar{V}$ at any point of a moving fluid.

$\therefore$ Volume of fluid crossing ds in Δ t seconds

= Volume contained in cylinder of base ds and slant height $\bar{V}\Delta t$

$= (\bar{V}\Delta t)\cdot\hat{n}\, ds = \bar{V}\cdot\hat{n}\, ds\, \Delta t$

Then, volume per second of fluid crossing

$= \bar{V}\cdot\hat{n}\, ds$

Step 3 :

$$\text{Volume Integral} = \iiint \nabla \cdot V \, dv$$

$$= \int_0^1 \int_0^1 \int_0^1 (4z - y) \, dx \, dy \, dz$$

$$= \int_0^1 \int_0^1 (2z^2 - yz)_0^1 \, dx \, dy$$

$$= \int_0^1 \int_0^1 [2 - y] \, dx \, dy$$

$$= \int_0^1 (2x - xy)_0^1 \, dy$$

$$= \int_0^1 (2 - y) \, dy$$

$$= \left[2y - \frac{y^2}{2}\right]_0^1$$

$$= 2 - \frac{1}{2}$$

$$= \frac{3}{2} \qquad \text{... (i)}$$

Step 4 : To evaluate the surface integral $\iint\limits_S \overline{F} \cdot \hat{N} \, ds$.

Consider all the six faces.

Over S_1	OALB	$\hat{N} = -\hat{k}$	ds = dx dy	z = 0	$\overline{F} \cdot \hat{N} = 0$
Over S_2	NPMC	$\hat{N} = \hat{k}$	ds = dx dy	z = 1	$\overline{F} \cdot \hat{N} = y$
Over S_3	OBMC	$\hat{N} = -\hat{i}$	ds = dy dz	x = 0	$\overline{F} \cdot \hat{N} = 0$
Over S_4	ALPN	$\hat{N} = \hat{i}$	ds = dy dz	x = 1	$\overline{F} \cdot \hat{N} = 4z$
Over S_5	OCNA	$\hat{N} = -\hat{j}$	ds = dx dz	y = 0	$\overline{F} \cdot \hat{N} = 0$
Over S_6	LBMP	$\hat{N} = \hat{j}$	ds = dx dz	y = 1	$\overline{F} \cdot \hat{N} = -1$

Step 5 : Consider the surface integral

$$\iint\limits_S \overline{F} \cdot \hat{N} \, ds = \iint\limits_{S_1} + \iint\limits_{S_2} + \iint\limits_{S_3} + \iint\limits_{S_4} + \iint\limits_{S_5} + \iint\limits_{S_6}$$

Step 6 : Substitute and put the limits

$$\iint_S \overline{F}\cdot\hat{N}\,ds = 0+\int_0^1\int_0^1 y\,dx\,dy+0+\int_0^1\int_0^1 4z\,dy\,dz+0+\int_0^1\int_0^1 -1\,dx\,dz$$

$$= [x]_0^1\left[\frac{y^2}{2}\right]_0^1+4\,[y]_0^1\left[\frac{z^2}{2}\right]_0^1-1\,[x]_0^1\,[z]_0^1$$

$$= [1]\left[\frac{1}{2}\right]+4[1]\left[\frac{1}{2}\right]-[1][1]$$

$$= \frac{1}{2}+2-1$$

$$= \frac{3}{2} \qquad \text{... (ii)}$$

From (i) and (ii) Gauss Divergence theorem is verified.

Example 13.65 : *Verify G.D.T. for* $\overline{F} = (x^2-yz)i+(y^2-zx)j+(z^2-xy)k$ *over the cube whose side is 'a'.*

Solution : Step 1 : Consider $\overline{F}$ and find $\nabla\cdot\overline{F}$

$$\nabla\cdot\overline{F} = \frac{\partial}{\partial x}(x^2-yz)+\frac{\partial}{\partial y}(y^2-zx)+\frac{\partial}{\partial z}(z^2-xy)$$

$$= 2x+2y+2z$$

Step 2 : Consider $\iiint\left(\nabla\cdot\overline{F}\right)dv$

$$= \iiint 2(x+y+z)\,dx\,dy\,dz$$

over the volume of cube whose side is 'a'.

Step 3 : Consider $\iiint 2x\,dx\,dy\,dz$

over the cube whose side is a.

$$= \int_0^a\int_0^a\int_0^a 2x\,dx\,dy\,dz$$

$$= 2\left[\frac{x^2}{2}\right]_0^a [y]_0^a\ [z]_0^a$$

$$= a^4$$

Step 4 : Similarly $\iiint 2y\,dx\,dy\,dz = a^4$

$$+\int_0^a \int_0^a xz\,dx\,dz + \int_0^a \int_0^a \left(a^2 - xz\right) dx\,dz$$

$$= \int_0^a \int_0^a a^2\,dx\,dy + \int_0^a \int_0^a a^2\,dy\,dz + \int_0^a \int_0^a a^2\,dx\,dz$$

$$= a^2 [x]_0^a [y]_0^a + a^2 [y]_0^a [z]_0^a + a^2 [x]_0^a [z]_0^a$$

$$= a^2 \cdot a \cdot a + a^2 \cdot a \cdot a + a^2 \cdot a \cdot a$$

$$\therefore \quad \iint_S \overline{F} \cdot \hat{N}\,ds = 3a^4 \qquad \text{... (ii)}$$

∴ From (i) and (ii)

$$\iiint_V \left(\nabla \cdot \overline{F}\right) dv = \iint_S \overline{F} \cdot \hat{N}\,ds$$

∴ Gauss divergence theorem is verified.

Example 13.66 : *Verify divergence theorem for $\overline{F} = 2x^2yi - y^2\hat{j} + 4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$ and $x = 2$.*

Solution : Divergence theorem state that

$$\oiint_S \overline{F} \cdot \hat{n}\,ds = \iiint_V \nabla \cdot \overline{F}\,dv$$

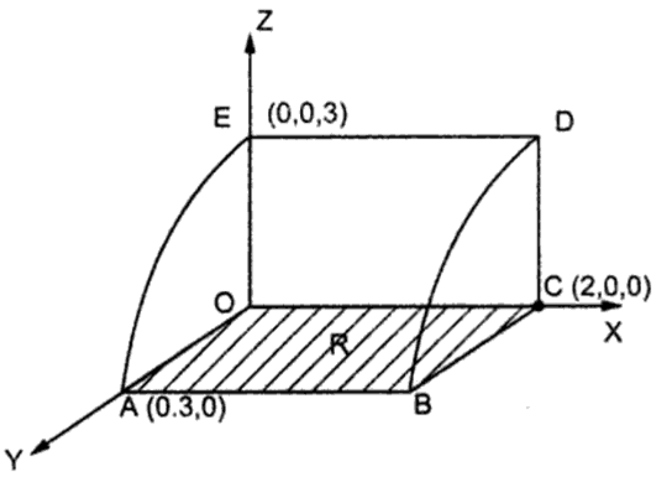


Fig. 13.44

Step 1 : To evaluate

$$\iiint_V \nabla \cdot \overline{F}\,dv$$

Step 2 : Consider

$$\nabla \cdot \overline{F} = 4xy - 2y + 8xz$$

Step 3 : For evaluating the triple integral firstly put the limits for z and integrate w.r.t.z.

$$\therefore \quad \iiint_V \nabla \cdot \overline{F}\,dv = \iint_R \int_0^{\sqrt{9-y^2}} (4xy - 2y + 8xz)\,dz\,dy\,dx$$

$$= \iint_R \left[(4xy - 2y)\,z + 4xz^2\right]_0^{\sqrt{9-y^2}} dy$$

$$= \iint_R (4xy-2y)\sqrt{9-y^2} + 4x\sqrt{(9-y^2)^2}\ dy\ dx$$

$$= \iint_R \left[(4xy-2y)\sqrt{9-yz} + 4x\,(9-y^2)\right] dx\ dy$$

Step 4 : Now put the limits for x and y to integrate this integral over the region R.

$$= \int_0^2 \int_0^3 (2x-1)\sqrt{9-y^2}\ (-2y)\ dy + 4\int_0^2 x\ dx \int_0^3 (9-y^2)\ dy$$

$$= -\int_0^2 (2x-1)\ dx \cdot \int_0^3 (9-y^2)\ (-2y)\ dy + 4\int_0^2 x\ dx \cdot \int_0^3 (9-y^2)\ dy$$

$$= -\left[\frac{(2x-1)^2}{4}\right]_0^2 \cdot \left[\frac{(9-y^2)^{3/2}}{3/2}\right]_0^3 + 4\left(\frac{x^2}{2}\right)_0^2 \left[9y - \frac{y^3}{3}\right]_0^3$$

$$= -\frac{1}{6}[9-1][0-27] + 8(27-9)$$

$$= 36 + 18 \times 8$$

$$= 36 + 144$$

$$= 180$$

Step 5 : To evaluate the surface integral consider all the surfaces.

Here $\quad S = OABC + BCD + OCDE + OAE + ABDE$

$$= S_1 + S_2 + S_3 + S_4 + S_5$$

Step 6 : To find the surface integral over each face.

Over S_1 : $\hat{n} = -\hat{k}$, $z = 0$, $ds = dx\ dy$

$$\overline{F}\cdot\hat{n} = 2xz^2 = 0$$

$$\therefore \quad \iint_{S_1} \overline{F}\cdot\hat{n}\ ds = 0$$

Over S_2 : $\hat{n} = \hat{i}$, $x = 2$, $ds = dy\ dz$

$$\overline{F}\cdot\hat{n} = 2x^2y = 8y$$

$$\iint_{S_2} \overline{F}\cdot\hat{n}\ ds = \int_0^{\pi/2} \int_0^3 8\ r\cos\theta\ r\ dr\ d\theta$$

$$= 8\int_0^{\pi/2} \cos\theta\ d\theta \int_0^3 r^2 dr$$

$$= 8\ (1)\ 9 = 72$$

$$\overline{F}\cdot\hat{n} = \frac{1}{a}[xy+yx] = \frac{2xy}{a}$$

$$\therefore \quad \iint_{S_2} \overline{F}\cdot\hat{n}\,ds = \int_0^{2\pi}\int_0^b \frac{2\,a\cos\phi\cdot a\sin\phi}{a}\,d\phi\,dz$$

$$= 2a\int_0^{2\pi}\cos\phi\sin\phi\,d\phi\int_0^b dz = 0$$

Step 7 : Adding all the integrals we get

$$\therefore \quad \iint_S \overline{F}\cdot\hat{n}\,ds = \pi a^2 b^2 \qquad \text{... (2)}$$

$\therefore$ From (1) and (2), divergence theorem is verified.

Example 13.68 : *Verify GDT $\overline{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$ over the closed region above XOY plane bounded by cone $z^2 = x^2+y^2$ and the plane $z = 4$.*

Solution : Step 1 : Consider $\overline{F}$ and find $\nabla\cdot\overline{F}$

$$\nabla\cdot\overline{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}\left(xyz^2\right) + \frac{\partial}{\partial z}(3z)$$

$$= 4z + xz^2 + 3$$

Step 2 : Consider $\iiint\left(\nabla\cdot\overline{F}\right)dv$ over the volume of cone $z^2 = x^2+y^2$, $z = 4$ above XOY plane.

Step 3 : Draw the Fig. 13.46.

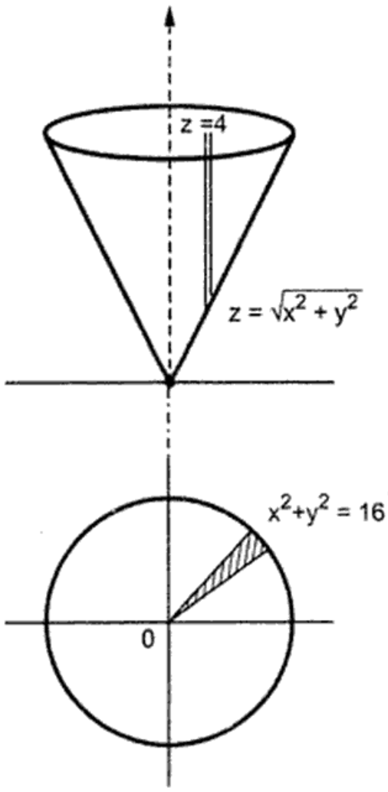


Fig. 13.46

Thus $\iint_{S_3} \overline{F} \cdot \hat{N} \, ds = \iint 2 \, ds$

$= 2 \text{ (area)}$

$= 2 \left[2\pi \cdot 2 \cdot 2\right]$

$= 16\pi$

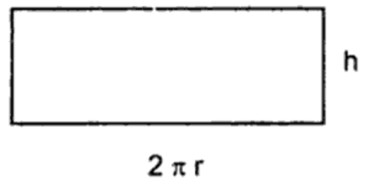


Fig. 13.50

Step 5 : Adding all the integrals

As $\iint_{S} \overline{F} \cdot \hat{N} \, ds = \iint_{S_1} \overline{F} \cdot \hat{N} \, ds + \iint_{S_2} \overline{F} \cdot \hat{N} \, ds + \iint_{S_3} \overline{F} \cdot \hat{N} \, ds$

$= 0 + 16\pi + 16\pi$

$= 32\pi$

$\therefore$ G.D.T. is verified.

Example 13.70 : *Verify Gauss Div. theorem* $\overline{F} = (x + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ *over the volume of tetrahedron bounded by co-ordinate planes and plane* $2x + y + 2z = 6$.

Solution : Step 1 : Consider $\overline{F}$ and find $\nabla \cdot \overline{F}$

$$\nabla \cdot \overline{F} = 1 - 0 + 2y$$

$$= 1 + 2y$$

Step 2 : Consider $\iiint \nabla \cdot \overline{F} \, dv$

$$= \iiint 1 + 2y \, dx \, dy \, dz$$

over the volume bounded by $x = 0$, $y = 0$, $z = 0$, $2x + y + 2z = 6$

Now consider $2x + y + 2z = 6$

$$\frac{x}{3} + \frac{y}{6} + \frac{z}{3} = 1$$

$\therefore$ Put $x = \frac{x}{3}$, $y = \frac{y}{6}$, $z = \frac{z}{3}$ then

$dx \, dy \, dz = 3 \cdot 6 \cdot 3 \, dx \, dy \, dz$ and the volume is bounded by $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

Thus the integral becomes

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1 + 2.6\,y) \; 3 \cdot 6 \cdot 3 \; dz \, dy \, dx$$

$$= 54 \int_0^1 \int_0^{1-x} (1 + 12\,y)(1 - x - y) \, dy \, dx$$

Put $1 - x = h$ for convenience

$$= 54\int_0^1 \int_0^h (1+12\,y)(h-y)\,dy\,dx$$

$$= 54\int_0^1 \int_0^h \left[(h-y)+12(h\,y-y^2)\right] dy\,dx$$

$$= 54\int_0^1 \left\{\frac{(h-y)^2}{-2} + 12\left[h\frac{y^2}{2}-\frac{y^3}{3}\right]\right\}_0^h dx$$

$$= 54\int_0^1 \left\{0+12\left[\frac{h^3}{2}-\frac{h^3}{3}\right]\right\} - \left\{\frac{h^2}{-2}+0\right\} dx$$

Put $h = 1 - x$

$$= 54\int_0^1 2(1-x)^3 + \frac{(1-x)^2}{2}\,dx$$

$$= 54\left\{\frac{2(1-x)^4}{-4} + \frac{(1-x)^3}{-3.2}\right\}_0^1$$

$$= 54\left\{[0-0]-\left[-\frac{2}{4}-\frac{1}{6}\right]\right\}$$

$$= 54\left\{\frac{1}{2}+\frac{1}{6}\right\}$$

$$= 54\cdot\frac{4}{6}$$

$$= 36$$

Step 3 : For double integral draw the Fig. 13.51.

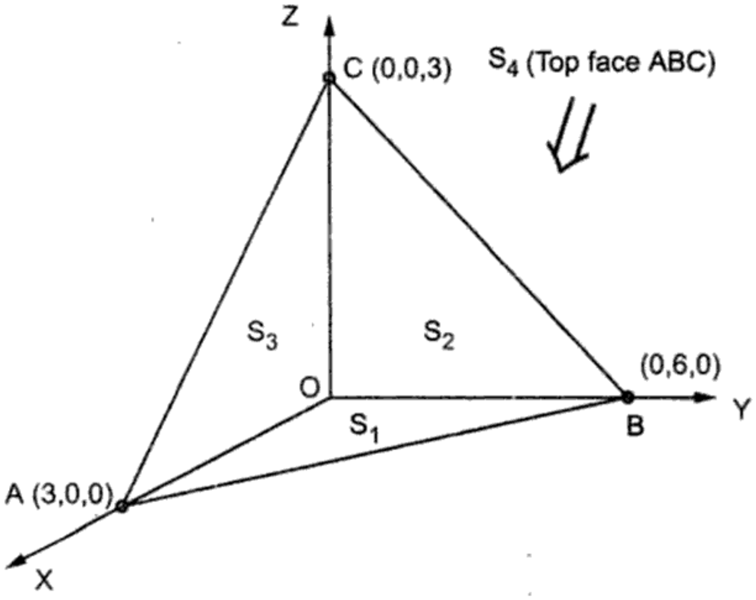


Fig. 13.51

$$= 3\left[-\cos\theta\right]_0^{\pi} (\phi)_0^{2\pi} \left(\frac{r^5}{5}\right)_0^4$$

$$= 3(2)\cdot 2\pi\cdot\frac{4^5}{5}$$

$$= \frac{3\pi\, 4^6}{5}$$

Example 13.75 : *Evaluate* $\iint (lx^2+my^2+nz^2)\,ds$ *where S is the surface of sphere* $(x-a)^2+(y-b)^2+(z-c)^2 = R^2$ *where l, m, n are direction cosines of outword normal to the surface.*

Solution : Step 1 : Given

$$\hat{N} = l\,\hat{i}+m\,\hat{j}+n\,\hat{k}$$

and $\quad \overline{F}\cdot\hat{N} = l\,x^2+m\,y^2+n\,z^2$

$\Rightarrow \quad \overline{F} = x^2\,\hat{i}+y^2\,\hat{j}+z^2\,\hat{k}$

$\therefore \quad \nabla\cdot\overline{F} = 2x + 2y + 2z$

Step 2 : By G.D.T.

$$\iint (l\,x^2+m\,y^2+n\,z^2)\,ds = \iiint 2(x+y+z)\,dx\,dy\,dz$$

over the volume of sphere $(x-a)^2+(y-b)^2+(z-c)^2 = R^2$

Step 3 : Consider $\iiint 2x\;dx\;dy\;dz$

Step 4 : Use spherical polar co-ordinates

$$x-a = r\sin\theta\cos\phi$$

$$y-b = r\sin\theta\sin\phi$$

$$z-c = r\cos\theta$$

$$dx\,dy\,dz = r^2\sin\theta\,dr\,d\phi\,d\theta$$

Step 5 : Use standard limits for full sphere of radius R

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{R} 2(a+r\sin\theta\cos\phi)\,r^2\sin\theta\,dr\,d\phi\,d\theta$$

Step 6 : Separate the integrals

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{R} 2a\,r^2\sin\theta\,dr\,d\phi\,d\theta + \int_0^{\pi}\int_0^{2\pi}\int_0^{R} r^3\sin^2\theta\cos\phi\,dr\,d\phi\,d\theta$$

$$= \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, 2a\left(\frac{r^3}{3}\right)_0^R + \int_0^{\pi} \sin^2\theta \, d\theta \int_0^{2\pi} \cos\phi \, d\phi\left(\frac{r^4}{4}\right)_0^R$$

Step 7 : As $\int_0^{2\pi} \cos\phi \, d\phi = 0 \quad \therefore$ second integral is zero.

Thus the integral becomes

$$= (-\cos\theta)_0^{\pi}(2\pi)\cdot 2a\,\frac{R^3}{3}$$

$$= 2\cdot 2\pi\cdot 2a\,\frac{R^3}{3}$$

$$= \frac{8\pi R^3 a}{3}$$

Similarly $\iiint 2y \, dx \, dy \, dz = \dfrac{8\pi R^3 b}{3}$

$$\iiint 2z \, dx \, dy \, dz = \frac{8\pi R^3 c}{3}$$

Step 8 : Adding all the integrals we get

$$\iint \overline{F}\cdot\hat{N}\, ds = \frac{8\pi R^3}{3}(a+b+c)$$

Example 13.76 : *Use G.D.T. to evaluate* $\iint_S \overline{F}\cdot\hat{N}\, ds$ $\overline{F} = y^2 z^2 \hat{i} + x^2 z^2 \hat{j} + x^2 y^2 \hat{k}$ *where S is the surface of hemisphere* $x^2+y^2+z^2 = 9$ *above XOY plane.*

Solution : Step 1 : Consider the hemisphere

S_1 = Surface of hemi sphere

S_2 = Surface of circle

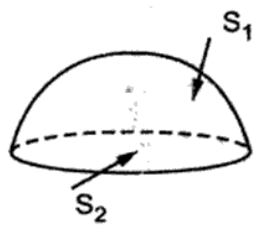


Fig. 13.52

Step 2 : $\therefore$ To find the integral over s_1 by G.D.T.

$$\iint_S \overline{F}\cdot\hat{N}\, ds = \iiint_V \nabla\cdot\overline{F}\, dv$$

Step 3 : But S is combination of S_1 and S_2

$$\therefore \iint_{S_1} \overline{F}\cdot\hat{N}\, ds + \iint_{S_2} \overline{F}\cdot\hat{N}\, ds = \iiint \nabla\cdot\overline{F}\, dv \qquad \text{... (i)}$$

Step 4 : Now $\nabla\cdot\overline{F} = \dfrac{\partial}{\partial x}(y^2 z^2) + \dfrac{\partial}{\partial y}(x^2 z^2) + \dfrac{\partial}{\partial z}(x^2 y^2)$

$$= 0 + 0 + 0$$

$$= 0$$

$$\nabla \cdot \overline{F} = 0 + 0 + 0$$
$$= 0$$

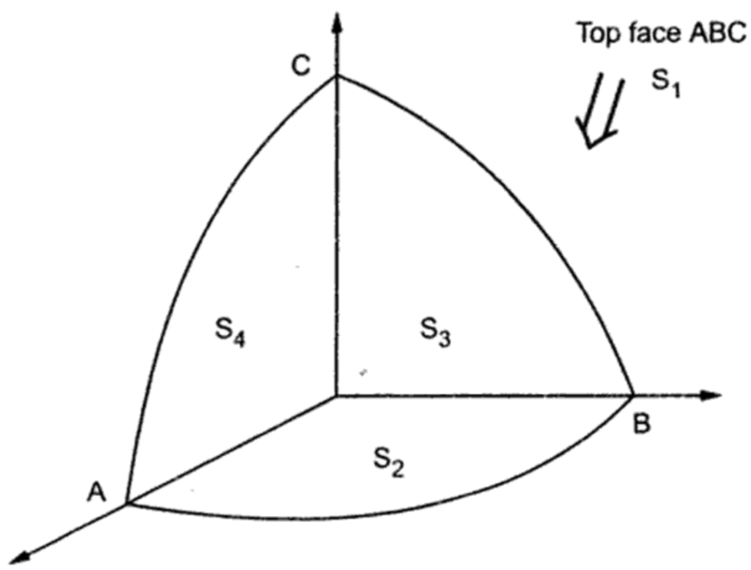


Fig. 13.53

Step 2 : Now by G.D.T.

$$\iint_S \overline{F} \cdot \hat{N}\, ds = \iiint \nabla \cdot \overline{F}\, dv$$

Step 3 : S is the combination of $S_1 + S_2 + S_3 + S_4$. Thus

$$\iint_{S_1} \overline{F} \cdot \hat{N}\, ds + \iint_{S_2} \overline{F} \cdot \hat{N}\, ds + \iint_{S_3} \overline{F} \cdot \hat{N}\, ds + \iint_{S_4} \overline{F} \cdot \hat{N}\, ds = 0$$

$$\therefore \quad \iint_{S_1} \overline{F} \cdot \hat{N}\, ds = -\left[\iint_{S_2} \overline{F} \cdot \hat{N}\, ds + \iint_{S_3} \overline{F} \cdot \hat{N}\, ds + \iint_{S_4} \overline{F} \cdot \hat{N}\, ds\right] \quad \text{... (1)}$$

Step 4 : Now consider S_2

For S_2 $\quad \hat{N} = -k, \quad z = 0, \quad ds = dx\, dy$

$$\overline{F} \cdot \hat{N} = -x^2 y^2$$

Step 5 : Consider the surface integral over S_2

$$\therefore \quad \iint_{S_2} \overline{F} \cdot \hat{N}\, ds = \iint -x^2 y^2\, dx\, dy$$

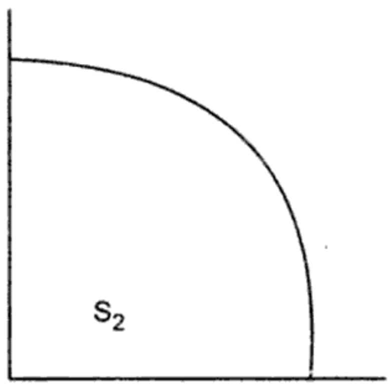


Fig. 13.54

Step 6 : Over positive quadrant of $x^2 + y^2 = a^2$

Put $x = r \cos \theta, \quad y = r \sin \theta, \quad dx\, dy = r\, dr\, d\theta$
and substitute the standard limits for positive quadrant of circle with radius a.

For S_2 $\quad \hat{N} = -k, \quad ds = dx\,dy, \quad z = 0, \quad \bar{F}\cdot\hat{N} = -xy$

$$\therefore \quad \iint_{S_2} \bar{F}\cdot\hat{N}\,ds = \iint -xy\,dx\,dy$$

Step 4 : Over the quadrant of $x^2 + y^2 = 1$

$\therefore$ Put

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$dx\,dy = r\,dr\,d\theta \quad \text{and substitute the standard limits.}$$

$$\int_0^{\pi/2}\int_0^1 -r\cos\theta\; r\sin\theta\; r\,dr\,d\theta$$

Step 5 : Integrate w.r.t. r

$$= -\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta \left[\frac{r^4}{4}\right]_0^1$$

Step 6 : Integrate w.r.t. θ use $\int_0^{\pi/2} \sin^m\theta\cos\theta\,d\theta = \frac{1}{m+1}$

$$= -\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = -\frac{1}{8}$$

Step 7 : Similarly

$$\iint_{S_3} \bar{F}\cdot\hat{N}\,ds = -\frac{1}{8}$$

$$\iint_{S_4} \bar{F}\cdot\hat{N}\,ds = -\frac{1}{8}$$

Step 8 : Thus from equation (1)

$$\iint_{S_1} \bar{F}\cdot\hat{N}\,ds = -\left[\iint_{S_2} \bar{F}\cdot\hat{N}\,ds + \iint_{S_3} \bar{F}\cdot\hat{N}\,ds + \iint_{S_4} \bar{F}\cdot\hat{N}\,ds\right]$$

$$= \frac{3}{8}$$

Example 13.79 : *Evaluate* $\iint xz^2\,dy\,dz + \left(x^2y - z^2\right)dz\,dx + \left(2xy + y^2z\right)dx\,dy$ *over the hemisphere* $x^2 + y^2 + z^2 = 4$ *above XOY plane.*

Solution : Step 1 :

$$\bar{F} = xz^2\,\hat{i} + \left(x^2y - z^2\right)\hat{j} + \left(2xy + y^2z\right)\hat{k}$$

$\therefore \quad \nabla \cdot \overline{F} = z^2 + x^2 + y^2$

Step 2 : Now by G.D.T.

$$\iint_S \overline{F} \cdot \hat{N}\, ds = \iiint (\nabla \cdot \overline{F})\, dv$$

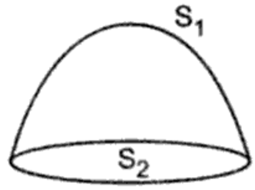


Fig. 13.56

Step 3 : The surface S is a combination of S_1 and S_2.

$$\therefore \iint_{S_1} \overline{F} \cdot \hat{N}\, ds + \iint_{S_2} \overline{F} \cdot \hat{N}\, ds = \iiint_V (\nabla \cdot \overline{F})\, dv$$

$$\iint_{S_1} \overline{F} \cdot \hat{N}\, ds = \iiint_V (\nabla \cdot \overline{F})\, dv - \iint_{S_2} \overline{F} \cdot \hat{N}\, ds \qquad \text{... (1)}$$

Step 4 : To find the volume integral

$$\iiint_V \nabla \cdot \overline{F}\, dv = \iiint_V (x^2 + y^2 + z^2)\, dx\, dy\, dz$$

Step 5 : Over the hemisphere $x^2 + y^2 + z^2 = 4$

Put

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

$$dx\, dy\, dz = r^2 \sin\theta\, dr\, d\phi\, d\theta$$

Step 6 : Substitute the standard limits for hemisphere of radius 2

$$= \int_0^{\pi/2} \int_0^{2\pi} \int_0^2 r^2\; r^2 \sin\theta\, dr\, d\phi\, d\theta$$

$$= \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{2\pi} d\phi \left(\frac{r^5}{5}\right)_0^2$$

$$= (1)(2\pi)\frac{2^5}{5} = \frac{2^6 \pi}{5}$$

Step 7 : Also for S_2 $\quad \hat{N} = -k, \quad ds = dx\, dy, \quad x^2 + y^2 = 4, \quad z = 0$

$$\therefore \quad \overline{F} \cdot \hat{N} = 2xy + y^2 z$$

$$= 2xy$$

$$\therefore \quad \iint_{S_2} \overline{F} \cdot \hat{N}\, ds = \iint 2xy\, dx\, dy$$

Step 8 : To integrate this over $x^2 + y^2 = 4$

Step 2 : Here

$$\overline{F}\cdot\hat{n} = \frac{1}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} \quad \text{(Given)}$$

Step 3 : Using $\phi = ax^2+by^2+cz^2 = 1$ find $\hat{n}$

$$\therefore \quad \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2ax\,\hat{i}+2by\,\hat{j}+2cz\,\hat{k}}{\sqrt{4a^2x^2+4b^2y^2+4c^2z^2}}$$

$$= \frac{ax\,\hat{i}+by\,\hat{j}+cz\,\hat{k}}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} \qquad \text{where } \phi = ax^2+by^2+cz^2 = 1$$

Step 4 : Choosing

$$\overline{F} = x\,\hat{i}+y\,\hat{j}+z\,\hat{k}$$

We get
$$\overline{F}\cdot\hat{n} = \frac{ax^2+by^2+cz^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

But on the surface of ellipsoid

$$ax^2+by^2+cz^2 = 1 \qquad \therefore \quad \overline{F}\cdot\hat{n} = \frac{1}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

Step 5 : As $\overline{F} = x\,\hat{i}+y\,\hat{j}+z\,\hat{k} \quad \therefore \quad \nabla\cdot\overline{F} = 3$

$$\therefore \quad \iint_S \frac{1}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}\,ds = \iiint_V \nabla\cdot\overline{F}\,dv$$

Substituting in (i) we get

$$= 3\iiint_V dv$$

$$= 3\,[\text{Volume of ellipsoid}]$$

$$= 3\cdot\frac{4}{3}\pi\cdot\frac{1}{\sqrt{a}}\cdot\frac{1}{\sqrt{b}}\cdot\frac{1}{\sqrt{c}}$$

$$= \frac{4\pi}{\sqrt{abc}}$$

Example 13.85 : *Let S be a closed surface and let $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$, be a position vector of any point in space, prove that $\oiint_S \frac{\hat{n}\cdot\bar{r}}{r^3}\,ds$ is equal to zero a) zero if O lies outside S b) 4π if O lies inside S.*

Solution : a) By the divergence theorem

$$\iint_S \overline{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\overline{F}\,dv$$

Let $$\overline{F} = \frac{\overline{r}}{r^3}$$

But $$\nabla\cdot\overline{F} = \nabla\cdot(r^{-3}\overline{r})$$

$$= \nabla r^{-3}\cdot\overline{r} + r^{-3}(\nabla\cdot\overline{r})$$

$$= -3\,r^{-5}\overline{r}\cdot\overline{r} + r^{-3}\,(3)$$

$$= 0$$

Every where within V if $r \neq 0$ in V, i.e. provided O is outside of V and thus outside of S,

$$\therefore \quad \iint_S \overline{F}\cdot ds = 0$$

b) If O is inside S, surround 'O' by a small sphere S_1 of radius a and let V_1 denote the volume enclosed between S and S_1.

$$\iint_{S+S_1} \frac{\overline{n}\cdot\overline{r}}{r^3}\,ds = \iint_S \frac{\hat{n}\cdot\overline{r}}{r^3}\,ds + \iint_S \frac{\overline{n}\cdot\overline{r}}{r^3}\,ds$$

$$= \iiint_{V_1} \nabla\cdot\frac{\overline{r}}{r^3}\,dv$$

$$= 0 \qquad \text{by (a)}$$

$$\therefore \quad \iint_S \frac{\hat{n}\cdot\overline{r}}{r^3}\,ds = -\iint_{S_1} \frac{\hat{n}\cdot\overline{r}}{r^3}\,ds$$

Now on S_1, $r = a$, $\hat{n} = -\frac{\overline{r}}{a}$

$$\frac{\hat{n}\cdot\overline{r}}{r^3} = \frac{\left(-\overline{r}/a\right)\cdot\overline{r}}{a^3} = -\frac{r^2}{a^4} = -\frac{1}{a^2}$$

$$\therefore \quad \iint_S \frac{\hat{n}\cdot\overline{r}}{r^3}\,ds = -\iint_{S_1} -\frac{1}{a^2}\,ds = \frac{1}{a^2}\iint_{S_1} ds$$

$$= \frac{1}{a^2}\,[\text{Surface area of sphere of radius a}]$$

$$= \frac{1}{a^2}\,[4\pi a^2] = 4\pi$$

Example 13.87 : *Evaluate* $\iint_S \bar{F} \cdot d\bar{s}$ *where* $\bar{F} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ *and S is the surface of cylinder* $x^2 + y^2 = 16$ *included in the first octant between z = 0 and z = 4.*

Solution : The above integral is difficult to solve using G.D.T.

$\therefore$ We evaluate the integral directly.

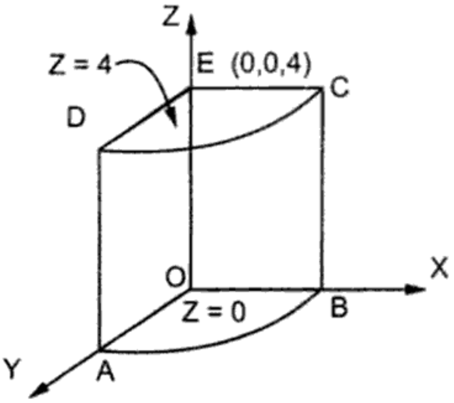


Fig. 13.58

Step 1 :

Let
$$\phi \equiv x^2 + y^2 - 16$$
$$\nabla \phi = 2x\hat{i} + 2y\hat{j}$$
$$\therefore \quad \hat{N} = \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{x^2 + y^2}} = \frac{x\hat{i} + y\hat{j}}{4}$$

Step 2 :

$$\therefore \quad \bar{F} \cdot \hat{N} = \left(z\hat{i} + x\hat{j} - 3y^2 z\hat{k}\right) \cdot \frac{\left(x\hat{i} + y\hat{j}\right)}{4}$$
$$= \frac{xz + xy - 0}{4}$$

Step 3 :

$$\therefore \quad \iint_S \bar{F} \cdot d\bar{s} = \iint_{ABCD} \bar{F} \cdot \hat{N}\, ds$$
$$= \frac{1}{4} \iint (xz + xy)\, ds$$

Step 4 : Use cylindrical polar co-ordinates

Put $x = \rho \cos\phi$, $y = \rho \sin\phi$, $z = z$

$$\therefore \quad ds = \rho\, d\phi\, dz = 4\, d\phi\, dz$$

Step 5 :

$$\iint_S \bar{F} \cdot d\bar{s} = \iint_S \frac{1}{4}(zx + xy)\, ds$$
$$= \frac{1}{4} \int_{\phi=0}^{\pi/2} \int_{z=0}^{4} (z\, 4\, \text{xos}\,\phi + 4\cos\phi \cdot 4\sin\phi)\, 4\, d\phi\, dz$$

... $\because$ surface in first octant between z = 0 to z = 4

$$= 4 \int_0^{\pi/2} d\phi \int_0^4 (z\cos\phi + 4\cos\phi\sin\phi)\, dz$$

$$= 4\int_0^{\pi/2} d\phi \left[\frac{z^2}{2}\cos\phi + 4\,z\cos\phi\sin\phi\right]_0^4$$

$$= 4\int_0^{\pi/2} (8\cos\phi + 16\cos\phi\sin\phi - 0)\,d\phi$$

$$= 4\left[8(1) + 16\,\frac{1}{(1+1)}\right] = 64$$

Example 13.88 : *Evaluate* $\iint\limits_S (z^2 - x)\,dy\,dz - xy\,dx\,dz + 3z\,dx\,dy$ *where S is the closed surface of the region bounded by* $z = 4 - y^2$, *planes* $x = 0$, $x = 3$ *and* $z = 0$. **[May-2004]**

Solution : $z = 4 - y^2$ is a parabolic cylinder.

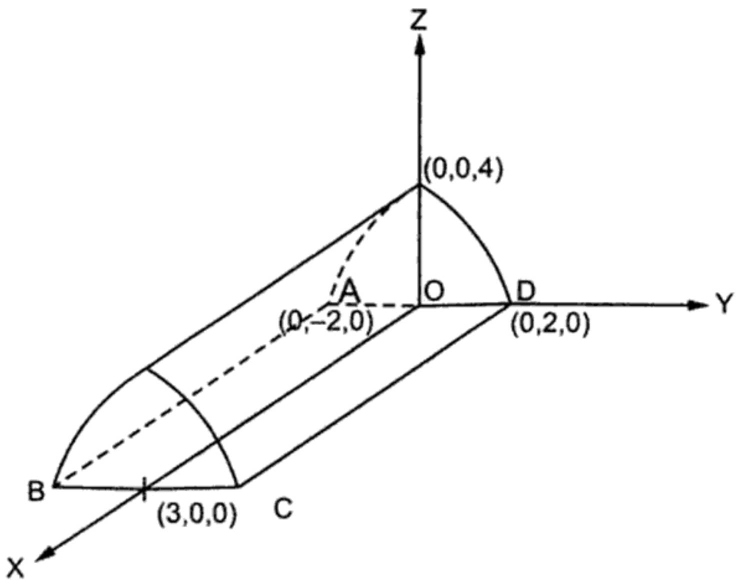


Fig. 13.59

Step 1 : i) Having generators parallel to X-axis (since variable x is absent in the equation).

ii) Symmetric about XOZ plane. (The parabola $z = 4 - y^2$ being symmetric about Z axis)

Step 2 : We have

$$\bar{F} = (z^2 - x)\,\hat{i} - x\,y\,\hat{j} + 3\,z\,\hat{k} \quad \text{... (1)}$$

$$\therefore \quad \nabla\cdot\bar{F} = \frac{\partial}{\partial x}(z^2 - x) + \frac{\partial}{\partial y}(-xy) + \frac{\partial}{\partial z}(3z) = -1 - x + 3 \quad \text{... using (1)}$$

$$\therefore \quad \nabla\cdot\bar{F} = 2 - x$$

Step 3 : $\iiint\limits_V \nabla\cdot\bar{F}\,dv = \iiint\limits_V (2 - x)\,dx\,dy\,dz$

S_1 = surface of paraboloid and S_2 = surface of circle in XOY plane.

$$\therefore \iint_{S_1} \bar{F}\cdot d\bar{s} + \iint_{S_2} \bar{F}\cdot d\bar{s} = \iiint_V \left(\nabla\cdot\bar{F}\right) dv$$

$$\therefore \qquad \iint_{S_1} \bar{F}\cdot ds = \iiint_V \left(\nabla\cdot\bar{F}\right) dv - \iint_{S_2} \bar{F}\cdot\hat{N}\, ds \qquad \text{... (i)}$$

Step 4 :

As $\qquad \bar{F} = x^2\,\hat{i} + 2\,x\,y\,\hat{j} + 3\,z^2\,\hat{k}$

$\therefore \qquad \nabla\cdot\bar{F} = 2x + 2x + 6z$

$$\therefore \qquad \iiint \nabla\cdot\bar{F}\, dv = \iiint (4x+6z)\, dx\, dy\, dz$$

Step 5 : Integrate w.r.t. z, z-varies from $z = 0$ to $z = 4 - x^2 - y^2$

$$= \int\int\int_0^{4-x^2-y^2} (4x+6z)\, dz\, dy\, dx$$

$$= \iint \left[4\,x\,z + 3\,z^2\right]_0^{4-x^2-y^2} dy\, dx$$

$$= \iint \left[4x\left(4-x^2-y^2\right) + 3\left(4-x^2-y^2\right)^2\right] dx\, dy$$

Step 6 : Integrate over the base circle $x^2+y^2=4$ put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$$\therefore \int_0^{2\pi}\int_0^{2} \left[4\,r\cos\theta\left(4-r^2\right) + 3\left(4-r^2\right)^2\right] r\, dr\, d\theta$$

Step 7 : Separate the limits

$$= \int_0^{2\pi}\int_0^{2} 4\cos\theta\left(4-r^2\right) r\, dr\, d\theta + \int_0^{2\pi}\int_0^{2} 3\left(4-r^2\right)^2 \cdot r\, dr\, d\theta$$

Step 8 : As $\int_0^{2\pi} \cos\theta\, d\theta = 0$ $\quad\therefore$ The first integral is zero

$$\therefore \qquad \iiint_V \left(\nabla\cdot\bar{F}\right) dv = \int_0^{2\pi} d\theta \cdot \int_0^{2} \frac{3}{-2}\left(4-r^2\right)^2 (-2r)\, dr$$

$$= [\theta]_0^{2\pi}\left(-\frac{3}{2}\right)\left[\frac{\left(4-r^2\right)^3}{3}\right]_0^2$$

$$= 2\pi\left(-\frac{3}{2}\right)\left[0 - \frac{4^3}{3}\right]$$

$$= \pi\cdot 4^3 = 64\pi$$

$$\nabla \phi = \frac{2x\hat{i}}{a^2} + \frac{2y\hat{j}}{b^2} + \frac{2z\hat{k}}{c^2}$$

$$\therefore \quad \hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{\frac{x}{a^2}\hat{i} + \frac{y}{b^2}\hat{j} + \frac{z}{c^2}\hat{k}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}$$

Step 5 : If $\bar{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ then

$$\iint p\left(\frac{x^4}{a^2} + \frac{y^4}{b^2} + \frac{z^4}{c^2}\right) ds = \iint \bar{F}\cdot\hat{n}\, ds \qquad \therefore \text{ by G.D. theorem}$$

$$= \iiint \nabla\cdot\bar{F}\, dV$$

$$= 3\iiint_V (x^2 + y^2 + z^2)\, dx\, dy\, dz$$

Step 6 : Over the volume of ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Step 7 : Consider $\iiint x^2\, dx\, dy\, dz$

over $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Step 8 : Use elliptical polar co-ordinates.

Put

$$x = a\, r \sin\theta \cos\phi$$

$$y = b\, r \sin\theta \sin\phi$$

$$z = c\, r \cos\theta$$

$$dx\, dy\, dz = abc\, r^2 \sin\theta\, dr\, d\phi\, d\theta$$

and use the standard limits.

Step 9 : $\int_0^{\pi}\int_0^{2\pi}\int_0^{1} a^2 r^2 \sin^2\theta \cos^2\phi \cdot abc\, r^2 \sin\theta\, dr\, d\phi\, d\theta$

$$a^3 bc \int_0^{\pi} \sin^3\theta\, d\theta \int_0^{2\pi} \cos^2\phi\, d\phi \int_0^{1} r^4\, dr$$

Step 10 : Use conversion formula

$$a^3 bc \cdot 2\int_0^{\pi/2} \sin^3\theta\, d\theta \cdot 4\int_0^{\pi/2} \cos^2\phi\, d\phi \left(\frac{r^5}{5}\right)_0^1$$

Step 11 : Use reduction formula

$$a^3 \cdot b \cdot c \cdot 2\left(\frac{2}{3}\right) 4\left(\frac{1}{2}\frac{\pi}{2}\right)\frac{1}{5}$$

$$= \frac{4\pi a^3 bc}{15}$$

Step 12 : Similarly we can get

$$\iiint y^2 \, dx\, dy\, dz = \frac{4\pi a b^3 c}{15}$$

$$\iiint z^2 \, dx\, dy\, dz = \frac{4\pi a b c^3}{15}$$

Step 13 : Substituting we get

$$I = 3\left[\frac{4\pi a^3 bc}{15} + \frac{4\pi a b^3 c}{15} + \frac{4\pi a b c^3}{15}\right]$$

$$= \frac{4\pi abc}{5}\left(a^2 + b^2 + c^2\right)$$

Exercise 13.4

1. *Verify divergence theorem for* $\overline{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ *taken over the volume bounded by* $x^2 + y^2 = 4$, $z = 0$ *and* $z = 2$. **[Ans. : Common value = 84 π]**
2. *Verify divergence theorem for* $\overline{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ *and s, the surface of the cube bounded by* $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$. **[Ans. : Common value = 24]**
3. *Verify divergence theorem for* $\overline{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$, *where V is the volume of the cone* $z = \sqrt{x^2 + y^2}$, $z = 4$. **[Ans. : Common value = 0]**
4. *Evaluate* $\iint_S \bar{r} \cdot \hat{n}\, ds$ *where S is a closed surface.*
5. *Prove that* $\iiint_V \nabla \times \overline{F}\, dv = \iint_S \hat{n} \times \overline{F}\, ds$
6. *Evaluate* $\iint_S \bar{r} \cdot \hat{n}\, ds$ *where*

   a) *S is the sphere of radius 2.* **[Ans. : $\frac{32\pi}{3}$]**

   b) *S is the surface of the cube bounded by* $x = -1, y = -1, z = -1, x = 1, y = 1, z = 1$. **[Ans. : 8]**

   c) *S is the surface of the paraboloid* $z = 4 - (x^2 + y^2)$ *and XY-plane.*
7. *Evaluate* $\iint_S 2x^2y\, dy\, dz - y^2 dz\, dx + 4xz^2 dx\, dy$ *over the curved surface of the cylinder* $y^2 + z^2 = 9$ *bounded by* $x = 0$ *and* $x = 2$. **[Ans. : 0]**

23. *Evaluate* $\iint_S \left[2xy^2\hat{i}+\left(xz^2-y^2\right)\hat{j}+z^3\hat{k}\right]\cdot d\bar{s}$ *over the cube bounded by* $x=0,\ x=2,\ y=0,\ y=2,$ $z=0,\ z=2.$ **[Ans. : 64/3]**

24. *Evaluate* $\iint_S \left(3x^2y\hat{i}+xy^2\hat{j}\right)\cdot d\bar{s}$ *over a closed surface of a triangular prism of unit length in the direction of z axis whose base is bounded by positive X and Y axes and the line* $x+y=1,\ z=0.$ **[Ans. : 1/3]**

25. *Evaluate* $\iint \left(x\hat{i}+2y\hat{j}+z^3\hat{k}\right)\cdot d\bar{s}$ *over the curved surface of cylinder* $x^2+y^2=9,\ z=0,\ z=3.$ **[Ans. : 81 π]**

26. *Evaluate* $\iint \bar{F}\cdot d\bar{s}$ *where* $\bar{F}=2y\hat{i}-z\hat{j}+x^2\hat{k}$ *over the surface of* $y^2=8x$ *is positive octant bounded by* $x=4$ *and* $z=6.$ **[Ans. : 132]**

27. *Evaluate* $\iint \bar{F}\cdot d\bar{s}$ *where* $\bar{F}=x^2\hat{i}-2y^2\hat{j}+42\hat{k}$ *and S is the surface bounded by* $y^2+z^2=4,\ x=0,$ $x=3.$ **[Ans. : 84 π]**

28. *Prove that* $\iiint_V \frac{1}{r^2}\,dv=\iint_S \frac{\bar{r}}{r^2}\cdot d\bar{s}.$

University Questions

Dec. - 98

1. *Use divergence theorem to evaluate :*

$$\iint_S \left[y^2z^2\,\hat{i}+z^2x^2\hat{j}+x^2y^2\hat{k}\right]d\bar{S}$$

where S is the upper part of the sphere $x^2+y^2+z^2=9$ *above the XOY plane.* **[8 Marks]**

2. *Apply Stoke's theorem to calculate :*

$$\int_C \left(4y\,dx+2z\,dy+6y\,dz\right)$$

where C is the curve of intersection of surface $x^2+y^2+z^2=6z$ *and the plane* $z=x+3.$ **[7 Marks]**

May - 99

1. *Evaluate :*

$$\oint_C \bar{F}\cdot d\bar{r}\quad \text{where}\quad \bar{F}=\frac{1}{x^2+y^2}\left(-y\bar{i}+x\bar{j}\right)$$

and 'C' is any closed curve containing the origin. **[7 Marks]**

2. *Verify Stokes theorem for* $\bar{F}=xz\bar{i}-y\bar{j}+x^2y\bar{k}$, *where S is the surface of the region bounded by* $y=0,\ z=0,\ 3x+y+3z=6$ *which is not included in the yz-plane.* **[7 Marks]**

3. *Evaluate :* $\iint_S \left(lx^2+my^2+nz^2\right)dS$

where S is the surface of the sphere

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2$$

l, m, n being the direction cosines of the outward drawn normal to the surface. **[8 Marks]**

Dec. - 99

1. *Evaluate :* $\iint_S xz^2\,dy\,dz + \left(x^2 y - z^2\right)dz\,dx + \left(2xy + y^2 x\right)dx\,dy,$

   where S is the surface enclosing a region bounded by hemisphere $x^2 + y^2 + z^2 = 4$ *above the XOY plane.* **[6 Marks]**

2. *Find the work done in moving particle from* $(0,1,-1)$ *to* $(\pi/2,-1,2)$ *in a force.*

   $\bar{F} = \left(y^2 \cos x + z^3\right)\bar{i} + (2y\sin x - 4)\,\bar{j} + \left(3xz^2 + 2\right)\bar{k}$ **[6 Marks]**

3. *Apply Stoke's theorem to calculate -*

   $\int_C 4y\,dx + 2z\,dy + 6y\,dz,$

   where C is the curve of intersection of $x^2 + y^2 + z^2 = 6z$ *and* $z = x + 3.$ **[7 Marks]**

May - 2000

1. *Find the work done in moving a particle along :*

   $x = a\cos\theta,\ \ y = a\sin\theta,\ \ z = b\,\theta$ *from* $\theta = \frac{\pi}{4}$ *to* $\theta = \frac{\pi}{2}$ *under a field of force given by :*

   $\bar{F} = -3a\sin^2\theta\cos\theta\,\bar{i} + a\left(2\sin\theta - 3\sin^3\theta\right)\bar{j} + b\sin 2\theta\,\bar{k}$ **[8 Marks]**

2. *Evaluate using Stokes theorem :* $\int_C \left(y\,dx + z\,dy + z\,dz\right)$ *C being intersection of*

   $x^2 + y^2 + z^2 = a^2,\ \ x + z = a.$ **[6 Marks]**

3. *Evaluate :* $\iint_S \left(l x^2 + m y^2 + n z^2\right) ds$ *where S is the surface of the sphere :*

   $(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2\ l,m,n$ *being the direction cosines of the outward drawn normal to the surface.* **[8 Marks]**

Dec. - 2000

1. *Evaluate :* $\iint_S \bar{F}\cdot d\bar{S}$

   where $\bar{F} = y^2 z^2\,\bar{i} + z^2 x^2\,\bar{j} + x^2 y^2\,\bar{k}$

   and S is the part of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ *which lies in the first octant.* **[8 Marks]**

2. *Verify Stokes theorem for* $\bar{F} = -y^3\,\bar{i} + x^3\,\bar{j}$ *and the closed curve 'C' is the boundary of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ **[7 Marks]**

May - 2001

1. *Evaluate* $\iint_S \left(x^3\,\bar{i} + y^3\,\bar{j} + z^3\,\bar{k}\right)\cdot d\bar{S}$

   where S is the surface of the sphere $x^2 + y^2 + z^2 = 16$ *(Use Divergence theorem).* **[8 Marks]**

2. *Apply Stoke's theorem to evaluate*

   $\int_C 4y\,dx + 2z\,dy + 6y\,dz$

   where C is curve of intersection of $x^2 + y^2 + z^2 = 6z$ *and* $z = x + 3.$ **[8 Marks]**

Dec. - 2003

1. *Find the work done in moving a particle along* $X = a \cos \theta$, $y = a \sin \theta$, $z = b\,\theta$ *from:* $\theta = \frac{\pi}{4}$ *to* $\theta = \frac{\pi}{2}$ *under a field of force given by*

   $\overline{F} = \left(-3a \sin^2 \theta \cos\theta\right) i + a\left(2\sin\theta - 3\sin^3\theta\right) j + \left(b \sin 2\theta\right) k.$ **[8 Marks]**

2. *Verify Stoke's theorem for* $\overline{F} = (2x - y) i - yz^2 j - y^2 z k$ *over the surface of hemisphere* $x^2 + y^2 + z^2 = 1$, $z \geq 0$. **[8 Marks]**

3. *Evaluate* $\iint_S \left(lx^2 + my^2 + nz^2\right) ds$ *where S is the surface of the sphere*

   $(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2$

   l, m, n being the direction cosines of the outward normal to the surface. **[8 Marks]**

May - 2004

1. *Evaluate :*

   $\iint_S \left[\left(x + y^2\right) i - 2xj + 2yz k\right] \cdot d\overline{S}$

   where S : $2x + y + 2z = 6$. **[5 Marks]**

2. *Verify Green's theorem in the plane x o y for*

   $\oint_C \left[\left(3x^2 - 8y^2\right) i + (4y - 6xy) j\right] \cdot d\overline{r},$

   where C is the boundary of the region defined by

   $y = \sqrt{x}$, $y = x^2$. **[6 Marks]**

3. *Verify Stoke's theorem for*

   $\overline{F} = yi + zj + xk$

   over the upper half surface $z = 1 - x^2 - y^2$ *bounded by its projection on x o y plane.* **[8 Marks]**

Dec. - 2004

1. *Find the work done in moving a particle once round the ellipse* $\frac{x^2}{25} + \frac{y^2}{16} = 1$, $z = 0$ *under the field of force given by,*

   $\overline{F} = (2x - y + z)\overline{i} + \left(x + y - z^2\right)\overline{j} + (3x - 2y + 4z)\overline{k}$ *Is the field conservative ?* **[8 Marks]**

2. *Verify Stoke's theorem for,*

   $\overline{F} = (y - z + 2)\overline{i} + (yz + 4)\overline{j} - xz\overline{k}$

   over the surface of a cube x = 0, y = 0, z = 0, x = 2, z = 2 above XOY plane (open at the bottom). **[8 Marks]**

3. *Use Gauss-Divergence theorem to evaluate*

   $\iint_S \left(y^2 z^2 \overline{i} + z^2 x^2 \overline{j} + x^2 y^2 \overline{k}\right) \cdot d\overline{s}$ *where S is the upper part of the sphere* $x^2 + y^2 + z^2 = 9$ *above XOY plane.* **[8 Marks]**

May - 2005

1. *A vector field is given by*

$\bar{F} = \sin y \bar{i} + x(1 + \cos y)\bar{j}$

Evaluate :

$\int_C \bar{F} \cdot d\bar{r}$

Where C is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

2. *Use the divergence theorem to evaluate :*

$\iint_S \left(y^2z^2\bar{i} + z^2x^2\bar{j} + x^2y^2\bar{k}\right) \cdot d\bar{s}$

where S is the upper part of the sphere $x^2 + y^2 + z^2 = 9$ *at x 0 y plane.*

3. *Verify Stoke's theorem for*

$\bar{F} = xy^2\bar{i} + y\bar{j} + z^2x\bar{k}$

for the surface of rectangular lamina bounded by $x = 0,\ yx = 1,\ y = 2,\ z = 0.$

4. *Verify Green's theorem for*

$\bar{F} = x^2\bar{i} + xy\,\bar{j}$

over the region R enclosed by $y = x^2$ *anf the line* $y = x.$

Dec. - 2005

1. *Find the work done in moving a particle along* $x = a\cos\theta,\ y = a\sin\theta,\ z = b\theta$ *from* $= \frac{\pi}{4}$ *to* $\theta = \frac{\pi}{2}$ *under a field of force given by :*

$\bar{F} = -3a\sin^2\theta\cos\theta\bar{i} + a\left(2\sin\theta - 3\sin^3\theta\right)\bar{j} + b\sin 2\theta\,\bar{k}.$ **[6 Marks]**

2. *Use the divergence theorem to evaluate :*

$\iint_S \left(y^2z^2\bar{i} + z^2x^2\bar{j} + x^2y^2\bar{k}\right) \cdot d\bar{S}$

where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ *in the positive octant.* **[6 Marks]**

3. *Evaluate :*

$\iint_S 2x^2y\,dydz - y^2\,dzdx + 4xz^2\,dxdy$

over the curved surface of the cylinder $y^2 + z^2 = a^2$, *bounded by* $x = 0$ *and* $x = h.$ **[6 Marks]**

4. *Evaluate :*

$\iint_S \left(\nabla \times \bar{F}\right) \cdot \hat{n}\,dS$

where :

$\bar{F} = (x - y)\bar{i} + \left(x^2 + yz\right)\bar{j} - 3xy^2\bar{k}$

and 'S' is the surface of the cone $z = 4 - \sqrt{x^2 + y^2}$ *above xoy plane.* **[6 Marks]**

14 Applications to Fluid Mechanics

14.1 Introduction

Vector calculus has a great utility in Hydrodynamics, which deals with the motion of fluids. Many difficult equations in Fluid Mechanics can be represented easily in compact vector form. We shall discuss here only the fundamental laws such as equation of continuity, Euler's equation of motion, equation of stream lines etc. We shall restrict to ideal fluids i.e. non-viscous and perfect fluids.

14.2 Stream Lines

Definition :

A stream line or line of flow of a fluid is a curve through the fluid such that at any instant of time, the tangent at any point of it, is along the velocity vector at that point.

Fluid flow is often represented by stream lines. Stream lines indicates the direction of motion at every instant.

With the help of equation of stream lines we can determine the pattern of flow of fluid at every instant, as the stream lines continuously changes with time, except in case of steady motion.

To find the equation of stream line consider an imaginary curve 'c' drawn in the fluid (Fig. 14.1) Consider any moving particle along the curve 'c' let P and Q be the positions of this particle at time t and $t+\delta t$.

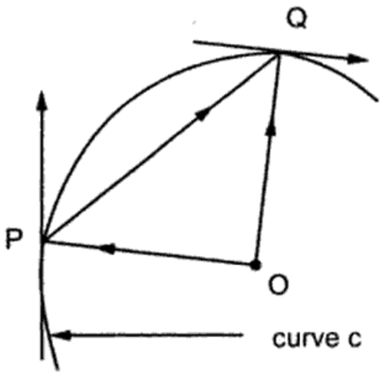


Fig. 14.1

14.3 Illustrations on Stream Lines

Example 14.1 : *Obtain the equation of stream lines in case of steady motion of fluid defined of* $\bar{q} = (x^2 + y^2)\hat{i} + 2xy\hat{j} + (x+y)^2\hat{k}$

Solution : The differential equations of stream lines are

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)^3 z}$$

Using property of ratio and proportion

$$\text{each ratio} = \frac{dx+dy}{x^2+y^2+2xy} = \frac{dx+dy}{(x+y)^2}$$

Also $$\text{each ratio} = \frac{dx-dy}{x^2+y^2-2xy} = \frac{dx-dy}{(x-y)^2}$$

equating we get

$$\frac{dx+dy}{(x+y)^2} = \frac{dx-dy}{(x-y)^2}$$

Let $x + y = u$, $x - y = v$

$\therefore$ $dx + dy = du$, $dx - dy = dv$

$$\therefore \quad \frac{du}{u^2} = \frac{dv}{v^2}$$

Integrating we get

$$\frac{-1}{u} = \frac{-1}{v} + c$$

$$\therefore \quad \frac{1}{x+y} = \frac{1}{x-y} - c$$

$$\therefore \quad \frac{1}{x+y} = \frac{1}{x-y} + c_1 \qquad \text{... (1)}$$

Also $$\frac{dx+dy}{(x+y)^2} = \frac{dz}{(x+y)^3 z}$$

i.e. $$\frac{dy}{u^2} = \frac{dz}{u^3 z}$$

$$\therefore \quad u\,du = \frac{dz}{z}$$

Integrating we get

$$\frac{u^2}{2} - \log z = c_2$$

$$\frac{(x+y)^2}{2} - \log z = c_2 \qquad \dots (2)$$

Thus the stream lines are curves of intersection of surfaces (1) and (2).

Example 14.2 : *Velocity distribution for a fluid is given by $u = -x$, $v = 2y$, $w = 3 - z$ find the equation of stream line passing through the point (1, 1, 2).*

Solution : The differential equation of stream line is

$$\frac{dx}{u} = \frac{du}{v} = \frac{dz}{w}$$

i.e.
$$\frac{dx}{-x} = \frac{dy}{2y} = \frac{dz}{3-z}$$

By combinations $\frac{dx}{-x} = \frac{dy}{2y}$

Integrating

$$\log x + \frac{1}{2}\log y = \log c_1$$

$$x\sqrt{y} = c_1$$

As the stream line passes through (1, 1, 2)

$$\therefore \quad 1\sqrt{1} = c_1$$

$$\Rightarrow \quad c_1 = 1$$

$$\therefore \quad x\sqrt{y} = 1 \qquad \dots (1)$$

Also by combinations

$$\frac{dx}{-x} = \frac{dz}{3-z}$$

$$-\log x = -\log(3-z) + \log c$$

$$\log x = \log(3-z) - \log c$$

$$\log x - \log(3-z) = -\log c$$

$$\log \frac{x}{3-z} = \log c_2$$

$$\frac{x}{3-z} = c_2$$

$$= \frac{xdx + ydy + zdz}{2z(x^2 + y^2 + z^2)}$$

$$\therefore \quad \frac{dx}{3xz} = \frac{xdx + ydy + zdz}{2z(x^2 + y^2 + z^2)}$$

$$\frac{2dx}{x} = \frac{3(xdx + ydy + zdz)}{(x^2 + y^2 + z^2)}$$

Put $\quad x^2 + y^2 + z^2 = u$

$$2xdx + 2ydy + 2zdz = du$$

$$\therefore \quad xdx + ydy + zdz = \frac{du}{2}$$

$$\therefore \quad \frac{2\,dx}{x} = \frac{3(du/2)}{u}$$

$$\frac{4\,dx}{x} = \frac{3\,du}{u}$$

Integrating we get

$$4 \log x = 3 \log u + \log c_2$$

$$\log x^4 = \log c_2 \cdot u^3$$

$$x^4 = c_2 \left(x^2 + y^2 + z^2\right)^3 \qquad \text{... (2)}$$

Thus the stream lines are curves of intersection of surfaces (1) and (2).

Example 14.4 : *Obtain the equation of stream lines in case of steady motion of fluid defined by* $\bar{q} = (y - xz)\hat{i} + (yz + x)\hat{j} + (x^2 + y^2)\hat{k}$.

Solution : The differential equation of stream lines are

$$\frac{dx}{y - xz} = \frac{dy}{x + yz} = \frac{dz}{x^2 + y^2}$$

Now by property of ratio and proportion

$$\text{each ratio} = \frac{ydx + xdy}{2y^2 - xyz + x^2 + xyz}$$

$$= \frac{ydx + xdy}{x^2 + y^2}$$

$$= \frac{d(xy)}{x^2 + y^2}$$

which is called as equation of continuity.

Also $$\nabla \cdot \rho \bar{q} = \nabla\rho \cdot \bar{q} + \rho(\nabla \cdot \bar{q})$$

$\therefore$ equation of continuity becomes

$$\boxed{\frac{\partial p}{\partial t} + \nabla\rho \cdot \bar{q} + p \cdot (\nabla\,\bar{q}) = 0} \quad \text{... (4)}$$

If the fluid is incompressible then ρ = constant, the equation of continuity becomes

$$\nabla \cdot (\rho\,\bar{q}) = 0$$

i.e. $$\rho[\nabla \cdot \bar{q}] = 0$$

i.e. $$\boxed{\nabla \cdot \bar{q} = 0} \quad \text{... (4)}$$

Note : If the motion is steady then the density does not change w.r.t. time i.e. $\frac{\partial \rho}{\partial t} = 0$

14.6 Irrotational Motion-The Scalar Velocity Potential

If $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$ then $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$ represents the equations of stream lines.

The fluid motion is said to be irrotational if $\nabla \times \bar{q} = 0$.

For irrotational fluid motion we can find a scalar potential ϕ such that

$$\bar{q} = -\nabla\phi$$

where ϕ is known as velocity potential

$$\therefore \quad ui + vj + wk = -\left[i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}\right]$$

$$\therefore \quad u = -\frac{\partial \phi}{\partial x},\ v = -\frac{\partial \phi}{\partial y},\ w = -\frac{\partial \phi}{\partial z}$$

Note :

i) In this case the equation of continuity $\nabla \cdot \bar{q} = 0$ becomes

$$\nabla \cdot (-\nabla\phi) = 0$$

i.e. $$\nabla^2\phi = 0$$

which is called as Laplace's equation.

$\therefore$ The velocity potential ϕ satisfies Laplace's equation.

ii) As $\bar{q} = ui + vj + wk$

$$\nabla \cdot \bar{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$\therefore$ Hence the equation of continuity in the cartesian form is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0$$

iii) As $\bar{q} = -\nabla\phi$ and $\nabla\phi$, by definition of gradient $\nabla\phi$ is perpendicular to the surfaces $\phi = c$, the stream lines intersect equipotential surfaces [i.e. ϕ = constant] orthogonally.

14.7 Acceleration of a Fluid Particle

The velocity $\bar{q}$ of a fluid particle depends upon its position i.e. (x, y, z) and time t.

$\therefore$ $\bar{q}$ is a function of x, y, z, t.

Hence $$\bar{a} = \frac{d\bar{q}}{dt}$$

$$= \frac{\partial\bar{q}}{\partial x}\frac{dx}{dt}+\frac{\partial\bar{q}}{\partial y}\frac{dy}{dt}+\frac{\partial\bar{q}}{\partial z}\frac{dz}{dt}+\frac{\partial\bar{q}}{\partial t}\frac{dt}{dt}$$

$$= \left[\frac{dx}{dt}\frac{\partial\bar{q}}{\partial x}+\frac{dy}{dt}\frac{\partial\bar{q}}{\partial y}+\frac{dz}{dt}\frac{\partial\bar{q}}{\partial z}\right]+\frac{\partial\bar{q}}{\partial t}$$

$$= \left[\frac{dx}{dt}\frac{\partial}{\partial x}+\frac{dy}{dt}\frac{\partial}{\partial y}+\frac{dz}{dt}\frac{\partial}{\partial z}\right]\bar{q}+\frac{\partial\bar{q}}{\partial t}$$

since $$\bar{q} = u\hat{i}+v\hat{j}+w\hat{k} = \frac{dx}{dt}\hat{i}+\frac{dy}{dt}\hat{j}+\frac{dz}{dt}\hat{k}$$

$\therefore$ $$\bar{a} = \left[u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right]\bar{q}+\frac{\partial\bar{q}}{\partial t}$$

$$\bar{a} = \left[\left(u\hat{i}+v\hat{j}+w\hat{k}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\right]\bar{q}+\frac{\partial\bar{q}}{\partial t}$$

i.e. $$\bar{a} = [\bar{q}\cdot\nabla]\bar{q}+\frac{\partial\bar{q}}{\partial t}$$

Thus the acceleration of the fluid particle is given by

$$\boxed{\bar{a} = (\bar{q}\cdot\nabla)\,\bar{q}+\frac{\partial\bar{q}}{\partial t}}$$

If $$\bar{a} = a_x\hat{i}+a_y\hat{j}+a_z\hat{k} \quad \text{then}$$

$$a_x\hat{i}+a_y\hat{j}+a_z\hat{k} = \left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)\left(u\hat{i}+v\hat{j}+w\hat{k}\right)+\frac{\partial}{\partial t}\left(u\hat{i}+v\hat{j}+w\hat{k}\right)$$

Equating the components we get

$$a_x = u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial w}{\partial t}$$

which gives the components of acceleration in scalar form.

14.8 Illustrations

Example 14.5 : *Given the velocity field $\bar{q}=(6+2yx+t^2)\hat{i}-(xy^2+10t)\hat{j}+25\hat{k}$ what is the acceleration of a particle at (3, 0, 2) at time t = 1.*

Solution : As $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$

$\therefore\ u = 6+2xy+t^2,\ v = -10xy^2-20\,t,\ w = 25$

Find the partial derivatives

$$\frac{\partial u}{\partial x} = 2y \qquad \frac{\partial v}{\partial x} = -10y^2 \qquad \frac{\partial w}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 2x \qquad \frac{\partial v}{\partial y} = -20\,xy \qquad \frac{\partial w}{\partial y} = 0$$

$$\frac{\partial u}{\partial z} = 0 \qquad \frac{\partial v}{\partial z} = 0 \qquad \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} = 2t \qquad \frac{\partial v}{\partial t} = -20 \qquad \frac{\partial w}{\partial t} = 0$$

Substituting (x, y, z) = (3, 0, 2) and t = 1 we get u = 6 + 0 + 1, v = – 0 – 10, w = 25

$$\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial y} = 6, \qquad \frac{\partial u}{\partial z} = 0, \qquad \frac{\partial u}{\partial t} = 2$$

$$\frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial v}{\partial z} = 0, \qquad \frac{\partial v}{\partial t} = -10$$

$$\frac{\partial w}{\partial x} = 0, \qquad \frac{\partial w}{\partial y} = 0 \qquad \frac{\partial w}{\partial z} = 0, \qquad \frac{\partial w}{\partial t} = 0$$

The acceleration components are given by

$$a_x = u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Substituting all the above calculated values we get

$$a_x = (7)(0) + (-10)(6) + (25)(0) + (2) = -58$$

$$a_y = (7)(0) + (-10)(0) + (25)(0) + (-10) = -10$$

$$a_z = (7)(0) + (-10)(0) + (25)(0) + (0) = 0$$

$$\therefore \quad |\bar{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{(58)^2 + (10)^2 + 0}$$

$$= 58.86 \text{ units}$$

14.9 Euler's Equation of Motion

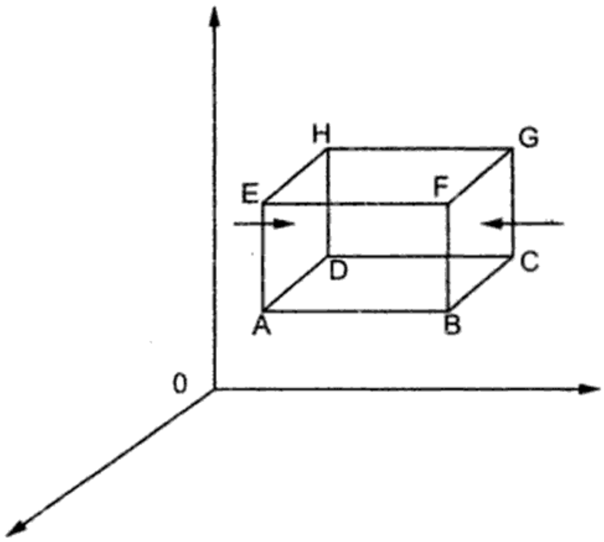


Fig. 14.3

Consider the volume element ABCDEFGH as shown in Fig. 14.3 whose edges are $\delta x, \delta y, \delta z$.

$\therefore$ Element of mass = $\delta m = \rho \cdot \delta x \cdot \delta y \cdot \delta z$

This infinitesimal mass is acted on by the forces

i) Normal pressure thrusts on the surface of the element due to surrounding fluid.

ii) The external force say $\bar{F}$ per unit mass.

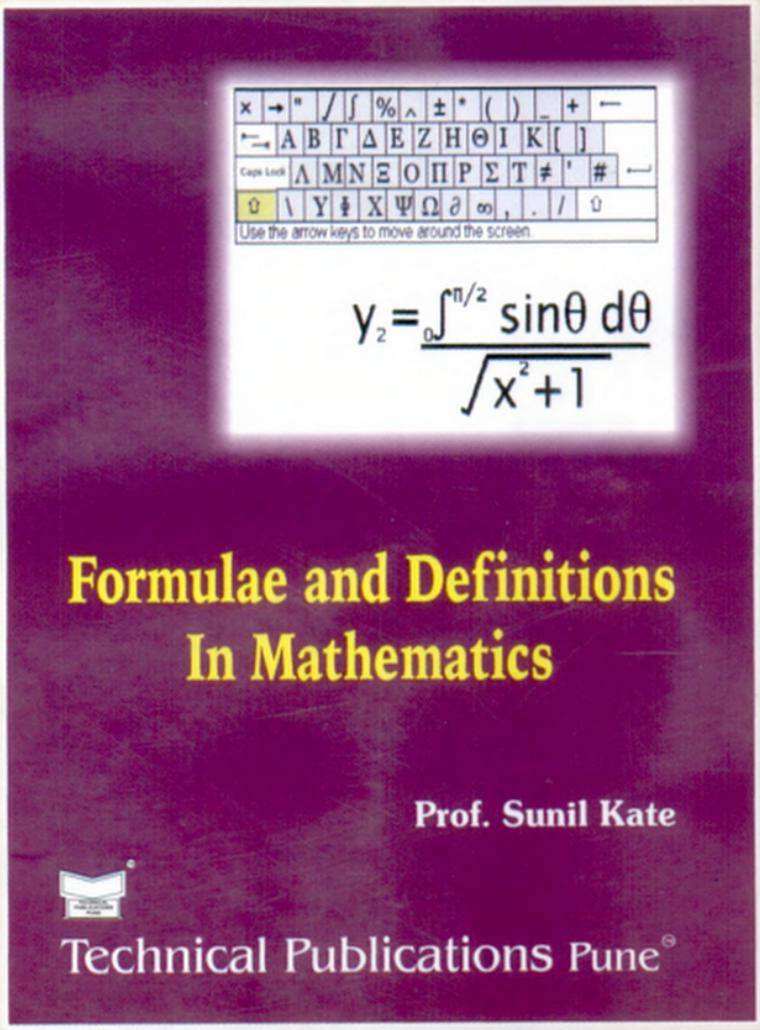
Let $p = p(x, y, z)$ be the pressure at the point A.

$\therefore$ Force due to pressure p on the face ADHE to y axis as shown in Fig. 14.3 is

$$= p(x, y, z)\,\delta x\,\delta z\, j$$

All Important Formulae in Engineering Mathematics are collectively available in a small book.

Price Rs. 30/-



TECHNICAL PUBLICATIONS PUNE

Technical Publications Pune

1, Amit Residency, 412, Shaniwar Peth,
Pune - 411030
(020) 24495496/97, Email : technical@vtubooks.com
Visit us at : www.vtubooks.com

Rs. 500/-
ISBN 81-8431-060-9
9 788184 310603